Solutions for Final Exam 380-452.

Q1

a) Expected payment with no deductible is 1000 which is the mean of a uniform ~~of~~ distribution on interval from 0 to 2000, that is, equals:

$$\int_0^{2000} x \cdot \frac{1}{2000}\, dx = \frac{1}{2000} \left.\frac{x^2}{2}\right|_0^{2000}$$

$$= 1000$$

& 25% of 1000 = 250.

Now, with deductible d, the amount paid on a loss of amount x is $Y = \begin{cases} 0 & x \leq d \\ x-d & x > d \end{cases}$

and the expected payment:

$$E[Y] = \int_d^{2000} (x-d) \frac{1}{2000}\, dx = 250$$

$$\Rightarrow \frac{(2000-d)^2}{4000} = 250$$

$$\Rightarrow \boxed{d = 1000}$$

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Q1 b) The overall expected loss is:

$$\int_0^1 x(1-x)\,dx = \frac{x^2}{2} - \frac{x^3}{3}\Big|_0^1 = \frac{1}{6}$$

with policy limit $\underline{\underline{u}}$, the expected insurance payment is $\int_0^u x \cdot f(x)\,dx + u \cdot [1 - F(u)]$. In this case,

$$F(u) = \int_0^u (1-x)\,dx = u - \frac{u^2}{2}$$

Therefore, the expected insurance payment is

$$\int_0^u x(1-x)\,dx + u\left[1 - u + \frac{u^2}{2}\right]$$

$$= \frac{u^2}{2} - \cancel{\frac{u^3}{3}} + u - u^2 + \cancel{\frac{u^3}{3}}$$

$$= u - \frac{u^2}{2}$$

In order for this to be one-half of expected total loss, we must have that:

$$u - \frac{u^2}{2} = \frac{1}{2} \cdot \frac{1}{6} = \frac{1}{12}$$

$$\Rightarrow -6u^2 + 12u - 1 = 0$$

$$\Rightarrow u^2 - 12u + 1 = 0$$

Solve by formula to get that:

$$\underline{\underline{u}} = \frac{12 \pm 10.95}{12} = 1.9125 \text{ or } 0.0875.$$

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Q2

a) $P_r\{X_0=1, X_1=0, X_2=2\}$

$$= P_1 P_{10} P_{02} = (0.3)(0.5)(0.3)$$

$$= 0.045$$

b) $P_r\{X_4=1 \mid X_0=0\}$

or $P_r\{X_5=1 \mid X_1=0$

$$= P_{01}^{4} = 0.4578$$

$$P^2 = \begin{bmatrix} 0.4 & 0.6 \\ 0.7 & 0.3 \end{bmatrix} \begin{bmatrix} 0.4 & 0.6 \\ 0.7 & 0.3 \end{bmatrix} = \begin{bmatrix} 0.58 & 0.42 \\ 0.49 & 0.51 \end{bmatrix}$$

$$P^4 = P^2 . P^2 = \begin{bmatrix} 0.58 & 0.42 \\ 0.49 & 0.51 \end{bmatrix} \begin{bmatrix} 0.58 & 0.42 \\ 0.49 & 0.51 \end{bmatrix}$$

$$= \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} 0.5422 & 0.4578 \\ 0.5341 & 0.4659 \end{bmatrix} \end{array}$$

Q3

a) $p = 0.3$ & $q = 1 - p = 0.7$

(i)

$$P = \begin{array}{cc} & \begin{array}{cccc} 0 & 1 & 2 & 3 \\ (2,0) & (1,0) & (1,1) & (0,1) \end{array} \\ \begin{array}{c} 0\ (2,0) \\ 1\ (1,0) \\ 2\ (1,1) \\ 3\ (0,1) \end{array} & \left\| \begin{array}{cccc} 0.7 & 0.3 & 0 & 0 \\ 0 & 0 & 0.7 & 0.3 \\ 0.7 & 0.3 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right\| \\ & \begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ \pi_0 & \pi_1 & \pi_2 & \pi_3 \end{array} \end{array}$$

(ii) we need to find $\underline{\underline{\pi_3}}$: The probability in the long run that neither Computer is operating.

