Course Notes: Math 209

Sequences

1. Definition of a Sequence

A **sequence** is a function from the natural numbers $\mathbb{N}$ to the real numbers $\mathbb{R}$. It is usually written as:

$$\{a_n\}_{n=1}^{\infty} = a_1, a_2, a_3, \dots$$

Each term a_n is called the *nth term* of the sequence.

Examples

- $a_n = \frac{1}{n} \Rightarrow \left\{1, \frac{1}{2}, \frac{1}{3}, \dots\right\}$
- $a_n = (-1)^n \Rightarrow \{-1, 1, -1, 1, \dots\}$
- $a_n = n^2 \Rightarrow \{1, 4, 9, 16, \dots\}$

2. Convergence of Sequences

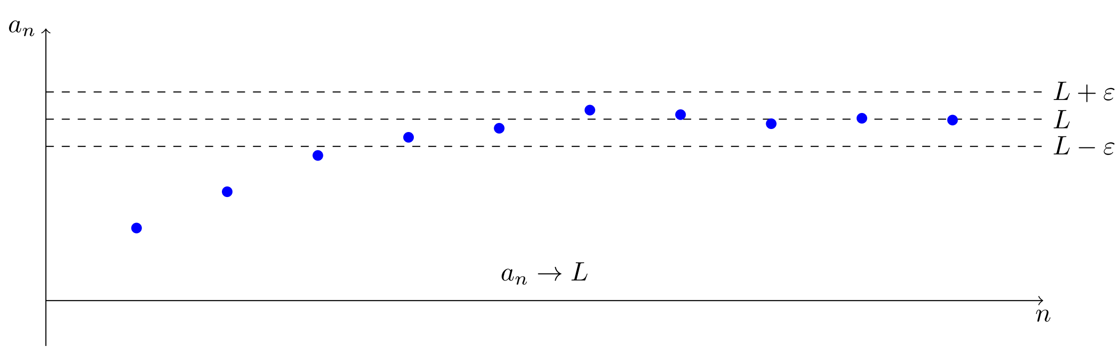
A sequence $\{a_n\}$ **converges** to a real number L if:

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N} \text{ such that } n \geq N \Rightarrow |a_n - L| < \varepsilon.$$

We denote this as $\lim_{n\to\infty} a_n = L$, or simply $a_n \to L$.

Geometric Interpretation:

To visualize this definition, we draw a horizontal band of width 2ε centered at L. This band represents the ε-neighborhood of the limit. The key idea is that, while a few early terms may fall outside this band, eventually all terms of the sequence lie inside it.



The dashed line at $y = L$ shows the limit of the sequence (a_n). The two dashed lines at $y = L + \varepsilon$ and $y = L - \varepsilon$ form a horizontal band around L, called the ε-neighborhood. This band represents a tolerance zone: how close the sequence terms must be to the limit. As shown, while a few early terms lie outside this band, from a certain index N onward, all terms lie within it — that is, they satisfy $|a_n - L| < \varepsilon$. This illustrates the formal definition of convergence: the terms of the sequence get arbitrarily close to the limit and eventually stay there.

Divergence

If a sequence does not converge, it is said to **diverge**. For example:

- $a_n = n$ diverges to infinity.
- Let $a_n = (-1)^n$. Then:
$$a_1 = -1,\ a_2 = 1,\ a_3 = -1,\ a_4 = 1,\ \ldots$$
This sequence does not converge. It keeps jumping between -1 and 1, so:
$$\lim_{n\to\infty} a_n \text{ does not exist.}$$
.

3. Properties of Convergent Sequences

Let $a_n \to a$ and $b_n \to b$. Then:

- $a_n + b_n \to a + b$
- $a_n b_n \to ab$
- If $b \neq 0$ and $b_n \neq 0$ for all n, then $\frac{a_n}{b_n} \to \frac{a}{b}$
- $|a_n| \to |a|$

Also, every convergent sequence is bounded.

4. Examples

Example 1:

$$a_n = \frac{2n^2 - 3n}{3n^2 + 5n + 3}$$

Highest powers dominate:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n^2}{3n^2} = \frac{2}{3}$$

Justification: Divide numerator and denominator by n^2:

$$a_n = \frac{2n^2 - 3n}{3n^2 + 5n + 3} = \frac{2 - \frac{3}{n}}{3 + \frac{5}{n} + \frac{3}{n^2}}.$$

Since

$$\frac{1}{n} \to 0 \quad \text{and} \quad \frac{1}{n^2} \to 0 \quad \text{as } n \to \infty,$$

we get:

$$\lim_{n\to\infty} a_n = \frac{2 - 0}{3 + 0 + 0} = \frac{2}{3}.$$

General Rule: For $a_n = \frac{P(n)}{Q(n)}$:

- If $\deg P < \deg Q$, then $\lim a_n = 0$
- If $\deg P = \deg Q$, then $\lim a_n$ equals the ratio of leading coefficients
- If $\deg P > \deg Q$, then $\lim a_n = \infty$ or $-\infty$ (divergent)

Example 2: Geometric and Power Sequences

Geometric sequence:

$$\lim_{n\to\infty} r^n = \begin{cases} 0, & \text{if } |r| < 1, \\ 1, & \text{if } r = 1, \\ \text{diverges}, & \text{if } |r| \geq 1 \text{ and } r \neq 1. \end{cases}$$

We can write:

$$r^n = e^{n \ln r}, \quad \text{for } r > 0.$$

- **If** $0 < r < 1$: then $\ln r < 0$ and $n \ln r \to -\infty$.

$$\Rightarrow \quad r^n = e^{n \ln r} \to 0.$$

- **If** $r = 1$: then $\ln r = 0$ and

$$\Rightarrow \quad r^n = e^0 = 1.$$

- **If** $r > 1$: then $\ln r > 0$ and $n \ln r \to \infty$.

$$\Rightarrow \quad r^n = e^{n \ln r} \to \infty.$$

- **If** $r < 0$: then r^n diverges and may oscillate, depending on the parity of n.

Power sequence (also called ”dual geometric”):

$$\lim_{n\to\infty} n^r = \begin{cases} 0, & \text{if } r < 0, \\ 1, & \text{if } r = 0, \\ \infty, & \text{if } r > 0. \end{cases}$$

- $n^0 = 1$ for all n,
- $n^r \to \infty$ if $r > 0$,
- $n^r = \dfrac{1}{n^{-r}} \to 0$ if $r < 0$.

Example 3:

Show that

$$\lim_{n\to\infty} n \cdot \sin\left(\frac{1}{n}\right) = 1.$$

As

$$\frac{1}{n} \to 0 \quad \text{and} \quad \lim_{x\to 0} \frac{\sin x}{x} = 1,$$

we rewrite:

$$n \cdot \sin\left(\frac{1}{n}\right) = \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} \to 1.$$

General Rule:

- If $\lim_{x\to 0} f(x) = L$, then $\lim_{n\to\infty} f\left(\frac{1}{n}\right) = L$.
- If $\lim_{x\to\infty} f(x) = L$, then $\lim_{n\to\infty} f(n) = L$.

Example 4

If $\alpha > 0$, then:

$$\lim_{n\to\infty} \alpha^{1/n} = 1.$$

Let $a_n = \alpha^{1/n}$. Then:

$$\ln a_n = \frac{\ln \alpha}{n} \to 0.$$

So,

$$a_n = e^{\ln a_n} \to e^0 = 1.$$

Example 5

The sequence $(n^{1/n})$ converges to 1:

$$\lim_{n\to\infty} n^{1/n} = 1.$$

Let $a_n = n^{1/n}$. Then:

$$\ln a_n = \frac{\ln n}{n}.$$

To evaluate the limit, apply L'Hôpital's Rule to:

$$\lim_{n\to\infty} \frac{\ln n}{n}.$$

Since both numerator and denominator tend to ∞, we differentiate top and bottom:

$$\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{1/n}{1} = 0.$$

Therefore:

$$\ln a_n \to 0 \quad \Rightarrow \quad a_n = e^{\ln a_n} \to e^0 = 1.$$

5. Monotone Sequences

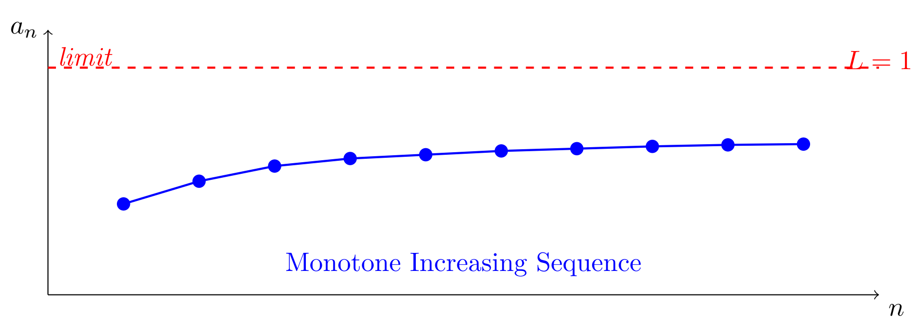
A sequence $\{a_n\}$ is:

- **Increasing** if $a_{n+1} \geq a_n$ for all n
- **Decreasing** if $a_{n+1} \leq a_n$ for all n
- **Monotonic** if it is either increasing or decreasing

Theorem (Monotone Convergence Theorem): If a sequence is monotonic and bounded, then it converges.

