

Course Notes: Math 209

Sequences

1. Definition of a Sequence

A **sequence** is a function from the natural numbers $\mathbb{N}$ to the real numbers $\mathbb{R}$. It is usually written as:

$$\{a_n\}_{n=1}^{\infty} = a_1, a_2, a_3, \dots$$

Each term a_n is called the *n*th term of the sequence.

Examples

- $a_n = \frac{1}{n} \Rightarrow \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$
- $a_n = (-1)^n \Rightarrow \{-1, 1, -1, 1, \dots\}$
- $a_n = n^2 \Rightarrow \{1, 4, 9, 16, \dots\}$

2. Convergence of Sequences

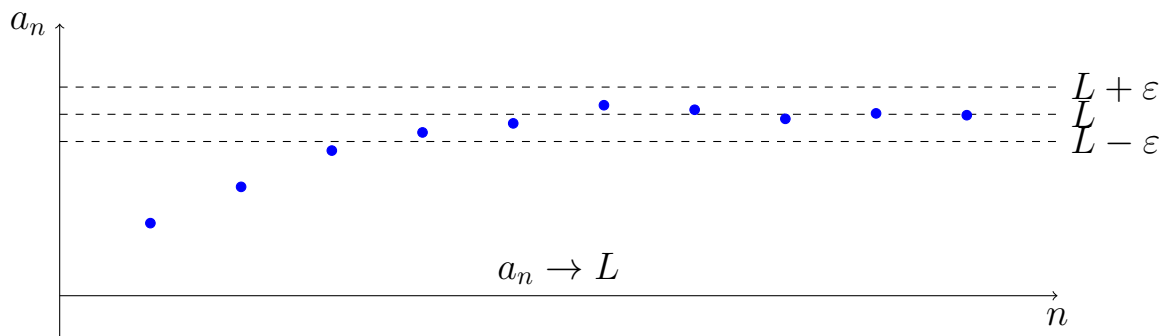
A sequence $\{a_n\}$ **converges** to a real number L if:

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ such that } n \geq N \Rightarrow |a_n - L| < \varepsilon.$$

We denote this as $\lim_{n \rightarrow \infty} a_n = L$, or simply $a_n \rightarrow L$.

Geometric Interpretation:

To visualize this definition, we draw a horizontal band of width 2ε centered at L . This band represents the ε -neighborhood of the limit. The key idea is that, while a few early terms may fall outside this band, eventually all terms of the sequence lie inside it.



The dashed line at $y = L$ shows the limit of the sequence (a_n) . The two dashed lines at $y = L + \varepsilon$ and $y = L - \varepsilon$ form a horizontal band around L , called the ε -neighborhood. This band represents a tolerance zone: how close the sequence terms must be to the limit. As shown, while a few early terms lie outside this band, from a certain index N onward, all terms lie within it — that is, they satisfy $|a_n - L| < \varepsilon$. This illustrates the formal definition of convergence: the terms of the sequence get arbitrarily close to the limit and eventually stay there.

Divergence

If a sequence does not converge, it is said to **diverge**. For example:

- $a_n = n$ diverges to infinity.
- Let $a_n = (-1)^n$. Then:

$$a_1 = -1, a_2 = 1, a_3 = -1, a_4 = 1, \dots$$

This sequence does not converge. It keeps jumping between -1 and 1 , so:

$$\lim_{n \rightarrow \infty} a_n \text{ does not exist.}$$

3. Properties of Convergent Sequences

Let $a_n \rightarrow a$ and $b_n \rightarrow b$. Then:

- $a_n + b_n \rightarrow a + b$
- $a_n b_n \rightarrow ab$

- If $b \neq 0$ and $b_n \neq 0$ for all n , then $\frac{a_n}{b_n} \rightarrow \frac{a}{b}$
- $|a_n| \rightarrow |a|$

Also, every convergent sequence is bounded.

4. Examples

Example 1:

$$a_n = \frac{2n^2 - 3n}{3n^2 + 5n + 3}$$

Highest powers dominate:

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{2n^2}{3n^2} = \frac{2}{3}$$

Justification: Divide numerator and denominator by n^2 :

$$a_n = \frac{2n^2 - 3n}{3n^2 + 5n + 3} = \frac{2 - \frac{3}{n}}{3 + \frac{5}{n} + \frac{3}{n^2}}.$$

Since

$$\frac{1}{n} \rightarrow 0 \quad \text{and} \quad \frac{1}{n^2} \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

we get:

$$\lim_{n \rightarrow \infty} a_n = \frac{2 - 0}{3 + 0 + 0} = \frac{2}{3}.$$

General Rule: For $a_n = \frac{P(n)}{Q(n)}$:

- If $\deg P < \deg Q$, then $\lim a_n = 0$
- If $\deg P = \deg Q$, then $\lim a_n$ equals the ratio of leading coefficients
- If $\deg P > \deg Q$, then $\lim a_n = \infty$ or $-\infty$ (divergent)

Example 2: Geometric and Power Sequences

Geometric Sequences

Let (a_n) be the geometric sequence $a_n = r^n$ for a fixed real r and $n \in \mathbb{N}$. Then

$$\lim_{n \rightarrow \infty} r^n = \begin{cases} 0, & \text{if } |r| < 1, \\ 1, & \text{if } r = 1, \\ \text{does not exist (oscillates between } \pm 1), & \text{if } r = -1, \\ +\infty, & \text{if } r > 1, \\ \text{diverges (unbounded and sign-alternating),} & \text{if } r < -1. \end{cases}$$

1) Case: $r > 0$. For $r > 0$, we may write

$$r^n = e^{n \ln r}.$$

- If $0 < r < 1$, then $\ln r < 0$, hence $n \ln r \rightarrow -\infty$ and thus $r^n = e^{n \ln r} \rightarrow 0$.
- If $r = 1$, then $r^n = 1$ for all n , so $\lim r^n = 1$.
- If $r > 1$, then $\ln r > 0$, hence $n \ln r \rightarrow +\infty$ and $r^n = e^{n \ln r} \rightarrow +\infty$ (diverges).

2) Case: $r < 0$. Write $r = -s$ with $s = |r| > 0$. Then

$$r^n = (-1)^n s^n \quad \text{and} \quad |r^n| = s^n.$$

- If $-1 < r < 0$, then $0 < s < 1$. From the positive-base case, $s^n \rightarrow 0$. Since $|r^n| = s^n \rightarrow 0$, we have $-s^n \leq r^n \leq s^n$ for all n , so by the squeeze theorem $r^n \rightarrow 0$.
- If $r = -1$ (i.e., $s = 1$), then $r^n = (-1)^n$ so the even subsequence equals 1 and the odd subsequence equals -1 . The two subsequences have different limits, hence $\lim r^n$ does not exist (oscillation).
- If $r < -1$, then $s = |r| > 1$ and $s^n \rightarrow \infty$. Consequently $|r^n| = s^n \rightarrow \infty$ while the factor $(-1)^n$ makes the signs alternate. The sequence is unbounded and has no limit.

This completes the classification.

Power sequence (also called "dual geometric"):

$$\lim_{n \rightarrow \infty} n^r = \begin{cases} 0, & \text{if } r < 0, \\ 1, & \text{if } r = 0, \\ \infty, & \text{if } r > 0. \end{cases}$$

- $n^0 = 1$ for all n ,
- $n^r \rightarrow \infty$ if $r > 0$,
- $n^r = \frac{1}{n^{-r}} \rightarrow 0$ if $r < 0$.

Example 3:

Show that

$$\lim_{n \rightarrow \infty} n \cdot \sin \left(\frac{1}{n} \right) = 1.$$

As

$$\frac{1}{n} \rightarrow 0 \quad \text{and} \quad \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1,$$

we rewrite:

$$n \cdot \sin \left(\frac{1}{n} \right) = \frac{\sin \left(\frac{1}{n} \right)}{\frac{1}{n}} \rightarrow 1.$$

General Rule:

- If $\lim_{x \rightarrow 0} f(x) = L$, then $\lim_{n \rightarrow \infty} f \left(\frac{1}{n} \right) = L$.
- If $\lim_{x \rightarrow \infty} f(x) = L$, then $\lim_{n \rightarrow \infty} f(n) = L$.

