KING SAUD UNIVERSITY
COLLEGE OF SCIENCES
DEPARTMENT OF MATHEMATICS
Semester 472 / MATH-244 (Linear Algebra) / Mid-term Exam 2

Max. Marks: 25 **Max. Time: $1\frac{1}{2}$ hrs.**

Note: Calculators are not allowed.

Question 1: [Marks: 10]
Which of the given choices are correct?

(i) If $B = \{u, v, w\}$ is a basis of $\mathbb{R}^3$ and $ru + sv + tw = 0$ for some scalars r, s, t, then:
a) $r = u, s = v, t = w$ b) $r = s = t = 0$ c) $u = v = w$ d) B is linearly dependent

Solution: b) (1 mark)

(ii) If a subset S of the vector space $\mathbb{R}^3$ spans $\mathbb{R}^3$ what must be true?
a) All vectors in S are non-zero.
b) S contains exactly three vectors.
c) S is linearly independent.
d) S contains at least three linearly independent vectors.

Solution: d) (1 mark)

(iii) If A is a matrix of size 5×7 then its:
a) rank is 5 b) rank is 7 c) rows are linearly independent d) columns are linearly dependent

Solution: d) (1 mark)

(iv) If $S = \{u, v, w\}$ is an ordered basis of a vector space V, then the coordinate vector $[u]_S$ with respect to the ordered basis S is:
a) $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ b) $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ c) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ d) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Solution: c) (1 mark)

(v) If $S = \{u, v, w, x\}$ is a subset of the vector space $\mathbb{R}^3$, then S must be:
a) a basis of $\mathbb{R}^3$ b) a subspace of $\mathbb{R}^3$ c) linearly dependent d) linearly independent

Solution: c) (1 mark)

(vi) Which of the following sets is a subspace of the vector space $\mathbb{R}^3$?
a) $\{(x, y, z) : x > 0\}$ b) $\{(x, y, z) : x + y + z = 1\}$
c) $\{(x, y, z) : x + y + z = 0\}$ d) $\{(x, y, z) : x^2 + y^2 + z^2 = 1\}$

Solution: c) (1 mark)

(vii) If a subset $S \neq \{0\}$ of a vector space V is linearly dependent, then:
a) The set S must span the vector space V.
b) No vector in S can be written as a linear combination of other vectors in S.
c) At least one vector in S is a linear combination of other vectors in S.
d) One vector in S must be 0.

Solution: c) (1 mark)

(viii) Any transition matrix P must satisfy that:
a) P is diagonal b) $\det(P) = 0$ c) P is invertible d) P is symmetric

Solution: c) (1 mark)

(ix) If A is a 3×6 matrix with linearly independent rows, then:
a) $\text{rank}(A) = 6$ b) $\text{rank}(A) = 3$ c) $\text{rank}(A) = 4$ d) $\text{rank}(A) = 5$

Solution: b) (1 mark)

(x) If a matrix A is of size 4×6 such that $\dim(\text{col}(A)) = 3$ then $\dim(N(A))$ is equal to:
a) ≤ 2 b) 1 c) 3 d) 0

Solution: c) (1 mark)

Question 2: [Marks: 2+2+3]

a) Give an example of a linearly dependent subset of the vector space $\mathbb{R}^3$ which spans $\mathbb{R}^3$.

Solution:
$\{(1,1,1),(1,0,0),(0,1,0),(0,0,1)\}$ is a subset of $\mathbb{R}^3$ which spans $\mathbb{R}^3$. (2 marks)

b) Let $\{u_1, u_2, u_3\}$ be a basis of a vector space V. Determine whether the set $\{2u_1, u_1 - 2u_2, 2u_1 + u_3\}$ is a basis of V or not.

Solution:
Assume $\alpha(2u_1) + \beta(u_1 - 2u_2) + \gamma(2u_1 + u_3) = 0$.
So, $(2\alpha + \beta + 2\gamma)u_1 - 2\beta u_2 + \gamma u_3 = 0$.
Because $\{u_1, u_2, u_3\}$ is a basis (and therefore linearly independent), we have:

$$\begin{aligned} 2\alpha + \beta + 2\gamma &= 0 \\ -2\beta &= 0 \\ \gamma &= 0 \end{aligned}$$

This implies $\alpha = \beta = \gamma = 0$.
Hence $\{2u_1, u_1 - 2u_2, 2u_1 + u_3\}$ is linearly independent.
Since $\dim(V) = 3$, we deduce that $\{2u_1, u_1 - 2u_2, 2u_1 + u_3\}$ is a basis of V. (2 marks)

c) Let $A = \{v_1 = (1,1,1), v_2 = (2,1,1), v_3 = (1,0,0)\}$. Find a basis B of $\text{span}(A)$, and then find a basis C of $\mathbb{R}^3$ such that $B \subset C$.