Example 1:

Let $a_n = 1 - \frac{1}{n}$. It is increasing and bounded above by 1, so the sequence a_n is convergent.



Explanation:

- The blue points represent terms a_n of a sequence.
- Each term is greater than or equal to the previous — the sequence is increasing.
- The red dashed line is the horizontal asymptote at $L = 1$, showing the limit.
- The sequence approaches the limit from below but never exceeds it.

Example 2: Sequence of partial sum

Let a_n be a sequence of non-negative numbers, meaning that $a_n \geq 0$ for every $n \in \mathbb{N}$. The sequence of partial sums associated with a_n is defined by:

$$S_N = \sum_{k=1}^{N} a_k = a_1 + a_2 + \cdots + a_N.$$

This means that :

$$\begin{aligned} S_1 &= a_1, \\ S_2 &= a_1 + a_2, \\ S_3 &= a_1 + a_2 + a_3, \\ &\vdots \end{aligned}$$

Since each term $a_n \geq 0$, adding a new term always makes the total sum stay the same or increase:

$$S_{N+1} - S_N = a_{N+1} \geq 0.$$

We now consider two possible situations:

- **Case 1:** The sequence (S_N) is bounded above.
  Since S_N is increasing and bounded, the Monotonic Convergence Theorem tells us that S_N converges to a finite limit.
- **Case 2:** The sequence (S_N) is not bounded above.
  In this case, the partial sums grow without limit, that is,
  $$\lim_{N\to\infty} S_N = \infty.$$

6. Squeeze Theorem

Theorem: Let $\{a_n\}, \{b_n\}, \{c_n\}$ be sequences such that:

$$a_n \leq b_n \leq c_n \text{ for all } n \geq N,$$

and if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then:

$$\lim_{n\to\infty} b_n = L.$$

Example 1:

Show that $\lim_{n\to\infty} \frac{\sin n}{n} = 0$.

Since $-1 \leq \sin n \leq 1$, we have:

$$-\frac{1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}.$$

Both bounds go to 0, so by the Squeeze Theorem:

$$\lim_{n\to\infty} \frac{\sin n}{n} = 0.$$

Infinite series

1. Introduction

An **infinite series** is the sum of the terms of a sequence:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

To analyze the convergence of this series, we consider the **sequence of partial sums**:

$$S_N = \sum_{k=1}^{N} a_k.$$

We say that the series $\sum_{n=1}^{\infty} a_n$ **converges** if the limit of the sequence $\{S_N\}$ exists and is finite. In that case, the value of the infinite sum is defined as:

$$\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} S_N.$$

If this limit does not exist or is infinite, the series is said to **diverge**.

2. Telescoping Series

Evaluate the infinite series:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}.$$

We first decompose the general term using partial fractions:

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1},$$

and then consider the partial sum:

$$S_N = \sum_{n=1}^{N} \left(\frac{1}{n} - \frac{1}{n+1}\right).$$

We expand the partial sum:

$$\begin{aligned} S_N = \left(\frac{\mathbf{1}}{\mathbf{1}} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots \\ + \left(\frac{1}{N} - \frac{\mathbf{1}}{\mathbf{N+1}}\right). \end{aligned}$$

After cancellation, only the first and the last term remain:

$$S_N = \mathbf{1} - \frac{\mathbf{1}}{\mathbf{N+1}}.$$

Taking the limit as $N \to \infty$, we find the sum of the infinite series:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \left(1 - \frac{1}{N+1}\right) = 1.$$

The series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges, and its sum is

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$$

General Rule (Telescoping Form):
If a sequence satisfies $a_n = b_n - b_{n+1}$, then the partial sum telescopes:

$$S_N = \sum_{n=1}^{N} a_n = b_1 - b_{N+1} \quad \Rightarrow \quad \sum_{n=1}^{\infty} a_n = b_1 - \lim_{N\to\infty} b_{N+1},$$

provided the limit exists.

Application to the Geometric Series $\sum_{n=0}^{\infty} r^n$:

$$\begin{aligned}(1-r)\sum_{n=0}^{N} r^n &= \sum_{n=0}^{N} \left(r^n - r^{n+1}\right) \\ &= r^0 - r^{N+1} = 1 - r^{N+1}.\end{aligned}$$

Dividing both sides by $1-r$, we obtain the formula for the partial sum:

$$\sum_{n=0}^{N} r^n = \frac{1-r^{N+1}}{1-r}, \quad r \neq 1.$$

Geometric Series Convergence ("r-Test"):

$$\sum_{n=0}^{\infty} r^n \text{ converges} \quad \Longleftrightarrow \quad |r| < 1.$$

Sum Formula:

$$\sum_{n=0}^{\infty} r^n = \begin{cases} \dfrac{1}{1-r}, & \text{if } |r| < 1, \\ \text{diverges}, & \text{if } |r| \geq 1. \end{cases}$$

Example:

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} = 2.$$

3. Basic Properties

- **Linearity:**

$$\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$

- **Multiplication by a constant:**

$$\sum_{n=1}^{\infty} c a_n = c \sum_{n=1}^{\infty} a_n$$

- **Necessary condition for convergence:**
  If $\sum_{n=1}^{\infty} a_n$ converges, then it must be that $a_n \to 0$ as $n \to \infty$. However, the converse is not true: the fact that $a_n \to 0$ does not guarantee convergence of the series.

  Counterexample: Let $a_n = \frac{1}{n}$. Then $a_n \to 0$, but the harmonic series

  $$\sum_{n=1}^{\infty} \frac{1}{n}$$

  diverges.

Justification: The necessary condition for convergence follows from the identity:

$$S_N = \sum_{k=1}^{N} a_k \quad \Longleftrightarrow \quad a_n = S_n - S_{n-1}, \quad \text{with } S_0 = 0,$$

and the fact that if $\sum a_n$ converges, then both S_n and S_{n-1} converge to the same limit, which forces $a_n \to 0$.

4. Convergence Tests for Positive Series

We are concerned with series $\sum_{n=1}^{\infty} a_n$, such that $a_n \geq 0$ for all n (positive series).

(a) nth-Term Test

If $\lim_{n\to\infty} a_n \neq 0$, the series diverges.

(b) Comparison Test

If $0 \leq a_n \leq b_n$ and $\sum b_n$ converges, then $\sum a_n$ also converges. If $\sum a_n$ diverges and $b_n \geq a_n \geq 0$, then $\sum b_n$ also diverges.

(c) Limit Comparison Test

Let $a_n, b_n > 0$. If:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0,$$

then either both series converge or both diverge.

(d) Ratio Test

$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$$

- If $L < 1$: the series converges.
- If $L > 1$ or $L = \infty$: the series diverges.
- If $L = 1$: the test is inconclusive.

(e) Root Test

$$L = \lim_{n\to\infty} \sqrt[n]{a_n}, \quad \text{same conclusions as the Ratio Test}$$

(f) Integral Test

Let $f : [1, \infty) \to \mathbb{R}$ be a function such that:

- f is continuous,
- $f(x) \geq 0$ for all $x \geq 1$,
- f is decreasing on $[1, \infty)$,
- $f(n) = a_n$ for all integers $n \geq 1$.

