[Draft]

**King Saud University**
**College of Sciences**
**Department of Mathematics**
**Semester 471 / Final Exam / MATH-244 (Linear Algebra)**

**Max. Marks: 40** **Time: 3 hours**

**Name:** **ID:** **Section:** **Signature:**

**Note**: ***Attempt all the five questions. Calculators are not allowed.***

**Question 1** [Marks 10]**:** Which of the given choices are correct?

(i) If a matrix $A$ is symmetric as well as skew-symmetric, then $A$ must be:
a) identity b) non-singular c) inverse of itself d) zero.

(ii) If $A$ is a 4×4 matrix such that $adj(A) = A^{-1}$, then the determinant $|adj(A)|$ is equal to:
a) $|A|^2$ b) $3|A|$ c) 4 d) $2|A|$.

(iii) If the matrix of coefficients in the linear system $AX = B$ is invertible, then the system must have:
a) infinitely many solutions b) a unique solution c) no solution d) zero solution.

(iv) Let $B = \{(2,1),(2,3)\}$ and $C = \{(0,1),(2,0)\}$ be ordered bases of a vector space $V$, then the transition matrix ${}_C\boldsymbol{P}_B$ from $B$ to $C$ is:
a) $\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$ b) $\begin{bmatrix} 1 & -3 \\ -1 & 1 \end{bmatrix}$ c) $\begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}$ d) $\begin{bmatrix} -2 & 6 \\ 2 & -2 \end{bmatrix}$.

(v) If $A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$, then $rank(A)$ is:
a) 0 b) 1 c) 2 d) 3.

(vi) If $A = \begin{bmatrix} -1 & 3 \\ 4 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 2 \\ 0 & 1 \end{bmatrix}$ are vectors in the vector space $M_2(\mathbb{R})$ with the standard inner product, then the angle $\theta$ between $A$ and $B$ is equal to:
a) $\theta = cos^{-1}(\frac{2}{3\sqrt{3}})$ b) $\theta = cos^{-1}\frac{\sqrt{2}}{30}$ c) $\theta = cos^{-1}\frac{2}{9\sqrt{10}}$ d) $\theta = cos^{-1}\frac{2}{\sqrt{30}}$ .

(vii) If $T: M_2(\mathbb{R}) \to \mathbb{R}^2$ is the linear transformation defined by $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a,b)$, $\forall\, a,b,c,d \in \mathbb{R}$, then $ker(T)$ is equal to:
a) $\left\{\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : a,b \in \mathbb{R}\right\}$ b) $\left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right\}$ c) $\left\{\begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix} : a,b \in \mathbb{R}\right\}$ d) $\{(0,0)\}$.

(viii) If $\{(-3r + 4s, r - s, r, s) \mid r,s \in \mathbb{R}\}$ is the solution set of the homogeneous system $A\boldsymbol{X} = 0$ and $T_A$ is the linear transformation given by $T_A(X) = AX$, then:
a) $nullity(T_A) = 2$ b) $nullity(A) = 0$ c) $rank(A^T) = 3$ (d) $ker(T_A) = \{0\}$.

(ix) If the linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^3$ is defined by $T(x_1,x_2) = (x_1 - x_2, x_2 - x_1, 3x_2)$, then the induced matrix $[T]_B^C$ with respect to the ordered basis $B = \{(2,0),(1,1)\}$ of $\mathbb{R}^2$ and the ordered standard basis $C$ of $\mathbb{R}^3$ is:
a) $\begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 0 & 3 \end{bmatrix}$ b) $\begin{bmatrix} 0 & 0 \\ -2 & 0 \\ 0 & 3 \end{bmatrix}$ c) $\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 3 \end{bmatrix}$ d) $\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$.

(x) If $A = \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}$, then the eigenvalues of $A^4$ are:
a) 16, 81 b) 2,3 c) 4, 9 d) −1, 2, 3.

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**Question 2** [Marks 2 + 2 + 2]:

(a) Let $A = \begin{bmatrix} 1 & 1 & -2 \\ 1 & 2 & -1 \\ 2 & 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & 1 \end{bmatrix}$. Find the invertible matrix $C$ that satisfies $AX^{-1} = B$.

(b) Find all the values of α for which the matrix $\begin{bmatrix} \alpha & 1 & 0 \\ \alpha+2 & 2 & 1 \\ \alpha^2 & 2 & 3 \end{bmatrix}$ is non-invertible.

(c) Consider the matrix $A = \begin{bmatrix} 0 & 2 & -1 & 1 \\ 1 & -3 & 0 & -1 \\ 1 & -1 & 1 & 0 \end{bmatrix}$. Find $rank(A)$ and $nullity(A^T)$.

**Question 3** [Marks 2 + (3 + 2 + 2)]:

(a) Consider the vector subspace $E = \{(x, y, z) \in \mathbb{R}^3 |\ x + y + z = 0\ ,\ 2x + 3y\ =\ 0\}$ of the vector space $\mathbb{R}^3$. Find a basis $B$ of $E$ and a basis $C$ of $\mathbb{R}^3$ containing $B$.

(b) Consider the vectors $u_1 = (1,1,0)$, $u_2 = (0,1,1)$ and $u_3 = (1,1,-1)$ in the Euclidean space $\mathbb{R}^3$.

(i) Use the Gram-Schmidt process to transform the basis $\{u_1, u_2, u_3\}$ into an orthonormal basis $\{v_1, v_2, v_3\}$ of $\mathbb{R}^3$.

(ii) Express the vector $u = (1, 2, -2)$ as the linear combination of $v_1, v_2, v_3$.

(iii) Find the angle θ between the vectors $u$ and $v_1$.

**Question 4** [Marks 2 + 3 + 2]:

Let $[T]_B = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$ be the matrix of the linear transformation $T : \longrightarrow \mathbb{R}^2$ relative to the ordered basis $B = \{v_1 = (1,3), v_2 = (-1,4)\}$. Find:

(i) $[T(v_1)]_B$ and $[T(v_2)]_B$ .

(ii) $T(v_1)$ and $T(v_2)$.

(iii) a formula for $T(x_1, x_2)$ for all $(x_1, x_2) \in \mathbb{R}^2$.

**Question 5** [Marks 4 + 2 + 2]:

Consider the matrix $A = \begin{bmatrix} 1 & 0 & 3 \\ 1 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$.

(i) Find the eigenvalues of $A$ and bases for the corresponding eigenspaces.

(iii) Is the matrix $A$ diagonalizable? Justify your answer.

(c) (iv) Compute $A^9$.

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