Rule use:
$$\begin{cases} \pi_j = \sum_{k=0}^{3} \pi_k P_{kj} & j = 0,1,2,3 \\ \& \quad \sum_{k=0}^{3} \pi_k = 1 \end{cases}$$

So:

$$\pi_0 = 0.7\pi_0 + 0.7\pi_2 \implies \pi_2 = \frac{0.3}{0.7}\pi_0$$

$$\pi_1 = 0.3\,\pi_0 + 0.3\,\pi_2 \implies \pi_1 = \frac{0.3 * 0.7 + (0.3)^2}{0.7}\pi_0$$

$$\pi_2 = 0.7\pi_1 + \pi_3 \implies \pi_3 = \frac{0.3 - (0.3)(0.7)^2 - (0.3)^2(0.7)}{(0.7)}\pi_0$$

$$(*)\ \& \quad \pi_0 + \pi_1 + \pi_2 + \pi_3 = 1$$

Substitute in (*)

$$\implies \pi_0 + \frac{0.3 * 0.7 + (0.3)^2}{0.7}\pi_0 + \frac{0.3}{0.7}\pi_0 + \frac{0.3 - (0.3)(0.7)^2 - (0.3)^2(0.7)}{0.7}\pi_0 = 1$$

$$\implies \boxed{\pi_0 = \frac{0.7}{1 + 0.3 + (0.3)^2}}, \quad \pi_1 = \frac{0.3}{1 + 0.3 + (0.3)^2}, \quad \pi_2 = \frac{0.3}{1 + 0.3 + (0.3)^2}$$

but we only need $\boxed{\underline{\underline{\Pi_3}} = \dfrac{(0.3)^2}{1+0.3+(0.3)^2} \simeq 0.0647}$

& we get that

by substituting in Π_3 formula using Π_0.

(iii) The availability that at least one computer is operating in the long run $= R_3$

$$= 1 - \Pi_3$$

$$= 1 - \frac{(0.3)^2}{1+(0.3)+(0.3)^2}$$

$$= \frac{1.3}{1+(0.3)+(0.3)^2}$$

$$\simeq \boxed{0.9353}$$

P3 b) State 3 is an absorbed state, so the mean time to reach state 3 is:

(**) $V = E[T \mid X_0 = i]$ $i = 0,1,2$ the non-absorbed states

and we need to find $v = E[T \mid X_0 = 0] = \vartheta_{03}$.

Set equations using (**):

$$\vartheta_{03} = 1 + P_{00}\vartheta_{03} + P_{01}\vartheta_{13} + P_{02}\vartheta_{23} \longrightarrow (1)$$

$$\vartheta_{13} = 1 + \cancel{P_{10}}^{0}\vartheta_{03} + P_{11}\vartheta_{13} + P_{12}\vartheta_{23} \longrightarrow (2)$$

$$\vartheta_{23} = 1 + \cancel{P_{20}}^{0}\vartheta_{03} + \cancel{P_{21}}^{0}\vartheta_{13} + P_{22}\vartheta_{23} \longrightarrow (3)$$

For short, let $\vartheta_0 = \vartheta_{03}$, $\vartheta_1 = \vartheta_{13}$ & $\vartheta_2 = \vartheta_{23}$,

$$\Rightarrow \begin{cases} \vartheta_0 = 1 + 0.3\,\vartheta_0 + 0.4\,\vartheta_1 + 0.1\,\vartheta_2 \\ \vartheta_1 = 1 + 0.2\,\vartheta_1 + 0.1\,\vartheta_2 \\ \vartheta_2 = 1 + 0.1\,\vartheta_2 \end{cases}$$