Example 4

If $\alpha > 0$, then:

$$\lim_{n \rightarrow \infty} \alpha^{1/n} = 1.$$

Let $a_n = \alpha^{1/n}$. Then:

$$\ln a_n = \frac{\ln \alpha}{n} \rightarrow 0.$$

So,

$$a_n = e^{\ln a_n} \rightarrow e^0 = 1.$$

Example 5

The sequence $(n^{1/n})$ converges to 1:

$$\lim_{n \rightarrow \infty} n^{1/n} = 1.$$

Let $a_n = n^{1/n}$. Then:

$$\ln a_n = \frac{\ln n}{n}.$$

To evaluate the limit, apply L'Hôpital's Rule to:

$$\lim_{n \rightarrow \infty} \frac{\ln n}{n}.$$

Since both numerator and denominator tend to ∞ , we differentiate top and bottom:

$$\lim_{n \rightarrow \infty} \frac{\ln n}{n} = \lim_{n \rightarrow \infty} \frac{1/n}{1} = 0.$$

Therefore:

$$\ln a_n \rightarrow 0 \quad \Rightarrow \quad a_n = e^{\ln a_n} \rightarrow e^0 = 1.$$

5. Monotone Sequences

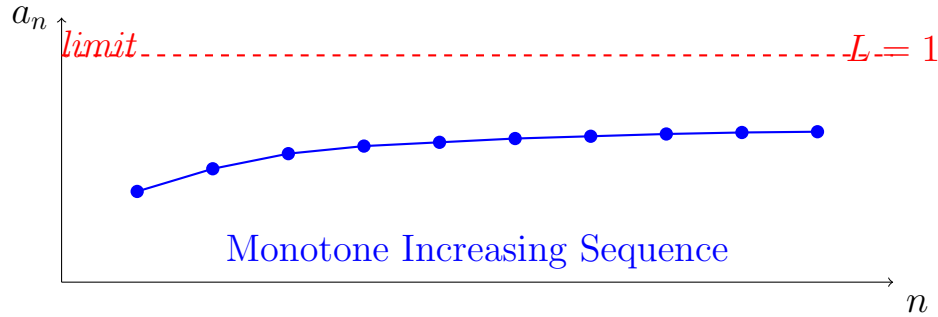
A sequence $\{a_n\}$ is:

- **Increasing** if $a_{n+1} \geq a_n$ for all n
- **Decreasing** if $a_{n+1} \leq a_n$ for all n
- **Monotonic** if it is either increasing or decreasing

Theorem (Monotone Convergence Theorem): If a sequence is monotonic and bounded, then it converges.

Example 1:

Let $a_n = 1 - \frac{1}{n}$. It is increasing and bounded above by 1, so the sequence a_n is convergent.



Explanation:

- The blue points represent terms a_n of a sequence.
- Each term is greater than or equal to the previous — the sequence is increasing.
- The red dashed line is the horizontal asymptote at $L = 1$, showing the limit.
- The sequence approaches the limit from below but never exceeds it.

Example 2: Sequence of partial sum

Let a_n be a sequence of non-negative numbers, meaning that $a_n \geq 0$ for every $n \in \mathbb{N}$. The sequence of partial sums associated with a_n is defined by:

$$S_N = \sum_{k=1}^N a_k = a_1 + a_2 + \cdots + a_N.$$

This means that :

$$\begin{aligned} S_1 &= a_1, \\ S_2 &= a_1 + a_2, \\ S_3 &= a_1 + a_2 + a_3, \\ &\vdots \end{aligned}$$

Since each term $a_n \geq 0$, adding a new term always makes the total sum stay the same or increase:

$$S_{N+1} - S_N = a_{N+1} \geq 0.$$

We now consider two possible situations:

- **Case 1:** The sequence (S_N) is bounded above.
Since S_N is increasing and bounded, the Monotonic Convergence Theorem tells us that S_N converges to a finite limit.
- **Case 2:** The sequence (S_N) is not bounded above.
In this case, the partial sums grow without limit, that is,

$$\lim_{N \rightarrow \infty} S_N = \infty.$$

6. Squeeze Theorem

Theorem: Let $\{a_n\}, \{b_n\}, \{c_n\}$ be sequences such that:

$$a_n \leq b_n \leq c_n \text{ for all } n \geq N,$$

and if $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_n = L$, then:

$$\lim_{n \rightarrow \infty} b_n = L.$$

Example 1:

Show that $\lim_{n \rightarrow \infty} \frac{\sin n}{n} = 0$.

Since $-1 \leq \sin n \leq 1$, we have:

$$-\frac{1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}.$$

Both bounds go to 0, so by the Squeeze Theorem:

$$\lim_{n \rightarrow \infty} \frac{\sin n}{n} = 0.$$

Exercises

Exercise 1

Decide whether the following sequences converge or diverge; if they converge, find the limit:

$$a_n = \frac{5n}{e^n}, \quad b_n = \frac{n^2}{2^n + 1}, \quad c_n = \frac{\ln n}{n}.$$

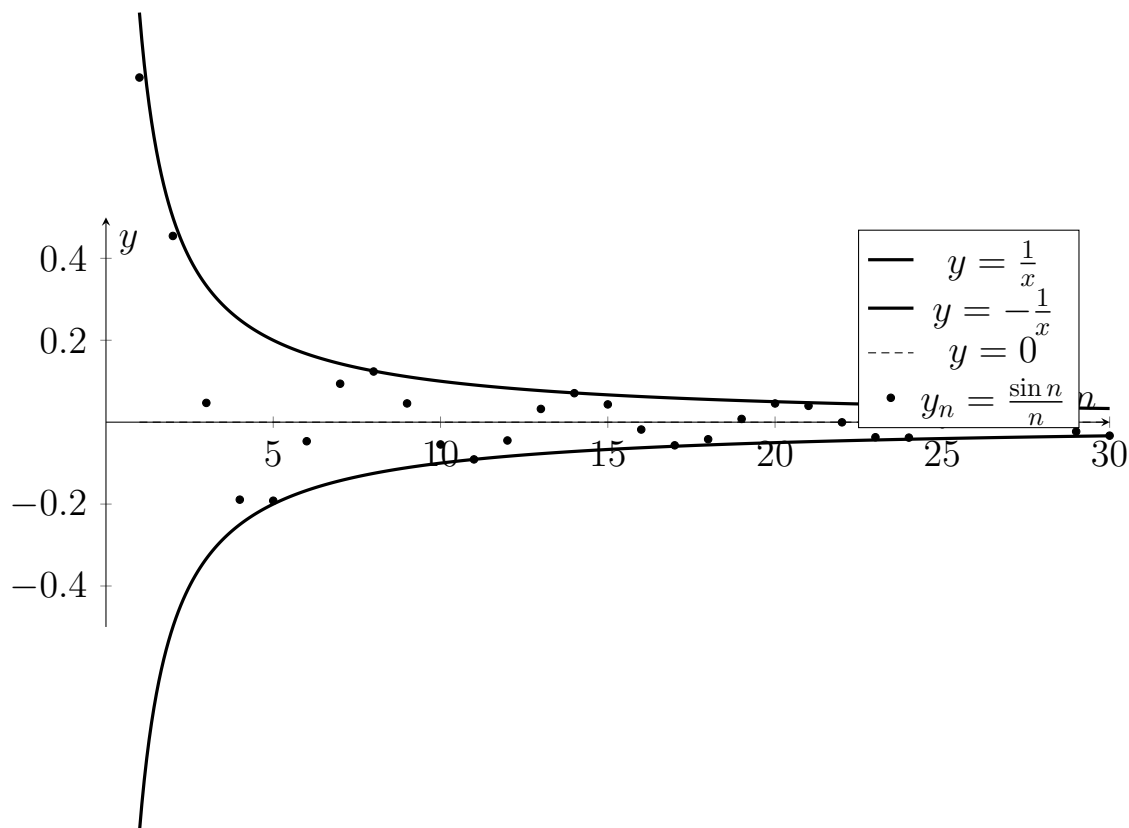


Figure 1: Points $y_n = \sin n/n$ squeezed between $y = \pm 1/x$ toward 0.

Solution

We analyze each sequence separately.