Then

$$\sum_{n=1}^{\infty} a_n \quad \text{converges} \quad \Longleftrightarrow \quad \int_1^{\infty} f(x)\,dx \quad \text{converges.}$$

Application: The p-Series Test
Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}, \quad \text{where } p > 0.$$

Let $f(x) = \frac{1}{x^p}$, which is continuous, positive, and decreasing for $x \geq 1$ when $p > 0$. We apply the integral test by evaluating

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^p}\,dx.$$

Case 1: $p \neq 1$

$$\int_1^t \frac{1}{x^p}\,dx = \left[\frac{x^{1-p}}{1-p}\right]_1^t = \frac{t^{1-p} - 1}{1-p}.$$

- If $p > 1$, then $1 - p < 0$, so $t^{1-p} \to 0$ as $t \to \infty$, and the integral converges to $\frac{1}{p-1}$.
- If $p < 1$, then $1 - p > 0$, so $t^{1-p} \to \infty$, and the integral diverges.

Case 2: $p = 1$

$$\int_1^t \frac{1}{x}\,dx = \ln t \quad \to \infty \quad \text{as } t \to \infty.$$

So the integral diverges.

Conclusion:

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \begin{cases} \text{Converges,} & \text{if } p > 1, \\ \text{Diverges,} & \text{if } p \leq 1. \end{cases}$$

By the integral test, the same result holds for the series:

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \begin{cases} \text{Converges,} & \text{if } p > 1, \\ \text{Diverges,} & \text{if } p \leq 1. \end{cases}$$

Examples:

- $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$: converges (since $p = 2 > 1$)
- $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$: diverges (harmonic series, $p = 1$)
- $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$: diverges (since $p = \frac{1}{2} < 1$)

6. Absolute and Conditional Convergence

- A series $\sum a_n$ is said to be **absolutely convergent** if the series of absolute values $\sum |a_n|$ converges.
- A series $\sum a_n$ is said to be **conditionally convergent** if $\sum a_n$ converges, but the series $\sum |a_n|$ diverges.

Theorem (Absolute Convergence Implies Convergence): If the series $\sum |a_n|$ converges, then the original series $\sum a_n$ also converges.

The converse is not true. That is, a convergent series $\sum a_n$ does not necessarily imply that $\sum |a_n|$ converges. A classical example is the alternating harmonic series:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots ,$$

which converges conditionally, since

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$$

diverges (harmonic series).

(g) Alternating Series Test (Leibniz Test)

If $a_n > 0$, decreasing, and $\lim a_n = 0$, then:

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n \text{ converges}$$

- $\sum \frac{(-1)^n}{n^2}$: absolutely convergent (use p-series test)
- $\sum \frac{(-1)^n}{\ln n}$: conditionally convergent (use Leibniz test)
- $\sum \frac{n!}{n^n}$: converges (use ratio or root test)

Power Series

A **real power series** centered at $x_0 \in \mathbb{R}$ is an infinite series of the form:

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

where $a_n \in \mathbb{R}$ and $x \in \mathbb{R}$. If $x_0 = 0$, the series is called a power series centered at the origin.

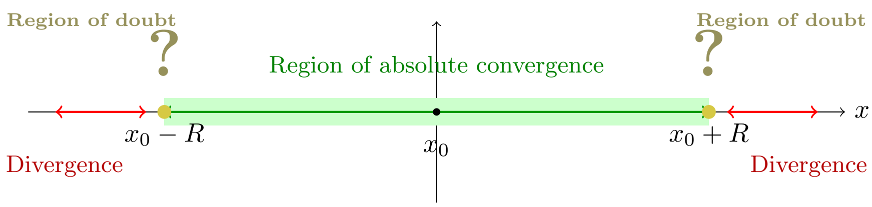
Radius and Interval of Convergence

Consider the real power series centered at $x_0 \in \mathbb{R}$:

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

There exists a non-negative number $R \in [0, \infty]$, called the **radius of convergence**, such that:

- The series **converges absolutely** for all $x \in \mathbb{R}$ such that $|x - x_0| < R$,
- The series **diverges** for all $x \in \mathbb{R}$ such that $|x - x_0| > R$.



The radius of convergence R can be computed using either the **Ratio Test** or the **Root Test**. Suppose one of the following limits exists (possibly infinite):

$$L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|, \quad \text{or} \quad L = \lim_{n \to \infty} \sqrt[n]{|a_n|}.$$

Then the **radius of convergence** is given by:

$$R = \begin{cases} \dfrac{1}{L} & \text{if } L \in (0, \infty), \\ \infty & \text{if } L = 0, \\ 0 & \text{if } L = \infty. \end{cases}$$

The corresponding **interval of convergence** is:

- If $R = \infty$, the series converges absolutely for all $x \in \mathbb{R}$; thus, the interval of convergence is $\mathbb{R}$.
- If $R \geq 0$, the series converges absolutely on the open interval $(x_0 - R,\, x_0 + R)$.

  At the boundary points $x = x_0 - R$ and $x = x_0 + R$, the behavior of the series is generally uncertain. Each endpoint must be tested separately: the series may converge conditionally , or it may diverge . There is no general rule; it depends on the specific form of the series.

Example 1: Geometric Series

Consider the geometric series:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots .$$

This is a power series centered at $x_0 = 0$, with $a_n = 1$ for all n.

We apply the ratio test to determine the values of $x \in \mathbb{R}$ for which the series converges:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}x^{n+1}}{a_n x^n} \right| = \lim_{n\to\infty} |x| = |x| \quad (x \neq 0).$$

According to the Ratio Test:

- The series converges if $|x| < 1$,
- The series diverges if $|x| > 1$,
- The test is inconclusive if $|x| = 1$.

Thus, the **radius of convergence** is:

$$R = 1.$$

Behavior at the Boundary

We must check convergence manually at the endpoints:

- At $x = 1$:

$$\sum_{n=0}^{\infty} 1^n = \sum_{n=0}^{\infty} 1 = 1 + 1 + 1 + \cdots \quad \Rightarrow \text{diverges.}$$

- At $x = -1$:

$$\sum_{n=0}^{\infty} (-1)^n = 1 - 1 + 1 - 1 + \cdots \quad \Rightarrow \text{diverges.}$$

  This is the Grandi series, which oscillates and does not converge.

Conclusion:

$$\sum_{n=0}^{\infty} x^n \text{ converges if and only if } |x| < 1.$$

Radius of convergence: $R = 1$
Interval of convergence: $(-1, 1)$

Example 2: Harmonic-Like Power Series

Consider the series:

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = \frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{3} + \cdots .$$

This is a power series centered at $x_0 = 0$, with $a_n = \frac{1}{n}$.

Step 1: Apply the Ratio Test

We apply the ratio test to determine for which values of $x \in \mathbb{R}$ the series converges:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}x^{n+1}}{a_n x^n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{n+1} \cdot \frac{n}{x^n} \right| = \lim_{n\to\infty} \left| x \cdot \frac{n}{n+1} \right| = |x|. \quad (x \neq 0)$$

Conclusion from Ratio Test:

- The series converges if $|x| < 1$,
- The series diverges if $|x| > 1$,
- The test is inconclusive at $|x| = 1$.

So, the **radius of convergence** is:

$$R = 1.$$

Step 2: Analyze the Boundary $|x| = 1$

We must test convergence at the endpoints.

- At $x = 1$:

$$\sum_{n=1}^{\infty} \frac{1}{n} = \text{harmonic series} \quad \Rightarrow \text{diverges.}$$

- At $x = -1$:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

  This is the **alternating harmonic series**, which satisfies the conditions of the Alternating Series Test:

  – $\frac{1}{n}$ is positive and decreasing,
  – $\lim_{n\to\infty} \frac{1}{n} = 0$,

$$\Rightarrow \text{Converges (conditionally).}$$

Conclusion:

$$\sum_{n=1}^{\infty} \frac{x^n}{n} \text{ converges for } x \in [-1, 1),$$

with:

- **Radius of convergence:** $R = 1$
- **Interval of convergence:** $[-1, 1)$

Example 3: Exponential Series

Consider the series:

$$\sum_{n=1}^{\infty} \frac{x^n}{n!} = \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots .$$

This is a power series centered at $x_0 = 0$, with $a_n = \frac{1}{n!}$.

Step 1: Apply the Ratio Test

We apply the ratio test to determine for which values of $x \in \mathbb{R}$ the series converges:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}x^{n+1}}{a_n x^n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n\to\infty} \left| \frac{x}{n+1} \right| = 0 \quad x \neq 0.$$

Conclusion from Ratio Test:

- Since the limit is 0 for all $x \in \mathbb{R}$, the series converges absolutely for every real number x.

