KING SAUD UNIVERSITY
COLLEGE OF SCIENCES
DEPARTMENT OF MATHEMATICS
Semester 472 / MATH-244 (Linear Algebra) / Mid-term Exam 1

Max. Marks: 25 **Max. Time: $1\frac{1}{2}$ hrs.**

Note: Calculators are not allowed.

Question 1: [Marks: 1+1+1+1+1]
Which of the given choices are correct?

(i) If A is a square matrix and $A^2 = 0$ and I is the identity matrix, then $(2A+I)^{-1}$ is equal to:
a) $I - A$ b) $I + A$ c) $I - 2A$ d) $2I - A$

Solution: c) (1 mark)

(ii) If $A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$, then the unique solution of the equation $A^{-1} = \alpha A + \beta I$ is:
a) $\alpha = \frac{1}{10}, \beta = -\frac{1}{5}$ b) $\alpha = \frac{1}{10}, \beta = -\frac{3}{10}$ c) $\alpha = -\frac{1}{10}, \beta = \frac{1}{2}$ d) $\alpha = \frac{1}{10}, \beta = -\frac{1}{2}$

Solution: c) (1 mark)

(iii) Which of the following matrices is not an elementary matrix:
a) $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ c) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$

Solution: b) (1 mark)

(iv) If $A\,\mathrm{adj}(A) = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, then determinant of the matrix A satisfies:
a) $|A| > 3$ b) $|A| < 5$ c) $|A| = 9$ d) $|A| = 27$

Solution: b) (1 mark)

(v) If I is a row echelon form of the matrix A, then the linear system $AX = B$ would have:
a) infinitely many solutions b) no solutions c) unique solution d) more than one solutions.

Solution: c) (1 mark)

Question 2: [Marks: 4+4+2]

a) Let A and B be two square matrices of the same size. Then:

i) Show that $(A+B)^2 = A^2 + 2AB + B^2$ if $(A+B)(A-B) = A^2 - B^2$.
Solution:
Assume $(A+B)(A-B) = A^2 - B^2$.
Expanding the left side yields $A^2 - AB + BA - B^2 = A^2 - B^2$.
Subtracting $A^2 - B^2$ from both sides gives $-AB + BA = 0$, which implies $AB = BA$.
(1 mark)
Now, expanding the left side of our target equation:
$(A+B)^2 = (A+B)(A+B) = A^2 + AB + BA + B^2$.

Since we established that $AB = BA$, we can substitute BA with AB to get:
$A^2 + AB + AB + B^2 = A^2 + 2AB + B^2$. (1 mark)

ii) Give an example of matrices A and B for which $(A+B)^2 \neq A^2 + 2AB + B^2$.

Solution:

Taking $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, we have $(A+B) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

Then, $(A+B)^2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$.

However, calculating the terms separately:

$A^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, $B^2 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, and $AB = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$.

So, $A^2 + 2AB + B^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ 4 & 4 \end{bmatrix}$.

Clearly, $\begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} \neq \begin{bmatrix} 6 & 4 \\ 4 & 4 \end{bmatrix}$. (2 marks)

b) Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix}$. Find all matrices B such that $AB = I$.

Solution:

Since the matrix A is of size 2×3 and $AB = I_2$, we assume that $B = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{bmatrix}$.

This gives us two linear systems to solve:

$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. (2 marks)

We can solve for both columns of B simultaneously by row reducing the combined augmented matrix $[A|I_2]$:

$$\left[\begin{array}{ccc|cc} 1 & 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 0 & 1 \end{array}\right]$$

Apply the row operation $R_2 \leftarrow R_2 - 2R_1$:

$$\sim \left[\begin{array}{ccc|cc} 1 & 1 & 1 & 1 & 0 \\ 0 & -1 & -2 & -2 & 1 \end{array}\right]$$

Apply $R_2 \leftarrow -R_2$, and then $R_1 \leftarrow R_1 - R_2$:

$$\sim \left[\begin{array}{ccc|cc} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|cc} 1 & 0 & -1 & -1 & 1 \\ 0 & 1 & 2 & 2 & -1 \end{array}\right]$$

This gives us two systems of equations in reduced row echelon form.
For the first column of B, we have $x_1 - x_3 = -1$ and $x_2 + 2x_3 = 2$.
Letting the free variable $x_3 = s$, we get $x_1 = -1 + s$ and $x_2 = 2 - 2s$.
For the second column of B, we have $y_1 - y_3 = 1$ and $y_2 + 2y_3 = -1$.
Letting the free variable $y_3 = t$, we get $y_1 = 1 + t$ and $y_2 = -1 - 2t$.

Thus, the required matrices are $B = \begin{bmatrix} -1+s & 1+t \\ 2-2s & -1-2t \\ s & t \end{bmatrix}$, $\forall s, t \in \mathbb{R}$. (2 marks)

c) Show that $\begin{vmatrix} a & a & b & b \\ b & a & a & b \\ b & b & a & a \\ b & b & b & a \end{vmatrix} = (a+b)(a-b)^3$.