(1) $a_n = \frac{5n}{e^n}.$

Consider the function $f(x) = \frac{5x}{e^x}$. Then $a_n = f(n)$. As $x \rightarrow \infty$, this is an indeterminate form $\frac{\infty}{\infty}$. By L'Hôpital's rule,

$$\lim_{x \rightarrow \infty} \frac{5x}{e^x} = \lim_{x \rightarrow \infty} \frac{5}{e^x} = 0.$$

Hence,

$$\lim_{n \rightarrow \infty} a_n = 0.$$

(2) $b_n = \frac{n^2}{2^n + 1}.$

Define $f(x) = \frac{x^2}{2^x + 1}$. As $x \rightarrow \infty$, this is of the form $\frac{\infty}{\infty}$. Applying

L'Hôpital's rule twice:

$$\lim_{x \rightarrow \infty} \frac{x^2}{2^x + 1} = \lim_{x \rightarrow \infty} \frac{2x}{2^x \ln 2} = \lim_{x \rightarrow \infty} \frac{2}{(\ln 2)^2 2^x} = 0.$$

Therefore,

$$\lim_{n \rightarrow \infty} b_n = 0.$$

$$(3) \quad c_n = \frac{\ln n}{n}.$$

Consider $f(x) = \frac{\ln x}{x}$. This is again an $\frac{\infty}{\infty}$ form. By L'Hôpital's rule:

$$\lim_{x \rightarrow \infty} \frac{\ln x}{x} = \lim_{x \rightarrow \infty} \frac{1/x}{1} = 0.$$

Thus,

$$\lim_{n \rightarrow \infty} c_n = 0.$$

Exercise 2

Discuss the convergence of the following sequences:

(i) $(-1.2)^n$

(ii) $\frac{1}{2^n}$

(iii) $\frac{\cos^2(n)}{3^n}$

(iv) $\frac{4n^3 + 5n + 1}{2n^3 + n^2 + 5}$

(v) $\left(1 + \frac{1}{n}\right)^n$

(vi) $n^{1/n}$

Solution

(i) $a_n = (-1.2)^n$. This is a geometric sequence with common ratio $r = -1.2$. For a geometric sequence $a_n = r^n$:

$$\begin{cases} |r| < 1 & \Rightarrow a_n \rightarrow 0, \\ |r| = 1 & \Rightarrow a_n \text{ oscillates or is constant,} \\ |r| > 1 & \Rightarrow |a_n| \rightarrow \infty \text{ (diverges).} \end{cases}$$

Since $|r| = 1.2 > 1$, the sequence diverges.

(ii) $b_n = \frac{1}{2^n} = (\frac{1}{2})^n$. This is a geometric sequence with common ratio $r = -1.2$. Therefore we have

$$b_n \rightarrow 0.$$

(iii) $c_n = \frac{\cos^2(n)}{3^n}$. The numerator satisfies $0 \leq \cos^2(n) \leq 1$, while the denominator grows without bound as $3^n \rightarrow \infty$. Therefore,

$$0 \leq c_n \leq \frac{1}{3^n} \rightarrow 0.$$

By the squeeze theorem, $c_n \rightarrow 0$.

(iv) $d_n = \frac{4n^3 + 5n + 1}{2n^3 + n^2 + 5}$. Dividing numerator and denominator by n^3 :

$$d_n = \frac{4 + \frac{5}{n^2} + \frac{1}{n^3}}{2 + \frac{1}{n} + \frac{5}{n^3}} \xrightarrow{n \rightarrow \infty} \frac{4}{2} = 2.$$

(v) $e_n = \left(1 + \frac{1}{n}\right)^n$. Take logarithms:

$$\ln(e_n) = n \ln\left(1 + \frac{1}{n}\right) = \frac{\ln(1 + 1/n)}{1/n}.$$

Let $x = 1/n \rightarrow 0^+$. Then

$$\lim_{n \rightarrow \infty} \ln(e_n) = \lim_{x \rightarrow 0^+} \frac{\ln(1 + x)}{x} = 1.$$

Thus $\lim_{n \rightarrow \infty} e_n = e$.

(vi) $f_n = n^{1/n}$. Take logarithms:

$$\ln(f_n) = \frac{\ln n}{n}.$$

As $n \rightarrow \infty$, this is $\frac{\infty}{\infty}$. By L'Hôpital's rule:

$$\lim_{x \rightarrow \infty} \frac{\ln x}{x} = \lim_{x \rightarrow \infty} \frac{1/x}{1} = 0.$$

So $\ln(f_n) \rightarrow 0$, which implies $f_n \rightarrow e^0 = 1$.

Exercise 3

Decide whether the following sequences converge or diverge; if they converge, find the limit:

$$a_n = \sqrt{n+1} - \sqrt{n}, \quad b_n = \frac{n^2}{2n-1} - \frac{n^2}{2n+1}.$$

Solution

(1) $a_n = \sqrt{n+1} - \sqrt{n}$.

Rationalize the difference:

$$a_n = \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}} = \frac{(n+1) - n}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}}.$$

Hence

$$0 < a_n = \frac{1}{\sqrt{n+1} + \sqrt{n}} \leq \frac{1}{2\sqrt{n}} \xrightarrow{n \rightarrow \infty} 0,$$

so by squeeze theorem $\lim_{n \rightarrow \infty} a_n = 0$.

(2) $b_n = \frac{n^2}{2n-1} - \frac{n^2}{2n+1}$.

Combine the fractions:

$$b_n = n^2 \left(\frac{1}{2n-1} - \frac{1}{2n+1} \right) = n^2 \frac{(2n+1) - (2n-1)}{(2n-1)(2n+1)} = \frac{2n^2}{4n^2 - 1}.$$

Therefore

$$\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{2n^2}{4n^2 - 1} = \frac{2}{4} = \frac{1}{2}.$$

Infinite series

1. Introduction

An **infinite series** is the sum of the terms of a sequence:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

To analyze the convergence of this series, we consider the **sequence of partial sums**:

$$S_N = \sum_{k=1}^N a_k.$$

We say that the series $\sum_{n=1}^{\infty} a_n$ **converges** if the limit of the sequence $\{S_N\}$ exists and is finite. In that case, the value of the infinite sum is defined as:

$$\sum_{n=1}^{\infty} a_n = \lim_{N \rightarrow \infty} S_N.$$

If this limit does not exist or is infinite, the series is said to **diverge**.

2. Telescoping Series

Evaluate the infinite series:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}.$$

We first decompose the general term using partial fractions:

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1},$$

and then consider the partial sum:

$$S_N = \sum_{n=1}^N \left(\frac{1}{n} - \frac{1}{n+1} \right).$$

We expand the partial sum:

$$\begin{aligned} S_N = & \left(\frac{\textcolor{blue}{1}}{\textcolor{blue}{1}} - \frac{\textcolor{red}{1}}{\textcolor{red}{2}} \right) + \left(\frac{\textcolor{red}{1}}{\textcolor{red}{2}} - \frac{\textcolor{green}{1}}{\textcolor{green}{3}} \right) + \left(\frac{\textcolor{green}{1}}{\textcolor{green}{3}} - \frac{\textcolor{orange}{1}}{\textcolor{orange}{4}} \right) + \cdots \\ & + \left(\frac{\textcolor{violet}{1}}{\textcolor{violet}{N}} - \frac{\textcolor{teal}{1}}{\textcolor{teal}{N+1}} \right). \end{aligned}$$

After cancellation, only the first and the last term remain:

$$S_N = \textcolor{blue}{1} - \frac{\textcolor{teal}{1}}{\textcolor{teal}{N+1}}.$$

Taking the limit as $N \rightarrow \infty$, we find the sum of the infinite series:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{N \rightarrow \infty} S_N = \lim_{N \rightarrow \infty} \left(1 - \frac{1}{N+1} \right) = 1.$$

The series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges, and its sum is

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$$

General Rule (Telescoping Form):

If a sequence satisfies $a_n = b_n - b_{n+1}$, then the partial sum telescopes:

$$S_N = \sum_{n=1}^N a_n = b_1 - b_{N+1} \quad \Rightarrow \quad \sum_{n=1}^{\infty} a_n = b_1 - \lim_{N \rightarrow \infty} b_{N+1},$$

provided the limit exists.

Exercise

Evaluate the series

$$\sum_{n=3}^{\infty} \frac{1}{n(n+1)}.$$

First, we must establish that the series converges. By recalling the result from the previous example,

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots = 1.$$

Hence

$$\begin{aligned} \sum_{n=3}^{\infty} \frac{1}{n(n+1)} &= \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \cdots \\ &= \left(\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \cdots \right) - \left(\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} \right) \\ &= \sum_{n=1}^{\infty} \frac{1}{n(n+1)} - \left(\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} \right) \\ &= 1 - \left(\frac{1}{2} + \frac{1}{6} \right). \end{aligned}$$

Compute explicitly with a common denominator 6:

$$\frac{1}{2} = \frac{3}{6}, \quad \frac{1}{6} = \frac{1}{6} \quad \Rightarrow \quad \frac{1}{2} + \frac{1}{6} = \frac{3}{6} + \frac{1}{6} = \frac{4}{6} = \frac{2}{3}.$$

So

$$\sum_{n=3}^{\infty} \frac{1}{n(n+1)} = 1 - \frac{2}{3} = \boxed{\frac{1}{3}}.$$