Conclusion:

- **Radius of convergence:** $R = \infty$
- **Interval of convergence:** $(-\infty, \infty)$

This series defines the exponential function:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Example 4: Root Test with a Logarithmic Denominator

Consider the power series:

$$\sum_{n=2}^{\infty} \frac{x^n}{n \log n}.$$

This is a power series centered at $x_0 = 0$, with coefficients $a_n = \frac{1}{n \log n}$ for $n \geq 2$.

We evaluate:

$$\lim_{n\to\infty} |a_n|^{1/n} = \limsup_{n\to\infty} \left(\frac{1}{n \log n} \right)^{1/n}.$$

To simplify this, consider the logarithm:

$$\ln \left(\left(\frac{1}{n \log n} \right)^{1/n} \right) = -\frac{1}{n} \ln(n \log n).$$

We now compute the limit using L'Hôpital's Rule:

$$\lim_{n\to\infty} -\frac{\ln(n \log n)}{n}.$$

Let $f(n) = \ln(n \log n)$ and $g(n) = n$. Since both tend to infinity, we apply L'Hôpital's Rule:

$$\lim_{n\to\infty} -\frac{\ln(n \log n)}{n} = -\lim_{n\to\infty} \frac{d}{dn}[\ln(n \log n)] \Big/ \frac{d}{dn}[n].(n \quad \text{is real number})$$

Differentiate numerator and denominator:

$$\frac{d}{dn}[\ln(n \log n)] = \frac{1}{n \log n} \cdot (\log n + 1) = \frac{\log n + 1}{n \log n}.$$

So the limit becomes:

$$-\lim_{n\to\infty} \frac{\log n + 1}{n \log n} = 0.$$

Therefore,

$$\lim_{n\to\infty} \left(\frac{1}{n \log n} \right)^{1/n} = e^0 = 1.$$

Conclusion from the Root Test:

$$\lim_{n\to\infty} |a_n x^n|^{1/n} = \left(\lim_{n\to\infty} |a_n|^{1/n}\right) \cdot |x| = |x|.$$

So:

- The series converges if $|x| < 1$,
- The series diverges if $|x| > 1$,
- The test is inconclusive at $|x| = 1$.

At $x = 1$:
We consider the series:

$$\sum_{n=2}^{\infty} \frac{1}{n \log n}.$$

To determine convergence, we apply the **integral test**.
Let

$$f(x) = \frac{1}{x \log x}, \quad \text{defined for } x \geq 2.$$

The function $f(x)$ is:

- positive on $[2, \infty)$,
- continuous on $[2, \infty)$,
- decreasing for $x \geq 3$ (since $\log x$ grows slowly).

We evaluate the improper integral:

$$\int_2^{\infty} \frac{1}{x \log x}\, dx.$$

Make the substitution $u = \log x$, so $du = \frac{1}{x} dx$. Then:

$$\int_2^{\infty} \frac{1}{x \log x}\, dx = \lim_{t\to\infty} \int_{\log 2}^{t} \frac{1}{u}\, du = \lim_{t\to\infty} [\log u]_{\log 2}^{t} = \infty.$$

Since the integral diverges, the integral test implies that the series also diverges:

$$\sum_{n=2}^{\infty} \frac{1}{n \log n} \quad \textbf{diverges}.$$

At $x = -1$:

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n \log n}$$

This is an alternating series. It converges conditionally by the Alternating Series Test since:

- $\frac{1}{n \log n}$ is positive and decreasing for $n \geq 3$,
- $\lim_{n\to\infty} \frac{1}{n \log n} = 0$.

Conclusion:

$$\sum_{n=2}^{\infty} \frac{x^n}{n \log n} \text{ converges for } x \in [-1, 1).$$

Radius of convergence: $R = 1$
Interval of convergence: $[-1, 1)$

Properties of Power Series

Let

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

be a real power series with radius of convergence $R > 0$. Then the function f has the following important properties on the open interval $(x_0 - R, x_0 + R)$:

- **Smoothness:** The function f is **infinitely differentiable** on $(x_0 - R, x_0 + R)$, i.e., all derivatives of f exist and are continuous on this interval. This follows from term-by-term differentiation being valid within the radius of convergence.

- **Term-by-term differentiation:** The derivative of f is obtained by differentiating term-by-term:

  $$f'(x) = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1},$$

  and this series has the same radius of convergence R. The term $n = 0$ vanishes in the derivative.

- **Term-by-term integration:** The indefinite integral of f is given by integrating term-by-term:

  $$\int f(x)\, dx = C + \sum_{n=0}^{\infty} \frac{a_n}{n+1} (x - x_0)^{n+1},$$

  where C is the constant of integration. This series also has radius of convergence R.

- **Preservation of convergence:** Both differentiation and integration **preserve the radius of convergence** R. That is, the derived and integrated series converge on the same interval $(x_0 - R, x_0 + R)$ as the original series.

- **Analyticity:** Within the interval of convergence, the function f is **analytic**. That is, not only is f smooth, but it equals its Taylor series expansion centered at x_0:

  $$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n.$$

Analytic Functions

A real function f is called **analytic at** x_0 if there exists a power series $\sum a_n (x - x_0)^n$ that converges to $f(x)$ for all x in some neighborhood of x_0.

In that case, f has a Taylor expansion:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n.$$

Important Notes

- A function can be infinitely differentiable but not analytic (e.g., $f(x) = e^{-1/x^2}$, with $f(0) = 0$, has all derivatives zero at 0, but is not identically zero).

- Power series converge uniformly on compact subintervals of the open interval of convergence.

- Within the interval of convergence, power series can be manipulated like polynomials (term-by-term addition, multiplication, etc.).

Representation of Functions by Power Series

Many functions can be expressed as power series within a suitable interval. A function f is said to be **represented by a power series** centered at $x_0 \in \mathbb{R}$ if there exists a sequence (a_n) such that:

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \quad \text{for all } x \text{ in some interval } (x_0 - R, x_0 + R),$$

where $R > 0$ is the radius of convergence.

Taylor Series

If f is infinitely differentiable at a point x_0, and the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

converges to $f(x)$ for $x \in (x_0 - R, x_0 + R)$, then f is said to be **analytic** at x_0. In this case, f is represented by its **Taylor series expansion**:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n.$$

Conditions and Notes

- Every function represented by a power series is analytic (infinitely differentiable and equal to its Taylor series).
- Not every infinitely differentiable function is analytic.
- The radius of convergence R can be found using the ratio or root test on the coefficients.

Common Examples

- **Exponential function:**

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \text{for all } x \in \mathbb{R}.$$

- **Geometric series:**

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad \text{for } |x| < 1.$$

- **Sine and cosine:**

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \qquad \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \quad \text{for all } x \in \mathbb{R}.$$

- **Natural logarithm:**

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}, \quad \text{for } |x| < 1.$$

- **Arctangent:**

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \quad \text{for } |x| \leq 1.$$

Fourier Series

Fourier series allow us to represent periodic functions as infinite sums of sines and cosines.

1. Periodic Functions

A function $f(t)$ is said to be **periodic** with period $L > 0$ if

$$f(t+L) = f(t) \quad \text{for all } t \in \mathbb{R}.$$

Example. Let $f(t) = \sin(t)$. Then f is periodic with period $L = 2\pi$, because:

$$f(t+2\pi) = \sin(t+2\pi) = \sin(t) = f(t).$$

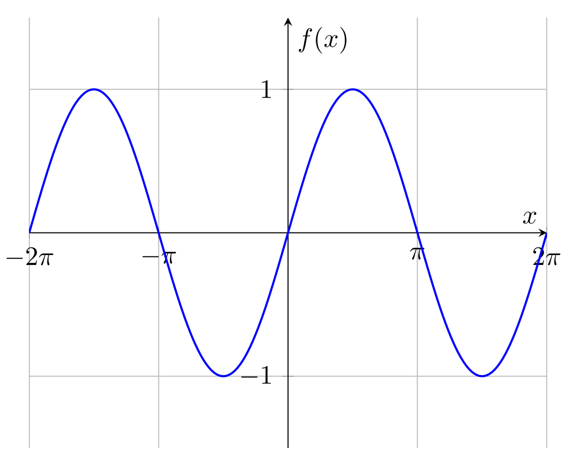
Note that $f(t) = \sin(t)$ also satisfies

$$f(t+4\pi) = \sin(t+4\pi) = \sin(t), \quad f(t+6\pi) = \sin(t+6\pi) = \sin(t), \quad \text{etc.}$$

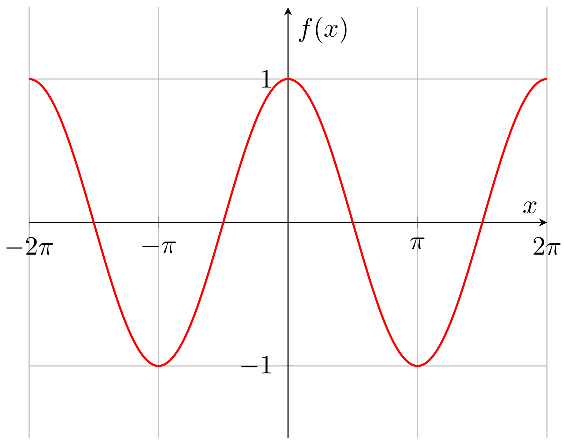
In fact, $\sin(t)$ has period $2n\pi$ for any positive integer $n = 1, 2, 3, \ldots$. However, the **smallest positive period** is 2π, which is called the **period**.