Solution:

Applying the row operation $R_1 \leftarrow R_1 + R_3$:

$$\begin{vmatrix} a & a & b & b \\ b & a & a & b \\ b & b & a & a \\ b & b & b & a \end{vmatrix} = \begin{vmatrix} a+b & a+b & a+b & a+b \\ b & a & a & b \\ b & b & a & a \\ b & b & b & a \end{vmatrix}$$

Taking the common factor $(a+b)$ out of the first row:

$$= (a+b) \begin{vmatrix} 1 & 1 & 1 & 1 \\ b & a & a & b \\ b & b & a & a \\ b & b & b & a \end{vmatrix}$$

Applying row operations $R_2 \leftarrow R_2 - bR_1$, $R_3 \leftarrow R_3 - bR_1$, and $R_4 \leftarrow R_4 - bR_1$:

$$= (a+b) \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & a-b & a-b & 0 \\ 0 & 0 & a-b & a-b \\ 0 & 0 & 0 & a-b \end{vmatrix}$$

Because the matrix is now upper triangular, its determinant is the product of its diagonal entries:

$= (a+b) \cdot 1 \cdot (a-b) \cdot (a-b) \cdot (a-b) = (a+b)(a-b)^3$. (2 marks)

Question 3: [Marks: 6+4]

a) Find the value/s of α such that the following linear system:

$$\begin{aligned} x + 2y - z &= 2 \\ x + 2y - (\alpha^2 - 3)z &= \alpha \\ x - 2y + 3z &= 1 \end{aligned}$$

has: (i) no solution (ii) unique solution (iii) infinitely many solutions.

Solution:

We start by setting up the augmented matrix for the system:

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 1 & 2 & 3-\alpha^2 & \alpha \\ 1 & -2 & 3 & 1 \end{array}\right]$$

Apply $R_2 \leftarrow R_2 - R_1$ and $R_3 \leftarrow R_3 - R_1$:

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 0 & 4-\alpha^2 & \alpha-2 \\ 0 & -4 & 4 & -1 \end{array}\right]$$

Swap rows R_2 and R_3 ($R_2 \leftrightarrow R_3$) to place the matrix into echelon form:

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & -4 & 4 & -1 \\ 0 & 0 & 4-\alpha^2 & \alpha-2 \end{array}\right].$$ (3 marks)

Now we analyze the last row: $(4-\alpha^2)z = \alpha - 2$, which factors to $(2-\alpha)(2+\alpha)z = \alpha - 2$.

(i) For **no solution**, the coefficient of z must be zero while the constant is non-zero.

Setting the coefficient to zero gives $\alpha = 2$ or $\alpha = -2$.

If we test $\alpha = -2$, the equation becomes $0z = -4$, which is a contradiction. Hence, $\alpha = -2$. (1 mark)

(ii) For a **unique solution**, we need a pivot in every column, meaning the coefficient of z cannot be zero.

This requires $4 - \alpha^2 \neq 0$, so $\alpha \notin \{-2, 2\}$. Therefore, $\alpha \in \mathbb{R} \setminus \{-2, 2\}$. (1 mark)

(iii) For **infinitely many solutions**, we need a row of all zeros (giving a free variable).

If we test $\alpha = 2$, the last equation becomes $0z = 0$, which is consistent. Hence, $\alpha = 2$. (1 mark)

b) Consider the following system of linear equations:

$$\begin{aligned} x + 2y - 3z &= 1 \\ x + y - z &= 2 \\ x + y + 2z &= 3 \end{aligned}$$

i) Find inverse A^{-1} of the matrix $A = \begin{bmatrix} 1 & 2 & -3 \\ 1 & 1 & -1 \\ 1 & 1 & 2 \end{bmatrix}$ by using $\operatorname{adj}(A)$.

Solution:

$|A| = -3$ and the Cofactor matrix $C = \begin{bmatrix} 3 & -3 & 0 \\ -7 & 5 & 1 \\ 1 & -2 & -1 \end{bmatrix}$. (1.5 marks)

So that $A^{-1} = \frac{1}{|A|}\operatorname{adj}(A) = \frac{1}{|A|}C^T = -\frac{1}{3}\begin{bmatrix} 3 & -7 & 1 \\ -3 & 5 & -2 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 7/3 & -1/3 \\ 1 & -5/3 & 2/3 \\ 0 & -1/3 & 1/3 \end{bmatrix}$. (1 mark)

ii) Use A^{-1} to solve the above linear system.

Solution:

Hence, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{3}\begin{bmatrix} 3 & -7 & 1 \\ -3 & 5 & -2 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = -\frac{1}{3}\begin{bmatrix} -8 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 8/3 \\ -1/3 \\ 1/3 \end{bmatrix}$. (1.5 marks)