Application to the Geometric Series $\sum_{n=0}^{\infty} r^n$:

$$\begin{aligned} (1-r) \sum_{n=0}^N r^n &= \sum_{n=0}^N (r^n - r^{n+1}) \\ &= (r^0 - r^1) + (r^1 - r^2) + \cdots + (r^N - r^{N+1}) \\ &= r^0 - r^{N+1} = 1 - r^{N+1}. \end{aligned}$$

Dividing both sides by $1 - r$, we obtain the formula for the partial sum:

$$\sum_{n=0}^N r^n = \frac{1 - r^{N+1}}{1 - r}, \quad r \neq 1.$$

Geometric Series Convergence ("r-Test"):

$$\sum_{n=0}^{\infty} r^n \text{ converges} \iff |r| < 1.$$

Sum Formula:

$$\sum_{n=0}^{\infty} r^n = \begin{cases} \frac{1}{1 - r}, & \text{if } |r| < 1, \\ \text{diverges,} & \text{if } |r| \geq 1. \end{cases}$$

Example:

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1 - \frac{1}{2}} = 2.$$

3. Basic Properties

• **Linearity:**

$$\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$

• **Multiplication by a constant:**

$$\sum_{n=1}^{\infty} ca_n = c \sum_{n=1}^{\infty} a_n$$

Exercise

Evaluate the series

$$(a) \quad \sum_{n=0}^{\infty} \frac{2^{n+1}}{3^{n-2}}, \quad (b) \quad \sum_{n=0}^{\infty} \frac{2^n + 3^n}{5^n}.$$

$$(a) \quad \sum_{n=0}^{\infty} \frac{2^{n+1}}{3^{n-2}}.$$

Rewrite the general term as a geometric term:

$$\frac{2^{n+1}}{3^{n-2}} = \frac{2^{n+1} 3^2}{3^n} = 9 \cdot \frac{2^{n+1}}{3^n} = 18 \cdot \frac{2^n}{3^n} = 18 \left(\frac{2}{3}\right)^n.$$

Hence

$$\sum_{n=0}^{\infty} \frac{2^{n+1}}{3^{n-2}} = \sum_{n=0}^{\infty} 18 \left(\frac{2}{3}\right)^n = 18 \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n.$$

The ratio is $r = \frac{2}{3}$ with $|r| < 1$, so

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \quad \Rightarrow \quad \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = \frac{1}{1-\frac{2}{3}} = \frac{1}{\frac{1}{3}} = 3.$$

Therefore

$$\sum_{n=0}^{\infty} \frac{2^{n+1}}{3^{n-2}} = 18 \cdot 3 = \boxed{54}.$$

$$(b) \quad \sum_{n=0}^{\infty} \frac{2^n + 3^n}{5^n}.$$

By linearity, split the series into two:

$$\sum_{n=0}^{\infty} \frac{2^n + 3^n}{5^n} = \sum_{n=0}^{\infty} \left(\frac{2}{5}\right)^n + \sum_{n=0}^{\infty} \left(\frac{3}{5}\right)^n.$$

Each is a geometric series with ratios $r_1 = \frac{2}{5}$ and $r_2 = \frac{3}{5}$, both satisfying $|r_i| < 1$, so they converge and

$$\sum_{n=0}^{\infty} \left(\frac{2}{5}\right)^n = \frac{1}{1-\frac{2}{5}} = \frac{5}{3}, \quad \sum_{n=0}^{\infty} \left(\frac{3}{5}\right)^n = \frac{1}{1-\frac{3}{5}} = \frac{5}{2}.$$

Therefore,

$$\sum_{n=0}^{\infty} \frac{2^n + 3^n}{5^n} = \frac{5}{3} + \frac{5}{2} = \frac{10}{6} + \frac{15}{6} = \boxed{\frac{25}{6}}.$$

Necessary condition for convergence

If the series $\sum_{n=1}^{\infty} a_n$ converges, then

$$\lim_{n \rightarrow \infty} a_n = 0.$$

The converse is *false*: having $a_n \rightarrow 0$ does not guarantee that the series converges.

Why? Let the partial sums be $S_n := \sum_{k=1}^n a_k$. Then

$$S_n - S_{n-1} = (a_1 + \cdots + a_n) - (a_1 + a_2 + \cdots + a_{n-1}) = a_n \quad (n \geq 2).$$

If $\sum_{n=1}^{\infty} a_n$ converges, the sequence (S_n) converges to the limit S ($S = \sum_{n=1}^{\infty} a_n$). Hence

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} (S_n - S_{n-1}) = \lim_{n \rightarrow \infty} S_n - \lim_{n \rightarrow \infty} S_{n-1} = S - S = 0.$$

4. Convergence Tests for Positive Series

We are concerned with series $\sum_{n=1}^{\infty} a_n$, such that $a_n \geq 0$ for all n (positive series).

(a) n^{th} -Term Test

If $\lim_{n \rightarrow \infty} a_n \neq 0$ (or the limit does not exist), then the series $\sum_{n=1}^{\infty} a_n$ *diverges*.

Examples.

1. $a_n = 1$. Here $\lim_{n \rightarrow \infty} a_n = 1 \neq 0$, so $\sum_{n=1}^{\infty} 1$ diverges. Partial sums: $S_1 = 1, S_2 = 2, S_3 = 3, \dots, S_N = N \rightarrow \infty$.
2. $a_n = (-1)^n$. The limit $\lim_{n \rightarrow \infty} (-1)^n$ *does not exist* (terms oscillate). Partial sums: $S_1 = -1, S_2 = 0, S_3 = -1, S_4 = 0, \dots$; (S_N) does not converge, so the series diverges.

3. Consider

$$\sum_{n=1}^{\infty} \frac{n}{n+1}.$$

By the n^{th} -term test:

$$\lim_{n \rightarrow \infty} \frac{n}{n+1} = \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{1}{n}} = 1 \neq 0,$$

so the series *diverges*.

Remark (why this test is only one-way). The condition $a_n \rightarrow 0$ is *necessary* but *not sufficient*: for example, $a_n = \frac{1}{n} \rightarrow 0$ but the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

(b) Integral Test

Let $f : [1, \infty) \rightarrow \mathbb{R}$ be a function such that:

- f is continuous,
- $f(x) \geq 0$ for all $x \geq 1$,
- f is decreasing on $[1, \infty)$,
- $f(n) = a_n$ for all integers $n \geq 1$.

Then

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \int_1^{\infty} f(x) dx \text{ converges}.$$

Application: The p -Series Test

Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}, \quad \text{where } p > 0.$$

Let $f(x) = \frac{1}{x^p}$, which is continuous, positive, and decreasing for $x \geq 1$ when $p > 0$. We apply the integral test by evaluating

$$\int_1^{\infty} \frac{1}{x^p} dx = \lim_{t \rightarrow \infty} \int_1^t \frac{1}{x^p} dx.$$

Case 1: $p \neq 1$

$$\int_1^t \frac{1}{x^p} dx = \left[\frac{x^{1-p}}{1-p} \right]_1^t = \frac{t^{1-p} - 1}{1-p}.$$

- If $p > 1$, then $1 - p < 0$, so $t^{1-p} \rightarrow 0$ as $t \rightarrow \infty$, and the integral converges to $\frac{1}{p-1}$.
- If $p < 1$, then $1 - p > 0$, so $t^{1-p} \rightarrow \infty$, and the integral diverges.

Case 2: $p = 1$

$$\int_1^t \frac{1}{x} dx = \ln t \rightarrow \infty \quad \text{as } t \rightarrow \infty.$$

So the integral diverges.

Conclusion:

$$\int_1^\infty \frac{1}{x^p} dx = \begin{cases} \text{Converges,} & \text{if } p > 1, \\ \text{Diverges,} & \text{if } p \leq 1. \end{cases}$$

By the integral test, the same result holds for the series:

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \begin{cases} \text{Converges,} & \text{if } p > 1, \\ \text{Diverges,} & \text{if } p \leq 1. \end{cases}$$

Examples:

- $\sum_{n=1}^{\infty} \frac{1}{n^2}$: converges (since $p = 2 > 1$)
- $\sum_{n=1}^{\infty} \frac{1}{n}$: diverges (harmonic series, $p = 1$)
- $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$: diverges (since $p = \frac{1}{2} < 1$)

(c) Comparison Test

If $0 \leq a_n \leq b_n$ and $\sum b_n$ converges, then $\sum a_n$ also converges. If $\sum a_n$ diverges and $b_n \geq a_n \geq 0$, then $\sum b_n$ also diverges.

Let

$$a_n = \frac{1}{n^2 + 1}, \quad b_n = \frac{1}{n^2} \quad (n \geq 1).$$

Then $0 \leq a_n \leq b_n$ since $n^2 + 1 \geq n^2$. The series $\sum_{n=1}^{\infty} b_n = \sum 1/n^2 + 1$ converges, hence by the Comparison Test,

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + 1} \text{ converges.}$$

(d) Limit Comparison Test

Let $a_n, b_n > 0$. If

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c \in (0, \infty),$$

then either both $\sum a_n$ and $\sum b_n$ converge, or both diverge.