Graphical Interpretation. A function with period L has a graph that repeats itself every L units. That is, the graph remains unchanged when shifted left or right by L.

1. Sine Function $\sin x$



Cosine Function: $\cos x$



Example: Periodic Extension of x^2

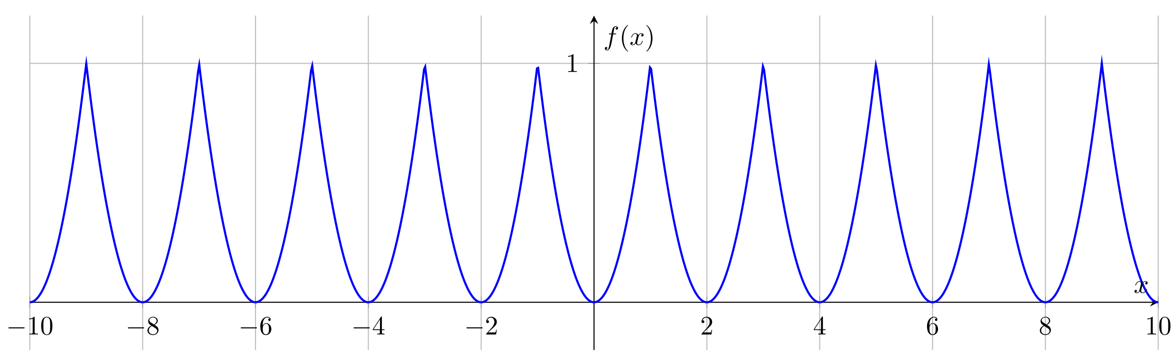
We define the function:

$$f(x) = x^2, \quad x \in (-1, 1),$$

and extend it to all real numbers by periodicity with period 2:

$$f(x + 2) = f(x), \quad \text{for all } x \in \mathbb{R}.$$

To repeat the shape of x^2 from $(-1, 1)$ across the real line, we shift the basic graph by multiples of 2. This means we "copy and paste" the curve x^2 into every interval of length 2.



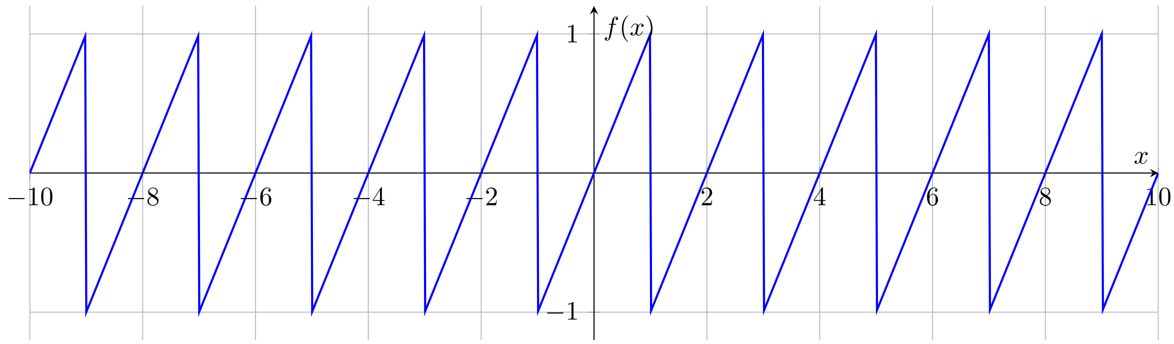
Example: Periodic Extension of x

We define the function:

$$f(x) = x, \quad x \in (-1, 1),$$

and extend it to all real numbers by periodicity with period 2:

$$f(x + 2) = f(x), \quad \text{for all } x \in \mathbb{R}.$$



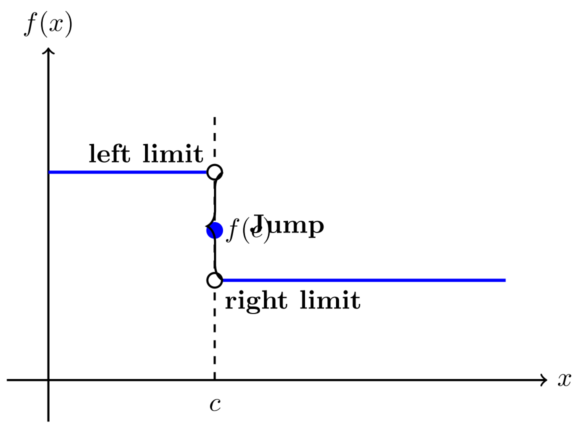
2. Piecewise Continuous Functions and Jump Discontinuities

A function f is called **piecewise continuous** on an interval $[a, b]$ if:

- It is continuous on parts of the interval, except at a finite number of **points of discontinuity**,
- At each point of discontinuity, the left-hand and right-hand limits exist and are finite.

When the two one-sided limits exist but are not equal, the function has a **jump discontinuity**.

Example: Jump Discontinuity



The function has a jump at $x = c$. Both one-sided limits exist but are not equal.

Orthogonality of Trigonometric Functions

The sine and cosine functions are orthogonal on $[-\pi, \pi]$:

$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,dx = \begin{cases} \pi & \text{if } m = n \neq 0, \\ 0 & \text{if } m \neq n, \end{cases}$$

$$\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,dx = \begin{cases} \pi & \text{if } m = n, \\ 0 & \text{if } m \neq n, \end{cases}$$

$$\int_{-\pi}^{\pi} \cos(mx)\sin(nx)\,dx = 0 \quad \text{for all } m, n.$$

These orthogonality relations are the foundation of Fourier coefficients.

3. Fourier Series of a Function

The Fourier series of a function $f(x)$ with period $2L$ is given by:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]$$

where the Fourier coefficients are:

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x)\, dx$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right)\, dx, \qquad b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right)\, dx$$

provided that these integrals exist.
The symbol $\sim$ means that the Fourier series represents an approximation of the function $f(x)$.

In general, the Fourier series of a function f is not exactly equal to f. However, under suitable conditions on f, the series converges to $f(x)$ in various senses — such as:

- **Pointwise convergence** at each point (if f is piecewise smooth),
- **Uniform convergence** on an interval (if f is continuous and periodic),
- **Convergence in norm**, such as in the L^2-sense (i.e., mean-square convergence).

4. Dirichlet's Theorem (Pointwise Convergence)

Let $f(x)$ be a function of period $2L$. Suppose that:

- f is piecewise continuous on $[-L, L]$,
- f' is piecewise continuous on $[-L, L]$.

Then the Fourier series of f converges at every point $x \in \mathbb{R}$ to the average of the left- and right-hand limits:

$$\frac{f(x^-) + f(x^+)}{2} = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]$$

where the **right-hand** and **left-hand** limits as:

$$f(x^+) = \lim_{\xi \to x^+} f(\xi), \quad f(x^-) = \lim_{\xi \to x^-} f(\xi)$$

Furthermore, if f is continuous at x, then the Fourier series converges to $f(x)$:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]$$

Summary

Let $S_N(x)$ denote the N-th partial sum of the Fourier series of a function f with period $2L$:

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]$$

This expression is known as a trigonometric polynomial of degree N.