Solution:
Clearly $v_1 = v_2 - v_3$.
The set $\{v_2, v_3\}$ is linearly independent.
Hence $B = \{v_2, v_3\}$ is a basis of $\text{span}(A)$. (2 marks)
We can extend B to a basis of $\mathbb{R}^3$. As $(0,1,0)$ is not in $\text{span}(B)$, then $C = \{v_2, v_3, (0,1,0)\}$ is a basis of $\mathbb{R}^3$ that contains B. (1 mark)

Question 3: [Marks: (2+3)+3]

a) Consider the two ordered bases $B = \{u_1 = (0,1,1), u_2 = (1,1,1), u_3 = (1,0,1)\}$ and $C = \{v_1 = (0,1,-1), v_2 = (1,1,0), v_3 = (-1,0,1)\}$ of the vector space $\mathbb{R}^3$, and the vector $v = (1,2,1) \in \mathbb{R}^3$.

(i) Find the coordinate vector $[v]_B$ of the vector v with respect to the ordered basis B.

Solution:

$[v]_B \implies \begin{pmatrix}1\\2\\1\end{pmatrix} = au_1 + bu_2 + cu_3 = a\begin{pmatrix}0\\1\\1\end{pmatrix} + b\begin{pmatrix}1\\1\\1\end{pmatrix} + c\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}b+c\\a+b\\a+b+c\end{pmatrix}$.

This gives the system:

$$\begin{aligned} b+c &= 1 \\ a+b &= 2 \\ a+b+c &= 1 \implies 2+c=1 \implies c=-1 \end{aligned}$$

Substituting $c=-1$ back gives $b=2$, and $a=0$.

So, $[v]_B = \begin{bmatrix}0\\2\\-1\end{bmatrix}$. (2 marks)

(ii) Construct the transition matrix $P_{B\to C}$ from the basis B to the basis C, and then use it to find the coordinate vector $[v]_C$ of the vector v with respect to the ordered basis C.

Solution:

We express the vectors of B as linear combinations of the vectors in C:

$$u_1 = \alpha v_1 + \beta v_2 + \gamma v_3 \implies \begin{cases}\beta-\gamma=0\\ \alpha+\beta=1\\ -\alpha+\gamma=1\end{cases} \implies \alpha=0, \beta=1, \gamma=1.$$

$$u_2 = \alpha v_1 + \beta v_2 + \gamma v_3 \implies \begin{cases}\beta-\gamma=1\\ \alpha+\beta=1\\ -\alpha+\gamma=1\end{cases} \implies \alpha=-1/2, \beta=3/2, \gamma=1/2.$$

$$u_3 = \alpha v_1 + \beta v_2 + \gamma v_3 \implies \begin{cases}\beta-\gamma=1\\ \alpha+\beta=0\\ -\alpha+\gamma=1\end{cases} \implies \alpha=-1, \beta=1, \gamma=0.$$

The transition matrix is $P_{B\to C} = \begin{bmatrix}0 & -1/2 & -1\\ 1 & 3/2 & 1\\ 1 & 1/2 & 0\end{bmatrix}$. (1.5 marks)

Now, $[v]_C = P_{B\to C}[v]_B = \begin{bmatrix}0 & -1/2 & -1\\ 1 & 3/2 & 1\\ 1 & 1/2 & 0\end{bmatrix}\begin{bmatrix}0\\2\\-1\end{bmatrix} = \begin{bmatrix}0\\2\\1\end{bmatrix}$. (1.5 marks)

b) Find the rank and nullity of the matrix $\begin{bmatrix}1 & 2 & 1 & 1\\ 2 & 1 & 1 & 1\\ 1 & 1 & 2 & 1\\ 1 & 1 & 1 & 2\end{bmatrix}$.

Solution:

By applying row operations, the matrix reduces to the following Echelon form:

$\sim \begin{bmatrix}1 & 2 & 1 & 1\\ 0 & 1 & 0 & 1\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 1\end{bmatrix}$. (2 marks)

From the reduced echelon form, we can see there are 4 leading 1s (pivots).
Therefore, Rank $=4$ and Nullity $=0$. (1 mark)