Method (quick).

- Pick a comparison series $\sum b_n$ you already know (typically a geometric or p -series).
- Compute $L = \lim_{n \rightarrow \infty} \frac{a_n}{b_n}$.
- If $0 < L < \infty$, the two series have the same behavior.
- If $L = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.
- If $L = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

Examples: rational terms (compare with a p -series)

Example 1 (convergent).

$$\sum_{n=1}^{\infty} \frac{2n^3 + 5}{9n^5 + 1}.$$

Compare with $\sum \frac{1}{n^2}$:

$$\lim_{n \rightarrow \infty} \frac{\frac{2n^3 + 5}{9n^5 + 1}}{1/n^2} = \lim_{n \rightarrow \infty} \frac{2n^5 + 5n^2}{9n^5 + 1} = \frac{2}{9} \in (0, \infty).$$

Since $\sum \frac{1}{n^2}$ converges, the given series converges.

Example 2 (divergent).

$$\sum_{n=1}^{\infty} \frac{7n + 4}{3n^2 + 1}.$$

Compare with $\sum \frac{1}{n}$:

$$\lim_{n \rightarrow \infty} \frac{\frac{7n + 4}{3n^2 + 1}}{1/n} = \lim_{n \rightarrow \infty} \frac{7n^2 + 4n}{3n^2 + 1} = \frac{7}{3} > 0.$$

Since $\sum \frac{1}{n}$ diverges, the given series diverges.

Examples: comparison with a geometric series

Example 3 (convergent).

$$\sum_{n=1}^{\infty} a_n, \quad a_n = \frac{3^n + 2}{6^n + 1}.$$

Compare with $b_n = \left(\frac{1}{2}\right)^n$:

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{3^n + 2}{6^n + 1}}{\left(\frac{1}{2}\right)^n} = \lim_{n \rightarrow \infty} \frac{(3^n + 2) 2^n}{6^n + 1} = \lim_{n \rightarrow \infty} \frac{6^n + 2 \cdot 2^n}{6^n + 1} = 1.$$

Since $\sum \left(\frac{1}{2}\right)^n$ converges, $\sum a_n$ converges.

Example 4 (divergent).

$$\sum_{n=1}^{\infty} c_n, \quad c_n = \frac{8^n - 1}{5^n}.$$

Compare with $d_n = \left(\frac{8}{5}\right)^n$:

$$\lim_{n \rightarrow \infty} \frac{c_n}{d_n} = \lim_{n \rightarrow \infty} \frac{8^n - 1}{\left(\frac{8}{5}\right)^n} = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{8^n}\right) = 1.$$

Since $\sum \left(\frac{8}{5}\right)^n$ diverges (ratio > 1), $\sum c_n$ diverges.

(For rational terms $\frac{\text{poly}(n)}{\text{poly}(n)}$, match the highest powers (compare with a p -series). For exponential terms (like $3^n, 6^n$), compare with a geometric series.)

(e) Ratio Test

$$L = \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n}$$

- If $L < 1$: the series converges.
- If $L > 1$ or $L = \infty$: the series diverges.
- If $L = 1$: the test is inconclusive.

Examples.

(a) **Convergent** ($L < 1$). $a_n = \frac{5^n}{n!}$.

$$\frac{a_{n+1}}{a_n} = \frac{5^{n+1}/(n+1)!}{5^n/n!} = \frac{5}{n+1} \xrightarrow{n \rightarrow \infty} 0 < 1.$$

Hence $\sum a_n$ converges.

(b) **Convergent** ($L < 1$). $a_n = \frac{n}{3^n}$.

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)/3^{n+1}}{n/3^n} = \frac{n+1}{n} \cdot \frac{1}{3} \xrightarrow{n \rightarrow \infty} \frac{1}{3} < 1.$$

Thus $\sum a_n$ converges.

(c) **Divergent** ($L > 1$). $a_n = \frac{2^n}{n}$.

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}/(n+1)}{2^n/n} = 2 \cdot \frac{n}{n+1} \xrightarrow{n \rightarrow \infty} 2 > 1.$$

Hence $\sum a_n$ diverges.

(d) **Divergent** ($L = \infty$). $a_n = n!$.

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{n!} = n+1 \xrightarrow{n \rightarrow \infty} \infty.$$

Therefore $\sum a_n$ diverges.

(e) **Inconclusive case** ($L = 1$).

$$a_n = \frac{1}{n} \Rightarrow \frac{a_{n+1}}{a_n} = \frac{n}{n+1} \rightarrow 1 \quad \text{but} \quad \sum \frac{1}{n} \text{ diverges;}$$

$$a_n = \frac{1}{n^2} \Rightarrow \frac{a_{n+1}}{a_n} = \frac{n^2}{(n+1)^2} \rightarrow 1 \quad \text{and} \quad \sum \frac{1}{n^2} \text{ converges.}$$

So when $L = 1$ the test gives no conclusion.

(f) Root Test

$$L = \lim_{n \rightarrow \infty} \sqrt[n]{a_n}, \quad \text{same conclusions as the Ratio Test.}$$

Examples.

(a) **Convergent** ($L < 1$). $a_n = \frac{n^5}{4^n}$.

$$\sqrt[n]{a_n} = \frac{\sqrt[n]{n^5}}{\sqrt[n]{4^n}} = \frac{n^{5/n}}{4} \xrightarrow{n \rightarrow \infty} \frac{1}{4} < 1.$$

Hence $\sum a_n$ converges.

(b) **Convergent** ($L < 1$). $a_n = \frac{n}{\left(\frac{5}{3}\right)^n} = \left(\frac{3}{5}\right)^n n$.

$$\sqrt[n]{a_n} = \left(\frac{3}{5}\right) n^{1/n} \xrightarrow{n \rightarrow \infty} \frac{3}{5} < 1,$$

so $\sum a_n$ converges.

(c) **Divergent** ($L > 1$). $a_n = \frac{2^n}{n^3}$.

$$\sqrt[n]{a_n} = \frac{2}{n^{3/n}} \xrightarrow{n \rightarrow \infty} 2 > 1,$$

hence $\sum a_n$ diverges.

(d) **Divergent** ($L = \infty$). $a_n = n!$.

$$\sqrt[n]{a_n} = \sqrt[n]{n!} \xrightarrow{n \rightarrow \infty} \infty (> 1),$$

therefore $\sum a_n$ diverges.

(e) **Inconclusive case** ($L = 1$).

$$a_n = \frac{1}{n} \quad \Rightarrow \quad \sqrt[n]{a_n} = n^{-1/n} \rightarrow 1 \text{ and } \sum \frac{1}{n} \text{ diverges;}$$

$$a_n = \frac{1}{n^2} \quad \Rightarrow \quad \sqrt[n]{a_n} = n^{-2/n} \rightarrow 1 \text{ but } \sum \frac{1}{n^2} \text{ converges.}$$

So when $L = 1$, the Root Test gives no conclusion.

6. Absolute and Conditional Convergence

- A series $\sum a_n$ is said to be **absolutely convergent** if the series of absolute values $\sum |a_n|$ converges.
- A series $\sum a_n$ is said to be **conditionally convergent** if $\sum a_n$ converges, but the series $\sum |a_n|$ diverges.

Theorem (Absolute Convergence Implies Convergence): If the series $\sum |a_n|$ converges, then the original series $\sum a_n$ also converges.

The converse is not true. That is, a convergent series $\sum a_n$ does not necessarily imply that $\sum |a_n|$ converges. A classical example is the alternating harmonic series:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots,$$

which converges conditionally, since

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$$

diverges (harmonic series).

(g) Alternating Series Test (Leibniz Test)

If $a_n > 0$, decreasing, and $\lim a_n = 0$, then:

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n \text{ converges}$$

- $\sum \frac{(-1)^n}{n^2}$: absolutely convergent (use p -series test)
- $\sum \frac{(-1)^n}{\ln n}$: conditionally convergent (use Leibniz test)
- $\sum \frac{n!}{n^n}$: converges (use ratio or root test)

Exercises

Decide whether each series converges or diverges. If it converges, state whether the convergence is absolute or conditional, and find the sum when possible.