- If f is **continuous** at a point x, then:

$$\lim_{N\to\infty} S_N(x) = f(x)$$

- If f has a **jump discontinuity** at x, then:

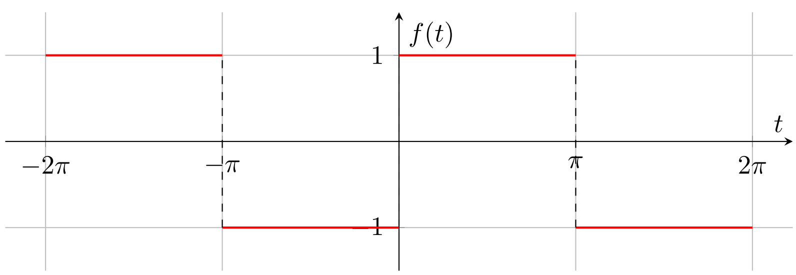
$$\lim_{N\to\infty} S_N(x) = \frac{1}{2}\left(f(x^-) + f(x^+)\right)$$

That is, the Fourier series converges to the midpoint of the jump.

Example: Fourier Series of a Square Wave

Let $f(t)$ be the square wave of period 2π, defined over one period by:

$$f(t) = sign(t) = \begin{cases} -1, & -\pi < t < 0 \\ 1, & 0 < t < \pi \end{cases}$$



We now compute the Fourier series of $f(t)$. Since f is piecewise smooth on $[-\pi, \pi]$, the Fourier series converges (by Dirichlet's Theorem) to:

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nt) + b_n \sin(nt)\right)$$

The Fourier coefficients are given by:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt$$

Step 1: Compute a_0

$$a_0 = \frac{1}{\pi}\left(\int_{-\pi}^{0} (-1)\,dt + \int_{0}^{\pi} 1\,dt\right) = \frac{1}{\pi}(-\pi + \pi) = 0$$

Step 2: Compute a_n

$$a_n = \frac{1}{\pi}\left(\int_{-\pi}^{0} (-1)\cos(nt)\,dt + \int_{0}^{\pi} \cos(nt)\,dt\right)$$

Since cosine is even:

$$a_n = \frac{1}{\pi}\left(-\int_{0}^{\pi} \cos(nt)\,dt + \int_{0}^{\pi} \cos(nt)\,dt\right) = 0$$

Step 3: Compute b_n

$$b_n = \frac{1}{\pi}\left(\int_{-\pi}^{0}(-1)\sin(nt)\,dt + \int_{0}^{\pi}\sin(nt)\,dt\right)$$

Since sine is odd:

$$b_n = \frac{1}{\pi}\left(-\int_{0}^{\pi}\sin(nt)\,dt + \int_{0}^{\pi}\sin(nt)\,dt\right) = \frac{2}{\pi}\int_{0}^{\pi}\sin(nt)\,dt$$

Now compute:

$$\int_{0}^{\pi}\sin(nt)\,dt = \begin{cases} 0, & \text{if } n \text{ is even} \\ \frac{2}{n}, & \text{if } n \text{ is odd} \end{cases}$$

Thus:

$$b_n = \begin{cases} 0, & \text{if } n \text{ is even} \\ \frac{4}{n\pi}, & \text{if } n \text{ is odd} \end{cases}$$

The Fourier series of $f(t)$ is:

$$\begin{aligned} f(t) &\sim \sum_{n=1\ n \text{ odd}}^{\infty} \frac{4}{n\pi}\sin(nt) \\ &\sim \sum_{n=0}^{\infty} \frac{4}{(2n+1)\pi}\sin((2n+1)t) \end{aligned}$$

Only the sine terms with odd indices appear in the expansion.

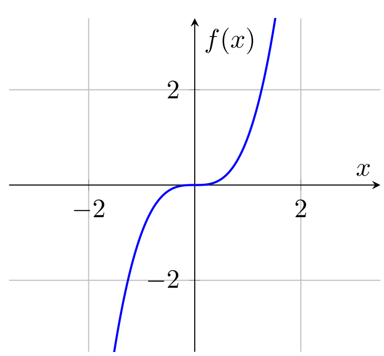
5. Symmetry : Even and Odd Functions

A function f is called **odd** if:

$$f(-x) = -f(x) \quad \text{for all } x \in \mathbb{R}.$$

Examples: $\text{sgn}(x)$, $\quad x$, $\quad x^3$, $\quad \sin(x)$.

Odd Function: $f(x) = x^3$

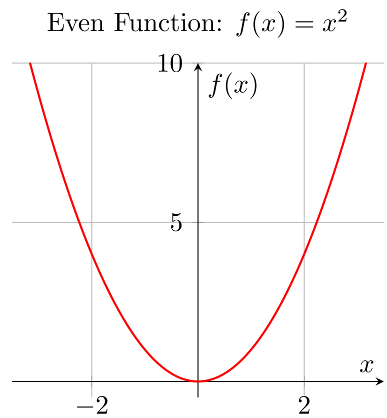


The graph is symmetric with respect to the origin.

A function f is called **even** if:

$$f(-x) = f(x) \quad \text{for all } x \in \mathbb{R}.$$

Examples: $|x|$, $\quad x^2$, $\quad \cos(x)$.



The graph is symmetric with respect to the y-axis.

Even–Odd Decomposition of a Function

Every function $f(x)$ defined on a symmetric interval $[-L, L]$ can be uniquely written as:

$$f(x) = f_{\text{even}}(x) + f_{\text{odd}}(x)$$

With:

$$f_{\text{even}}(x) = \frac{f(x) + f(-x)}{2}, \qquad f_{\text{odd}}(x) = \frac{f(x) - f(-x)}{2}$$

These satisfy:

- f_{even} is even,
- f_{odd} is odd,
- Their sum gives $f(x)$ exactly.

Example: Decompose $f(x) = e^x$

Let $f(x) = e^x$. Then:

$$f_{\text{even}}(x) = \frac{e^x + e^{-x}}{2} = \cosh(x) \quad \text{(even part)}$$

$$f_{\text{odd}}(x) = \frac{e^x - e^{-x}}{2} = \sinh(x) \quad \text{(odd part)}$$

So:

$$e^x = \cosh(x) + \sinh(x)$$

Even and Odd Decomposition in Fourier Series

According to Dirichlet's Theorem, any function $f(x)$ (under suitable conditions) can be written as the sum of an even and an odd function. This is reflected in the structure of the Fourier series:

$$f(x) \sim \underbrace{\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)}_{\text{even part}} + \underbrace{\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)}_{\text{odd part}}$$

- The **even part** consists of cosine terms and the constant term. Since cosine is even, this part is symmetric about the y-axis.
- The **odd part** consists of sine terms. Since sine is odd, this part is symmetric about the origin.

Consequence: Uniqueness of Even–Odd Decomposition

Since the Fourier series naturally separates into an even part (cosine terms) and an odd part (sine terms), we can make the following conclusions:

- If f is **even**, then its odd part must be identically zero. This happens if and only if all sine coefficients vanish:
$$b_n = 0 \quad \text{for all } n \geq 1.$$
Thus, the Fourier series contains only cosine terms (and possibly a_0).

- If f is **odd**, then its even part must be zero. This occurs only when all cosine coefficients vanish, including the constant term:
$$a_0 = 0, \quad a_n = 0 \quad \text{for all } n \geq 1.$$
Therefore, the Fourier series contains only sine terms.

Summary:

- If f is **even**, then all sine coefficients vanish: $b_n = 0$ for all $n \geq 1$.
- If f is **odd**, then all cosine coefficients vanish (including the constant term): $a_0 = 0$, $a_n = 0$ for all $n \geq 1$.

Common Fourier Series

- **Square Wave (odd function):**
$$f(x) = \begin{cases} 1, & 0 < x < \pi, \\ -1, & -\pi < x < 0, \end{cases} \qquad \Rightarrow \qquad f(x) \sim \frac{4}{\pi} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{\sin(nx)}{n}$$
This is a classic example of a piecewise constant, odd function.

- **Sawtooth Wave (odd function):**
$$f(x) = x \quad \text{on } [-\pi, \pi] \qquad \Rightarrow \qquad f(x) \sim -2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin(nx)$$
This function is odd and continuous, but non-smooth at the endpoints.

- **Absolute Value (even function):**
$$f(x) = |x| \quad \text{on } [-\pi, \pi] \qquad \Rightarrow \qquad f(x) \sim \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n 4}{\pi n^2} \cos(nx)$$
Being even, the Fourier series contains only cosine terms.