Solve for $\vartheta_0 = \vartheta_{03}$:

$$10\times \quad \left.\begin{aligned} 10\vartheta_0 &= 10 + 3\vartheta_0 + 4\vartheta_1 + \vartheta_2 \\ 10\vartheta_1 &= 10 + 2\vartheta_1 + \vartheta_2 \\ 10\vartheta_2 &= 10 + \vartheta_2 \end{aligned}\right\} \Rightarrow$$

by back substitution

$$\vartheta_2 = \frac{10}{9}$$
$$\Downarrow$$
$$\vartheta_1 = \frac{25}{18}$$
$$\Downarrow$$
$$\boxed{\vartheta_0 = \frac{150}{63} \simeq 2.381}$$

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Q4

a) $P_r\left(X(s+t)-X(s)=k\right)=\dfrac{(\lambda t)^k e^{-\lambda t}}{k!}$, $k=0,1,\dots$

for a Poisson Process $X(t)$ of rate λ.

in the question $\lambda=3$.

(i) $P_r\{X(2)=3\}$

$= P_r\{\underset{s+t}{X(2)}-\underset{s}{X(0)}=\underset{k}{3}\}$ $\{$As $X(0)=0\}$

$= \dfrac{(3*2)^3 \, e^{-3*2}}{3!}$

$= \dfrac{6^3 e^{-6}}{3!} = 6^2 e^{-6} = 36\, e^{-6}$

(ii) $P_r\{X(2)=3$ and $X(4)=7\}$

$= P_r\{\underbrace{X(2)-X(0)=3,\ X(4)-X(2)=4}_{\text{indep. increments}}\}$

$= P_r\{X(2)-X(0)=3\} * P_r\{\underbrace{\underset{s+t}{X(4)}-\underset{s}{X(2)}=\underset{k}{4}}_{t=2,\ k=4,\ \lambda=3}\}$

$= 6^2 e^{-6} * \dfrac{6^4 e^{-6}}{4!}$

$= \dfrac{6^6 e^{-12}}{4!}$

Q4 a) (iii) $P_r\{X(4)=7 \mid X(2)=3\}$

$$= P_r\{\underbrace{X(4)-X(2)=4}_{} \mid \underbrace{X(2)-X(0)=3}_{\text{indep.}}\}$$

$$= P_r\{X(4)-X(2)=4\}$$

$$= \frac{6^4 e^{-6}}{4!}.$$

Q4 b) $P_0(t) = e^{-\lambda_0 t} = e^{-2t}$.

$n=1$, $P_1(t) = \lambda_0 \left[B_{0,1} e^{-\lambda_0 t} + B_{1,1} e^{-\lambda_1 t}\right]$

$$B_{0,1} = \frac{1}{\lambda_1-\lambda_0} = \frac{1}{4-2} = \frac{1}{2}$$

$$B_{1,1} = \frac{1}{\lambda_0-\lambda_1} = -\frac{1}{2}$$

$$\Rightarrow P_1(t) = 2\left[\frac{1}{2}e^{-2t} - \frac{1}{2}e^{-4t}\right] = e^{-2t} - e^{-4t}$$

$n=2$, $P_2(t) = \lambda_0\lambda_1\left[B_{0,2}e^{-\lambda_0 t} + B_{1,2}e^{-\lambda_1 t} + B_{2,2}e^{-\lambda_2 t}\right]$

$$B_{0,2} = \frac{1}{\lambda_1-\lambda_0}\cdot\frac{1}{\lambda_2-\lambda_0} = \frac{1}{4-2}\cdot\frac{1}{3-2} = \frac{1}{2}$$

$$B_{1,2} = \frac{1}{\lambda_0-\lambda_1}\cdot\frac{1}{\lambda_2-\lambda_1} = \frac{1}{2-4}\cdot\frac{1}{3-4} = \frac{1}{2}$$

$$B_{2,2} = \frac{1}{\lambda_0-\lambda_2}\cdot\frac{1}{\lambda_1-\lambda_2} = \frac{1}{2-3}\cdot\frac{1}{4-3} = -1$$

$$\Rightarrow P_2(t) = 8\left[\frac{1}{2}e^{-2t} + \frac{1}{2}e^{-4t} - e^{-3t}\right] = 4e^{-2t} + 4e^{-4t} - 8e^{-3t}$$