1. $\sum_{n=1}^{\infty} \frac{1}{n^p} \quad (p > 0)$

2. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p} \quad (p > 0)$

3. $\sum_{n=2}^{\infty} \frac{\ln n}{n^2}$

4. $\sum_{n=1}^{\infty} \frac{n}{n^2 + 1}$

5. $\sum_{n=1}^{\infty} \frac{n^2 + 3}{n^3 - 1}$

6. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

$$7. \sum_{n=0}^{\infty} \left(\frac{3}{5}\right)^n$$

$$8. \sum_{n=1}^{\infty} \frac{n}{3^n}$$

$$9. \sum_{n=0}^{\infty} \frac{2^n}{n!}$$

$$10. \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

$$11. \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

$$12. \sum_{n=1}^{\infty} \left(\frac{3n+1}{2n-1}\right)^n$$

$$13. \sum_{n=1}^{\infty} \frac{\left(1 + \frac{1}{n}\right)^n}{n^2}$$

$$14. \sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$$

$$15. \sum_{n=2}^{\infty} (-1)^{n-1} \frac{\ln n}{n}$$

$$16. \sum_{n=1}^{\infty} \frac{n^p}{2^n} \quad (p \in \mathbb{R})$$

$$17. \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p} \quad (p > 0)$$

Solutions

$$1. \sum_{n=1}^{\infty} \frac{1}{n^p}$$

Integral test. For $p > 0$, set $f(x) = x^{-p}$, positive and decreasing on $[1, \infty)$. Then

$$\int_1^{\infty} x^{-p} dx = \begin{cases} \left[\frac{x^{1-p}}{1-p} \right]_1^{\infty} = \frac{1}{p-1}, & p > 1, \\ \infty, & p \leq 1. \end{cases}$$

Hence $\sum n^{-p}$ converges $\iff p > 1$; diverges for $p \leq 1$.

$$2. \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p} \quad (\text{use the integral test})$$

Let $f(x) = \frac{1}{x(\ln x)^p}$ for $x \geq 2$. Then f is positive and decreasing for x large. With $u = \ln x$ so $du = dx/x$,

$$\int_2^{\infty} \frac{dx}{x(\ln x)^p} = \int_{\ln 2}^{\infty} \frac{du}{u^p} = \begin{cases} \left[\frac{u^{1-p}}{1-p} \right]_{\ln 2}^{\infty} = \frac{(\ln 2)^{1-p}}{p-1} < \infty, & p > 1, \\ \infty, & p \leq 1 \quad (\text{for } p = 1 \text{ it is } \infty). \end{cases}$$

Therefore $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ converges iff $p > 1$, and diverges for $0 < p \leq 1$.

$$3. \sum_{n=2}^{\infty} \frac{\ln n}{n^2}$$

Integral test with computation. Let $f(x) = \frac{\ln x}{x^2}$ for $x \geq 2$. Then

$$\int_2^{\infty} \frac{\ln x}{x^2} dx \stackrel{u=\ln x}{=} \int_{\ln 2}^{\infty} u e^{-u} du = \left[-u e^{-u} \right]_{\ln 2}^{\infty} - \int_{\ln 2}^{\infty} (-e^{-u}) du = \frac{\ln 2 + 1}{2} < \infty.$$

(Equivalently, integration by parts: $u = \ln x$, $dv = x^{-2}dx$ gives $-(\ln x + 1)/x$.) Hence the series converges absolutely.

4. $\sum_{n=1}^{\infty} \frac{n}{n^2 + 1}$

Limit comparison with $b_n = \frac{1}{n}$:

$$\lim_{n \rightarrow \infty} \frac{\frac{n}{n^2+1}}{1/n} = \lim_{n \rightarrow \infty} \frac{n^2}{n^2 + 1} = 1 \in (0, \infty).$$

Since $\sum \frac{1}{n}$ diverges, the given series diverges.

5. $\sum_{n=1}^{\infty} \frac{n^2 + 3}{n^3 - 1}$

Limit comparison with $b_n = \frac{1}{n}$:

$$\lim_{n \rightarrow \infty} \frac{\frac{n^2+3}{n^3-1}}{1/n} = \lim_{n \rightarrow \infty} \frac{n^3 + 3n}{n^3 - 1} = 1.$$

Hence it behaves like $\sum \frac{1}{n}$ and diverges.

6. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

Telescoping via partial fractions. Note

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$

Thus $S_N = \sum_{n=1}^N \left(\frac{1}{n} - \frac{1}{n+1} \right) = 1 - \frac{1}{N+1} \xrightarrow[N \rightarrow \infty]{} 1$. So the series converges

and $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$.

7. $\sum_{n=0}^{\infty} \left(\frac{3}{5} \right)^n$

Geometric with ratio $r = \frac{3}{5}$, $|r| < 1$. Sum:

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r} = \frac{1}{1-\frac{3}{5}} = \frac{5}{2}.$$

$$8. \sum_{n=1}^{\infty} \frac{n}{3^n}$$

Ratio test:

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)/3^{n+1}}{n/3^n} = \frac{n+1}{3n} \xrightarrow{n \rightarrow \infty} \frac{1}{3} < 1,$$

hence absolutely convergent. *Exact sum:* For $|r| < 1$,

$$\sum_{n=1}^{\infty} nr^n = \frac{r}{(1-r)^2}.$$

With $r = \frac{1}{3}$, the sum equals $\frac{1/3}{(2/3)^2} = \frac{3}{4}$.

$$9. \sum_{n=0}^{\infty} \frac{2^n}{n!}$$

Ratio test:

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}/(n+1)!}{2^n/n!} = \frac{2}{n+1} \rightarrow 0 < 1.$$

So the series converges absolutely. In fact $\sum_{n=0}^{\infty} \frac{2^n}{n!} = e^2$.

$$10. \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \text{ (alternating harmonic)}$$

Let $a_n = \frac{1}{n}$. Then $a_{n+1} \leq a_n$ and $a_n \rightarrow 0$. By Leibniz, the series converges. Absolute series $\sum \frac{1}{n}$ diverges $\Rightarrow$ convergence is *conditional*.

$$11. \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

$a_n = \frac{1}{\sqrt{n}}$ is decreasing to 0. By Leibniz, the series converges. But $\sum \frac{1}{\sqrt{n}}$ diverges (p -series with $p = \frac{1}{2}$), so not absolute; *conditional*.

$$12. \sum_{n=1}^{\infty} \left(\frac{3n+1}{2n-1} \right)^n$$

Root test:

$$\sqrt[n]{\left(\frac{3n+1}{2n-1}\right)^n} = \frac{3n+1}{2n-1} \xrightarrow{n \rightarrow \infty} \frac{3}{2} > 1.$$

Hence the terms do not tend to 0 (indeed grow), so the series diverges.

$$13. \sum_{n=1}^{\infty} \frac{\left(1 + \frac{1}{n}\right)^n}{n^2}$$

We know $\left(1 + \frac{1}{n}\right)^n \nearrow e$ and is bounded by e . Thus

$$0 \leq \frac{\left(1 + \frac{1}{n}\right)^n}{n^2} \leq \frac{e}{n^2},$$

and $\sum \frac{e}{n^2} = e \sum \frac{1}{n^2} < \infty$. By comparison, the series converges absolutely.

$$14. \sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$$

Why $\sin(1/n) > 0$. For $n \geq 1$ we have $0 < \frac{1}{n} \leq 1 < \frac{\pi}{2}$. Since $\sin x > 0$ for all $x \in (0, \pi)$ (and in particular on $(0, \frac{\pi}{2})$), it follows that

$$\sin\left(\frac{1}{n}\right) > 0 \quad \text{for every } n \geq 1.$$

Convergence test (limit comparison with the harmonic series). Compute

$$\lim_{n \rightarrow \infty} \frac{\sin(1/n)}{1/n} = \lim_{x \downarrow 0} \frac{\sin x}{x} = 1.$$

By the Limit Comparison Test with $\sum_{n=1}^{\infty} \frac{1}{n}$ (which diverges), the series $\sum_{n=1}^{\infty} \sin(1/n)$ also diverges.