- **Quadratic Function (even function):**
$$f(x) = x^2 \quad \text{on } [-\pi, \pi] \qquad \Rightarrow \qquad f(x) \sim \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos(nx)$$
Since x^2 is even, the series contains only cosine terms. This is a smooth function, so the series converges rapidly.

Remarks

- Fourier series enable efficient approximation of functions by truncating to a finite number of terms.
- They are crucial in solving partial differential equations (e.g., heat, wave, and Laplace equations).
- Parseval's identity relates the sum of squares of the coefficients to the L^2-norm of the function:

$$\frac{1}{\pi}\int_{-\pi}^{\pi}|f(x)|^2dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right).$$

Fourier Integrals

Motivation

While Fourier series represent periodic functions as sums of sines and cosines, many functions in practice are defined on the entire real line and are not periodic. In such cases, we use the **Fourier integral** to represent non-periodic functions using integrals rather than infinite sums.

Fourier Integral Representation

Let $f(x)$ be a real-valued function defined on $(-\infty, \infty)$. Suppose that f satisfies the following conditions:

(i) f and f' are piecewise continuous on every bounded interval of $\mathbb{R}$,

(ii) f is **absolutely integrable** on $\mathbb{R}$, i.e.,

$$\int_{-\infty}^{\infty} |f(x)|\, dx \quad \text{converges} \quad \text{(as an improper integral).}$$

Then the **Fourier integral representation** of f is given by:

$$f(x) = \frac{1}{\pi} \int_0^{\infty} [A(\lambda)\cos(\lambda x) + B(\lambda)\sin(\lambda x)]\, d\lambda, \tag{1}$$

where the Fourier cosine and sine transforms are defined as:

$$A(\lambda) = \int_{-\infty}^{\infty} f(t)\cos(\lambda t)\, dt, \qquad B(\lambda) = \int_{-\infty}^{\infty} f(t)\sin(\lambda t)\, dt.$$

The identity (1) holds at every point $x \in \mathbb{R}$ where f is continuous. If x is a point of discontinuity, the Fourier integral converges to the average of the left- and right-hand limits:

$$\frac{1}{2}\left(f(x^-) + f(x^+)\right) = \frac{1}{\pi} \int_0^{\infty} [A(\lambda)\cos(\lambda x) + B(\lambda)\sin(\lambda x)]\, d\lambda.$$

Symmetry

- If f is **even**, then:

$$f(x) = \int_0^{\infty} A(\omega)\cos(\omega x)\, d\omega.$$

- If f is **odd**, then:

$$f(x) = \int_0^{\infty} B(\omega)\sin(\omega x)\, d\omega.$$

Example: Rectangular Pulse Function

Define:

$$f(x) = \begin{cases} 1, & |x| \le a, \\ 0, & |x| > a. \end{cases}$$

Then the Fourier transform is:

$$F(\omega) = \frac{1}{2\pi} \int_{-a}^{a} e^{-i\omega t} dt = \frac{\sin(a\omega)}{\pi\omega},$$

and

$$f(x) = \int_{-\infty}^{\infty} \frac{\sin(a\omega)}{\pi\omega} e^{i\omega x} \, d\omega.$$

This is the classical **sinc function** representation.

First Order Differential Equations

Introduction

In this chapter, we study methods for solving **first-order differential equations**. The most general form is:

$$\frac{dy}{dt} = f(y, t)$$

There is no general formula for solving this equation. Instead, we explore several special cases.

Topics Covered

- Separable and linear equations
- Exact equations and integrating factors
- Direction fields and qualitative behavior
- Existence and uniqueness theorems
- Applications in population models, mixing problems, and Newton's law of cooling

Linear First-Order Differential Equations

A linear first-order differential equation has the form:

$$\frac{dy}{dt} + p(t)y = g(t)$$

where $p(t)$ and $g(t)$ are continuous.

Solution Method

1. Compute the **integrating factor**:

$$\mu(t) = e^{\int p(t)\,dt}$$

2. Multiply the entire equation by $\mu(t)$:

$$\mu(t)\frac{dy}{dt} + \mu(t)p(t)y = \mu(t)g(t)$$

3. Recognize the left side as the derivative of a product:

$$\frac{d}{dt}[\mu(t)y(t)] = \mu(t)g(t)$$

4. Integrate both sides:

$$\mu(t)y(t) = \int \mu(t)g(t)\,dt + C$$

5. Solve for $y(t)$:

$$y(t) = \frac{1}{\mu(t)} \left(\int \mu(t) g(t)\, dt + C \right)$$

Summary

The general solution is:

$$y(t) = \frac{1}{e^{\int p(t)\, dt}} \left(\int e^{\int p(t)\, dt} g(t)\, dt + C \right)$$

However, it's often easier to use the process above rather than memorizing the formula.

Example1

Solve the IVP:

$$ty' + 2y = t^2 - t + 1, \quad y(1) = \frac{1}{2}$$

Step 1: Standard form.
Divide by t (assume $t > 0$):

$$y' + \frac{2}{t} y = t - 1 + \frac{1}{t}$$

Step 2: Integrating factor.

$$\mu(t) = e^{\int \frac{2}{t}\, dt} = e^{2 \ln |t|} = t^2$$

Step 3: Multiply through by $\mu(t) = t^2$:

$$t^2 y' + 2ty = t^3 - t^2 + t \quad \Rightarrow \quad \frac{d}{dt}(t^2 y) = t^3 - t^2 + t$$

Step 4: Integrate both sides.

$$t^2 y = \int (t^3 - t^2 + t)\, dt = \frac{1}{4} t^4 - \frac{1}{3} t^3 + \frac{1}{2} t^2 + C$$

Step 5: Solve for $y(t)$.

$$y(t) = \frac{1}{4} t^2 - \frac{1}{3} t + \frac{1}{2} + \frac{C}{t^2}$$

Step 6: Apply initial condition.

$$y(1) = \frac{1}{4} - \frac{1}{3} + \frac{1}{2} + C = \frac{1}{2} \Rightarrow C = \frac{1}{12}$$

Final Solution:

$$\boxed{y(t) = \frac{1}{4} t^2 - \frac{1}{3} t + \frac{1}{2} + \frac{1}{12 t^2}}$$

Example 2

Solve the initial value problem:

$$\cos(x) y' + \sin(x) y = 2 \cos^3(x) \sin(x) - 1, \quad y\left(\frac{\pi}{4}\right) = 3\sqrt{2}, \quad 0 \leq x < \frac{\pi}{2}$$

Dividing both sides by $\cos(x)$, we obtain:

$$y' + \tan(x) y = 2 \cos^2(x) \sin(x) - \sec(x)$$

The integrating factor is:

$$\mu(x) = e^{\int \tan(x)\, dx} = \sec(x)$$

Multiplying the equation by the integrating factor:

$$\sec(x)y' + \sec(x)\tan(x)y = 2\sec(x)\cos^2(x)\sin(x) - \sec^2(x)$$

$$\frac{d}{dx}[\sec(x)y] = 2\cos(x)\sin(x) - \sec^2(x)$$

Integrating both sides:

$$\sec(x)y = \int (2\cos(x)\sin(x) - \sec^2(x))\,dx = \int \sin(2x)\,dx - \int \sec^2(x)\,dx$$

$$\sec(x)y = -\frac{1}{2}\cos(2x) - \tan(x) + C$$

Solving for $y(x)$:

$$y(x) = \cos(x)\left(-\frac{1}{2}\cos(2x) - \tan(x) + C\right)$$

$$y(x) = -\frac{1}{2}\cos(x)\cos(2x) - \cos(x)\tan(x) + C\cos(x)$$

$$y(x) = -\frac{1}{2}\cos(x)\cos(2x) - \sin(x) + C\cos(x)$$

Using the initial condition $y\left(\frac{\pi}{4}\right) = 3\sqrt{2}$, we have:

$$3\sqrt{2} = -\frac{1}{2}\cdot\frac{\sqrt{2}}{2}\cdot 0 - \frac{\sqrt{2}}{2} + C\cdot\frac{\sqrt{2}}{2} \Rightarrow 3\sqrt{2} = -\frac{\sqrt{2}}{2} + C\cdot\frac{\sqrt{2}}{2} \Rightarrow C = 7$$

Thus, the solution is:

$$\boxed{y(x) = -\frac{1}{2}\cos(x)\cos(2x) - \sin(x) + 7\cos(x)}$$

Separable Equations

We now consider nonlinear first-order differential equations, starting with separable equations. A differential equation is separable if it can be written as:

$$N(y)\frac{dy}{dx} = M(x)$$

This means all y-terms are with dy and all x-terms are with dx. Rearranging gives:

$$N(y)\,dy = M(x)\,dx$$

We solve by integrating both sides:

$$\int N(y)\,dy = \int M(x)\,dx$$

This gives an *implicit solution*, which may or may not be solvable for an *explicit solution* $y = y(x)$.