Q5) a) Using that: $P_r\{B(s+t) \leq y \mid B(s) = x\}$

$$= \Phi\left(\frac{y-x}{\sqrt{t}}\right)$$

$$\Rightarrow P_r\{B(\overset{s+t}{8}) \leq \overset{y}{6} \mid B(\overset{s}{0}) = \overset{x}{2}\} = \Phi\left(\frac{6-2}{\sqrt{8}}\right)$$

$$= \Phi\left(\frac{4}{2\sqrt{2}}\right)$$

$$= \Phi(1.41)$$

$$= 0.9207$$

b) For a BM with $B(0)=0$, we know that $B(t) \sim N(0,t)$

Now, by definition, $Cov[B(t), B(s)] = E[B(t)B(s)] - \underbrace{E[B(t)]}_{0} E[B(s)]$

$$\Rightarrow Cov[B(t), B(s)] = E[B(t)B(s)] \text{ for any } s,t > 0.$$

Now for $0 < t < s$:

$$B(s) = B(t) + B(s) - B(t)$$

$$E[B(t)B(s)] = E[B(t)(B(t) + B(s) - B(t))]$$

$$= E[B^2(t) + B(t)(B(s) - B(t))]$$

$$= \underbrace{E[B^2(t)]}_{t} + E[B(t).(B(s)-B(t))]$$

indep. incre. as $t<s$ by properties of BM

because $B(t) \sim N(0,t)$ $\Rightarrow Var[B(t)] = t$ $E[B^2(t)] - (\cancel{E[B(t)]})^2$

$$= t + \underbrace{E[B(t)]}_{0}.E[B(s) - B(t)]$$

$$= t = \min(t, s).$$

Similarly for if $0 < s < t$. Result is proved!

Q5) c) A Continous time process $(X(t), t \geq 0)$ is a martingale if for any $t > 0$:

① $E|X(t)| < \infty$

② $E[X(t) \mid \mathcal{F}_s] = X(s)$, for all $0 \leq s < t$.

To prove $X(t) = B^2(t) - t$ is a martingale

1. show $E|X(t)| < \infty$?

$$E|B^2(t) - t| \leq E|B^2(t)| + E|t|$$
$$= E(B^2(t)) + t = t + t = 2t < \infty$$

2. For $0 \leq s < t$, $E[X(t) \mid \mathcal{F}_s] = X(s)$?

$$\text{L.H.S.} = E[X(t) \mid \mathcal{F}_s]$$
$$= E[B^2(t) - t \mid \mathcal{F}_s]$$
$$= E[\{B(t) - B(s) + B(s)\}^2 - t \mid \mathcal{F}_s]$$
$$= E[(B(t) - B(s))^2 + B^2(s) + 2(B(t) - B(s))B(s) - t \mid \mathcal{F}_s]$$
$$= E[\underbrace{(B(t) - B(s))^2}_{\text{independent of } \mathcal{F}_s} \mid \mathcal{F}_s] + E[\underbrace{B^2(s)}_{\text{determined by } \mathcal{F}_s} \mid \mathcal{F}_s] + E[2(B(t) - B(s))B(s) \mid \mathcal{F}_s] - t$$

because of

$$= \underbrace{E[(B(t) - B(s))^2]}_{= t-s} + B^2(s) + 2B(s)\underbrace{E[B(t) - B(s)]}_{= E[B(t)] - E[B(s)] = 0 - 0} - t$$

$\boxed{\text{because } B(t) - B(s) \sim N(0, t-s)}$

$$= t - s + B^2(s) - t$$
$$= B^2(s) - s = X(s) = \text{R.H.S.}$$

$\therefore X(t)$ is a martingale