Explicit lower bound (to see divergence to $+\infty$). From the limit above, there exists N such that for all $n \geq N$,

$$\frac{\sin(1/n)}{1/n} \geq \frac{1}{2} \implies \sin(1/n) \geq \frac{1}{2n}.$$

Hence

$$\sum_{n=1}^{\infty} \sin(1/n) \geq \sum_{n=N}^{\infty} \sin(1/n) \geq \frac{1}{2} \sum_{n=N}^{\infty} \frac{1}{n} = +\infty,$$

so the series diverges (to $+\infty$).

$$15. \sum_{n=2}^{\infty} (-1)^{n-1} \frac{\ln n}{n}$$

$a_n = \frac{\ln n}{n}$ is decreasing to 0 for $n \geq e$ since

$$\frac{d}{dx} \left(\frac{\ln x}{x} \right) = \frac{1 - \ln x}{x^2} < 0 \quad (x > e).$$

By Leibniz, $\sum (-1)^{n-1} \frac{\ln n}{n}$ converges. Absolute divergence:

$$\int_2^{\infty} \frac{\ln x}{x} dx = \left[\frac{1}{2} (\ln x)^2 \right]_2^{\infty} = \infty \Rightarrow \sum \frac{\ln n}{n} \text{ diverges.}$$

$$16. \sum_{n=1}^{\infty} \frac{n^p}{2^n} \quad (p \in \mathbb{R})$$

Ratio test:

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^p / 2^{n+1}}{n^p / 2^n} = \frac{(n+1)^p}{2n^p} = \frac{1}{2} \left(1 + \frac{1}{n} \right)^p \xrightarrow{n \rightarrow \infty} \frac{1}{2} < 1.$$

Hence the series converges absolutely for every real p .

$$17. \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p} \quad (p > 0)$$

$a_n = \frac{1}{n^p}$ decreases to 0 for $p > 0$. By Leibniz, the alternating series converges for all $p > 0$. Absolute convergence occurs iff $p > 1$ (since $\sum 1/n^p$ converges exactly for $p > 1$); for $0 < p \leq 1$ the convergence is conditional.

Power Series

A **real power series** centered at $x_0 \in \mathbb{R}$ is an infinite series of the form:

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n,$$

where $a_n \in \mathbb{R}$ and $x \in \mathbb{R}$. If $x_0 = 0$, the series is called a power series centered at the origin.

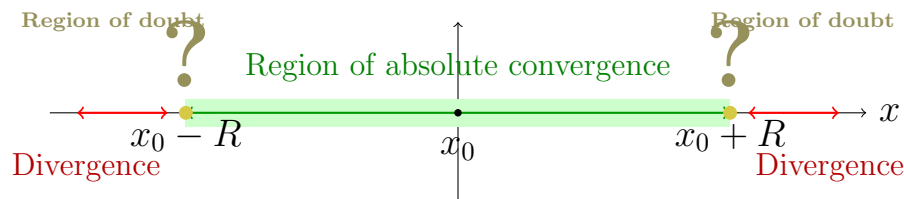
Radius and Interval of Convergence

Consider the real power series centered at $x_0 \in \mathbb{R}$:

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n.$$

There exists a non-negative number $R \in [0, \infty]$, called the **radius of convergence**, such that:

- The series **converges absolutely** for all $x \in \mathbb{R}$ such that $|x - x_0| < R$,
- The series **diverges** for all $x \in \mathbb{R}$ such that $|x - x_0| > R$.



The radius of convergence R can be computed using either the **Ratio Test** or the **Root Test**. Suppose one of the following limits exists (possibly

infinite):

$$L = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|, \quad \text{or} \quad L = \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|}.$$

Then the **radius of convergence** is given by:

$$R = \begin{cases} \frac{1}{L} & \text{if } L \in (0, \infty), \\ \infty & \text{if } L = 0, \\ 0 & \text{if } L = \infty. \end{cases}$$

The corresponding **interval of convergence** is:

- If $R = \infty$, the series converges absolutely for all $x \in \mathbb{R}$; thus, the interval of convergence is $\mathbb{R}$.
- If $R \geq 0$, the series converges absolutely on the open interval $(x_0 - R, x_0 + R)$.

At the boundary points $x = x_0 - R$ and $x = x_0 + R$, the behavior of the series is generally uncertain. Each endpoint must be tested separately: the series may **converge conditionally**, or it may **diverge**. There is no general rule; it depends on the specific form of the series.

Example 1: Geometric Series

Consider the geometric series:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots.$$

This is a power series centered at $x_0 = 0$, with $a_n = 1$ for all n .

We apply the ratio test to determine the values of $x \in \mathbb{R}$ for which the series converges:

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}x^{n+1}}{a_nx^n} \right| = \lim_{n \rightarrow \infty} |x| = |x| \quad (x \neq 0).$$

According to the Ratio Test:

- The series converges if $|x| < 1$,

- The series diverges if $|x| > 1$,
- The test is inconclusive if $|x| = 1$.

Thus, the **radius of convergence** is:

$$R = 1.$$

Behavior at the Boundary

We must check convergence manually at the endpoints:

- At $x = 1$:

$$\sum_{n=0}^{\infty} 1^n = \sum_{n=0}^{\infty} 1 = 1 + 1 + 1 + \cdots \Rightarrow \text{diverges.}$$

- At $x = -1$:

$$\sum_{n=0}^{\infty} (-1)^n = 1 - 1 + 1 - 1 + \cdots \Rightarrow \text{diverges.}$$

This is the Grandi series, which oscillates and does not converge.

Conclusion:

$$\sum_{n=0}^{\infty} x^n \text{ converges if and only if } |x| < 1.$$

Radius of convergence: $R = 1$

Interval of convergence: $(-1, 1)$

Example 2: Harmonic-Like Power Series

Consider the series:

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = \frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{3} + \cdots .$$

This is a power series centered at $x_0 = 0$, with $a_n = \frac{1}{n}$.

Step 1: Apply the Ratio Test

We apply the ratio test to determine for which values of $x \in \mathbb{R}$ the series converges:

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}x^{n+1}}{a_nx^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{n+1} \cdot \frac{n}{x^n} \right| = \lim_{n \rightarrow \infty} \left| x \cdot \frac{n}{n+1} \right| = |x|. \quad (x \neq 0)$$

Conclusion from Ratio Test:

- The series converges if $|x| < 1$,
- The series diverges if $|x| > 1$,
- The test is inconclusive at $|x| = 1$.

So, the **radius of convergence** is:

$$R = 1.$$

Step 2: Analyze the Boundary $|x| = 1$

We must test convergence at the endpoints.

- At $x = 1$:

$$\sum_{n=1}^{\infty} \frac{1}{n} = \text{harmonic series} \Rightarrow \text{diverges.}$$

- At $x = -1$:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = - \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

This is the **alternating harmonic series**, which satisfies the conditions of the Alternating Series Test:

- $\frac{1}{n}$ is positive and decreasing,
- $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$,

$\Rightarrow$ Converges (conditionally).

Conclusion:

$$\sum_{n=1}^{\infty} \frac{x^n}{n} \text{ converges for } x \in [-1, 1),$$

with:

- **Radius of convergence:** $R = 1$
- **Interval of convergence:** $[-1, 1)$

Example 3: Exponential Series

Consider the series:

$$\sum_{n=1}^{\infty} \frac{x^n}{n!} = \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots .$$

This is a power series centered at $x_0 = 0$, with $a_n = \frac{1}{n!}$.

Step 1: Apply the Ratio Test

We apply the ratio test to determine for which values of $x \in \mathbb{R}$ the series converges:

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}x^{n+1}}{a_nx^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x}{n+1} \right| = 0 \quad x \neq 0.$$

Conclusion from Ratio Test:

- Since the limit is 0 for all $x \in \mathbb{R}$, the series converges absolutely for every real number x .

Conclusion:

- **Radius of convergence:** $R = \infty$
- **Interval of convergence:** $(-\infty, \infty)$

This series defines the exponential function:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Example 4: Root Test with a Logarithmic Denominator

Consider the power series:

$$\sum_{n=2}^{\infty} \frac{x^n}{n \log n}.$$

This is a power series centered at $x_0 = 0$, with coefficients $a_n = \frac{1}{n \log n}$ for $n \geq 2$.

We evaluate:

$$\lim_{n \rightarrow \infty} |a_n|^{1/n} = \limsup_{n \rightarrow \infty} \left(\frac{1}{n \log n} \right)^{1/n}.$$

To simplify this, consider the logarithm:

$$\ln \left(\left(\frac{1}{n \log n} \right)^{1/n} \right) = -\frac{1}{n} \ln(n \log n).$$

We now compute the limit using L'Hôpital's Rule:

$$\lim_{n \rightarrow \infty} -\frac{\ln(n \log n)}{n}.$$

Let $f(n) = \ln(n \log n)$ and $g(n) = n$. Since both tend to infinity, we apply L'Hôpital's Rule:

$$\lim_{n \rightarrow \infty} -\frac{\ln(n \log n)}{n} = -\lim_{n \rightarrow \infty} \frac{d}{dn} [\ln(n \log n)] \bigg/ \frac{d}{dn} [n]. \quad (n \text{ is real number})$$

Differentiate numerator and denominator:

$$\frac{d}{dn} [\ln(n \log n)] = \frac{1}{n \log n} \cdot (\log n + 1) = \frac{\log n + 1}{n \log n}.$$

So the limit becomes:

$$-\lim_{n \rightarrow \infty} \frac{\log n + 1}{n \log n} = 0.$$

Therefore,

$$\lim_{n \rightarrow \infty} \left(\frac{1}{n \log n} \right)^{1/n} = e^0 = 1.$$

Conclusion from the Root Test:

$$\lim_{n \rightarrow \infty} |a_n x^n|^{1/n} = \left(\lim_{n \rightarrow \infty} |a_n|^{1/n} \right) \cdot |x| = |x|.$$

So:

- The series converges if $|x| < 1$,
- The series diverges if $|x| > 1$,
- The test is inconclusive at $|x| = 1$.