Be aware of the **interval of validity**: the solution is only valid where it is defined (no division by zero, negative logs, etc.).

Most separable equations can be solved using this straightforward technique. We begin with a simple example to illustrate the method.

Example 1

Solve the differential equation:

$$\frac{dy}{dx} = 6y^2x, \quad y(1) = \frac{1}{25}$$

This is a separable equation:

$$y^{-2}dy = 6x\,dx$$

Integrating both sides:

$$\int y^{-2}dy = \int 6x\,dx \quad \Rightarrow \quad -\frac{1}{y} = 3x^2 + C$$

Apply the initial condition $y(1) = \frac{1}{25}$:

$$-\frac{1}{1/25} = 3(1)^2 + C \quad \Rightarrow \quad -25 = 3 + C \Rightarrow C = -28$$

So the solution is:

$$-\frac{1}{y} = 3x^2 - 28 \quad \Rightarrow \quad y(x) = \frac{1}{28 - 3x^2}$$

Interval of Validity:
To avoid division by zero, we require:

$$28 - 3x^2 \neq 0 \quad \Rightarrow \quad x \neq \pm\sqrt{\frac{28}{3}} \approx \pm 3.055$$

The valid intervals are:

$$(-\infty, -\sqrt{28/3}), \quad (-\sqrt{28/3}, \sqrt{28/3}), \quad (\sqrt{28/3}, \infty)$$

Since the initial condition is at $x = 1$, the correct interval of validity is:

$$\boxed{-\sqrt{28/3} < x < \sqrt{28/3}}$$

Note: With different initial conditions, the solution remains the same, but the interval of validity changes:

- If $y(-4) = -\frac{1}{20}$, then the interval is $(-\infty, -\sqrt{28/3})$
- If $y(6) = -\frac{1}{80}$, then the interval is $(\sqrt{28/3}, \infty)$

Example 2

Solve the IVP:

$$\frac{dy}{dx} = \frac{3x^2 + 4x - 4}{2y - 4}, \quad y(1) = 3$$

This is a separable equation. Rearranging and integrating:

$$(2y - 4)dy = (3x^2 + 4x - 4)dx$$

$$\int (2y - 4)\,dy = \int (3x^2 + 4x - 4)\,dx \Rightarrow y^2 - 4y = x^3 + 2x^2 - 4x + C$$

Apply the initial condition $y(1) = 3$:

$$3^2 - 4(3) = 1^3 + 2(1)^2 - 4(1) + C \Rightarrow 9 - 12 = 1 + 2 - 4 + C \Rightarrow -3 = -1 + C \Rightarrow C = -2$$

So the implicit solution is:

$$y^2 - 4y = x^3 + 2x^2 - 4x - 2$$

Solve explicitly using the quadratic formula:

$$y = \frac{4 \pm \sqrt{16 + 4(x^3 + 2x^2 - 4x - 2)}}{2} = 2 \pm \sqrt{x^3 + 2x^2 - 4x + 2}$$

Use the initial condition to determine the correct sign:

$$y(1) = 2 \pm \sqrt{1 + 2 - 4 + 2} = 2 \pm \sqrt{1} = 2 \pm 1 \Rightarrow y(1) = 3 \Rightarrow \text{use } +$$

Thus, the explicit solution is:

$$\boxed{y(x) = 2 + \sqrt{x^3 + 2x^2 - 4x + 2}}$$

Interval of Validity:
We require the argument of the square root to be non-negative:

$$x^3 + 2x^2 - 4x + 2 \geq 0$$

Graphing, the real root is approximately $x \approx -3.36523$, so the interval of validity is:

$$\boxed{x \geq -3.36523}$$

which contains $x = 1$, satisfying the initial condition.

Exact Equations

We now explore a new class of first-order differential equations: **exact equations**. Before diving into the solution method, we demonstrate the concept using an example.

Example 1

$$2xy - 9x^2 + (2y + x^2 + 1)\frac{dy}{dx} = 0$$

Assume a function $\Psi(x, y)$ exists such that:

$$\Psi(x, y) = y^2 + (x^2 + 1)y - 3x^3$$

Then,

$$\frac{\partial \Psi}{\partial x} = 2xy - 9x^2, \quad \frac{\partial \Psi}{\partial y} = 2y + x^2 + 1$$

So the equation becomes:

$$\frac{d}{dx}[\Psi(x, y(x))] = 0 \Rightarrow \Psi(x, y) = C$$

Implicit solution:

$$y^2 + (x^2 + 1)y - 3x^3 = C$$

General Form

A differential equation is exact if:

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0$$

and there exists a function $\Psi(x, y)$ such that:

$$\frac{\partial \Psi}{\partial x} = M, \quad \frac{\partial \Psi}{\partial y} = N$$

Then the solution is:

$$\Psi(x, y) = C$$

To test for exactness:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Example 2:

$$2xy - 9x^2 + (2y + x^2 + 1)\frac{dy}{dx} = 0, \quad y(0) = -3$$

$$M = 2xy - 9x^2 \Rightarrow M_y = 2x, \quad N = 2y + x^2 + 1 \Rightarrow N_x = 2x$$

Exact equation.

Integrate M with respect to x:

$$\Psi(x, y) = \int (2xy - 9x^2)\, dx = x^2y - 3x^3 + h(y)$$

Differentiate with respect to y:

$$\frac{\partial \Psi}{\partial y} = x^2 + h'(y) = 2y + x^2 + 1 \Rightarrow h'(y) = 2y + 1 \Rightarrow h(y) = y^2 + y$$

So:

$$\Psi(x, y) = x^2y - 3x^3 + y^2 + y$$

Implicit solution:

$$x^2y - 3x^3 + y^2 + y = C$$

Apply initial condition $y(0) = -3$:

$$0 + 9 - 3 = C \Rightarrow C = 6$$

$$y^2 + (x^2 + 1)y - 3x^3 = 6$$

Solve using quadratic formula:

$$y = -(x^2 + 1) \pm \frac{\sqrt{(x^2 + 1)^2 + 12x^3 + 24}}{2}$$

Choose $-$ sign based on initial condition:

$$y(x) = -(x^2 + 1) - \frac{\sqrt{x^4 + 12x^3 + 2x^2 + 25}}{2}$$

Interval of validity:

$$x^4 + 12x^3 + 2x^2 + 25 \geq 0 \Rightarrow x \geq -1.396911133$$

Example 3

$$2xy^2 + 4 = 2(3 - x^2y)y', \quad y(-1) = 8$$

Rewriting:

$$2xy^2 + 4 + 2(x^2y - 3)y' = 0$$

$$M = 2xy^2 + 4, \quad N = 2x^2y - 6$$

$$M_y = 4xy, \quad N_x = 4xy \Rightarrow \text{Exact}$$

Integrate N w.r.t. y:

$$\Psi(x, y) = \int (2x^2y - 6)dy = x^2y^2 - 6y + h(x)$$

Differentiate w.r.t. x:

$$\frac{\partial \Psi}{\partial x} = 2xy^2 + h'(x) = 2xy^2 + 4 \Rightarrow h'(x) = 4 \Rightarrow h(x) = 4x$$

$$\Psi(x, y) = x^2y^2 - 6y + 4x$$

Implicit solution:

$$x^2y^2 - 6y + 4x = C$$

Apply initial condition:

$$64 - 48 - 4 = C \Rightarrow C = 12$$

$$x^2y^2 - 6y + 4x - 12 = 0$$

Solve for y:

$$y = \frac{6 \pm \sqrt{36 + 48x^2 - 16x^3}}{2x^2} = 3 \pm \frac{\sqrt{9 + 12x^2 - 4x^3}}{x^2}$$

Use initial condition to choose "+":

$$y(x) = 3 + \frac{\sqrt{9 + 12x^2 - 4x^3}}{x^2}$$

Interval of validity:

$$x \neq 0, \quad 9 + 12x^2 - 4x^3 > 0 \Rightarrow x \in (-\infty, 0) \text{ since } x = -1 \text{ is valid}$$

$$\boxed{(-\infty, 0)}$$