At $x = 1$:

We consider the series:

$$\sum_{n=2}^{\infty} \frac{1}{n \log n}.$$

To determine convergence, we apply the **integral test**.

Let

$$f(x) = \frac{1}{x \log x}, \quad \text{defined for } x \geq 2.$$

The function $f(x)$ is:

- positive on $[2, \infty)$,
- continuous on $[2, \infty)$,
- decreasing for $x \geq 3$ (since $\log x$ grows slowly).

We evaluate the improper integral:

$$\int_2^{\infty} \frac{1}{x \log x} dx.$$

Make the substitution $u = \log x$, so $du = \frac{1}{x} dx$. Then:

$$\int_2^{\infty} \frac{1}{x \log x} dx = \lim_{t \rightarrow \infty} \int_{\log 2}^t \frac{1}{u} du = \lim_{t \rightarrow \infty} [\log u]_{\log 2}^t = \infty.$$

Since the integral diverges, the integral test implies that the series also diverges:

$$\sum_{n=2}^{\infty} \frac{1}{n \log n} \quad \textbf{diverges}.$$

At $x = -1$:

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n \log n}$$

This is an alternating series. It converges conditionally by the Alternating Series Test since:

- $\frac{1}{n \log n}$ is positive and decreasing for $n \geq 3$,
- $\lim_{n \rightarrow \infty} \frac{1}{n \log n} = 0$.

Conclusion:

$$\sum_{n=2}^{\infty} \frac{x^n}{n \log n} \text{ converges for } x \in [-1, 1).$$

Radius of convergence: $R = 1$

Interval of convergence: $[-1, 1)$

Properties of Power Series

Let

$$f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$$

be a real power series with radius of convergence $R > 0$. Then the function f has the following important properties on the open interval $(x_0 - R, x_0 + R)$:

- **Smoothness:** The function f is **infinitely differentiable** on $(x_0 - R, x_0 + R)$, i.e., all derivatives of f exist and are continuous on this interval. This follows from term-by-term differentiation being valid within the radius of convergence.
- **Term-by-term differentiation:** The derivative of f is obtained by differentiating term-by-term:

$$f'(x) = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1},$$

and this series has the same radius of convergence R . The term $n = 0$ vanishes in the derivative.

- **Term-by-term integration:** The indefinite integral of f is given by integrating term-by-term:

$$\int f(x) dx = C + \sum_{n=0}^{\infty} \frac{a_n}{n+1} (x - x_0)^{n+1},$$

where C is the constant of integration. This series also has radius of convergence R .

- **Preservation of convergence:** Both differentiation and integration **preserve the radius of convergence** R . That is, the derived and integrated series converge on the same interval $(x_0 - R, x_0 + R)$ as the original series.
- **Analyticity:** Within the interval of convergence, the function f is **analytic**. That is, not only is f smooth, but it equals its Taylor series expansion centered at x_0 :

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n.$$

Analytic Functions

A real function f is called **analytic at** x_0 if there exists a power series $\sum a_n(x - x_0)^n$ that converges to $f(x)$ for all x in some neighborhood of x_0 .

In that case, f has a Taylor expansion:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n.$$

Important Notes

- A function can be infinitely differentiable but not analytic (e.g., $f(x) = e^{-1/x^2}$, with $f(0) = 0$, has all derivatives zero at 0, but is not identically zero).
- Power series converge uniformly on compact subintervals of the open interval of convergence.
- Within the interval of convergence, power series can be manipulated like polynomials (term-by-term addition, multiplication, etc.).

Representation of Functions by Power Series

Many functions can be expressed as power series within a suitable interval. A function f is said to be **represented by a power series** centered at $x_0 \in \mathbb{R}$ if there exists a sequence (a_n) such that:

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \quad \text{for all } x \text{ in some interval } (x_0 - R, x_0 + R),$$

where $R > 0$ is the radius of convergence.

Maclaurin formula (Taylor at 0).

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \quad \text{for } |x| < R,$$

Let $f(x) = \frac{1}{1-x} = (1-x)^{-1}$. Compute the first few derivatives explicitly:

$$\begin{aligned} f(x) &= (1-x)^{-1}, \\ f'(x) &= (+1) \cdot (1-x)^{-2}, \\ f''(x) &= (+1) \cdot 2! (1-x)^{-3}, \\ f^{(3)}(x) &= (+1) \cdot 3! (1-x)^{-4}, \\ &\vdots \end{aligned}$$

This suggests

$$f^{(n)}(x) = n! (1-x)^{-(n+1)} \quad \text{for all } n \geq 0.$$

Evaluating at $x = 0$ gives

$$f^{(n)}(0) = n! (1-0)^{-(n+1)} = n!.$$

Therefore, the Maclaurin series of f is

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} x^n.$$

The *radius of convergence* is

$$R = 1,$$

so the series converges for $|x| < 1$. (Endpoint behavior: at $x = 1$, terms are all 1 and the series diverges; at $x = -1$, terms alternate ± 1 and also diverge.)

One often rewrites a function as $\frac{1}{1 - g(x)}$ and substitutes

$$\frac{1}{1 - g(x)} = \sum_{n=0}^{\infty} (g(x))^n \quad (|g(x)| < 1).$$

- $\frac{1}{2 + 3x}$. Write $\frac{1}{2 + 3x} = \frac{1}{2} \frac{1}{1 + \frac{3}{2}x} = \frac{1}{2} \frac{1}{1 - (-\frac{3}{2}x)}$, so

$$g(x) = -\frac{3}{2}x, \quad |g(x)| < 1 \Leftrightarrow |x| < \frac{2}{3}.$$

Then

$$\frac{1}{2 + 3x} = \sum_{n=0}^{\infty} \frac{(-1)^n 3^n}{2^{n+1}} x^n.$$

- $\frac{1}{1 + x^2}$. View as $\frac{1}{1 - (-x^2)}$, hence

$$g(x) = -x^2, \quad |g(x)| < 1 \Leftrightarrow |x| < 1,$$

and

$$\frac{1}{1 + x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}.$$

Differentiation rule for power series. If a power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

has radius of convergence $R > 0$, then for every $|x| < R$ it is differentiable term-by-term and

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1},$$

with the *same* radius R .

Apply to the geometric series.

For $|x| < 1$,

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

Differentiate both sides term-by-term on $|x| < 1$:

$$\frac{d}{dx} \left(\frac{1}{1-x} \right) = \sum_{n=0}^{\infty} \frac{d}{dx} (x^n) \implies \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}.$$

(We also know directly that $\frac{d}{dx} \frac{1}{1-x} = \frac{1}{(1-x)^2}$ by calculus.)

Reindex to standard powers of x . The series $\sum_{n=1}^{\infty} n x^{n-1}$ is correct, but it is often nicer to write powers as x^n . Let $m = n - 1$ (so $n = m + 1$ and $m \geq 0$):

$$\sum_{n=1}^{\infty} n x^{n-1} = \sum_{m=0}^{\infty} (m+1) x^m = \sum_{n=0}^{\infty} (n+1) x^n.$$

Therefore, for $|x| < 1$,

$$\boxed{\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1) x^n.}$$

Differentiate the previous identity once more (still valid for $|x| < 1$):

$$\frac{d}{dx} \left(\frac{1}{(1-x)^2} \right) = \frac{d}{dx} \left(\sum_{n=1}^{\infty} n x^{n-1} \right) \implies \frac{2}{(1-x)^3} = \sum_{n=1}^{\infty} n(n-1) x^{n-2}.$$

Divide by 2:

$$\frac{1}{(1-x)^3} = \frac{1}{2} \sum_{n=2}^{\infty} n(n-1) x^{n-2}.$$

Reindex to express in powers x^m : let $m = n - 2$ (so $n = m + 2$, $m \geq 0$):

$$\frac{1}{(1-x)^3} = \frac{1}{2} \sum_{m=0}^{\infty} (m+2)(m+1) x^m$$

Integration rule for power series .

If a power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

has radius of convergence $R > 0$, then for every $|x| < R$ it is integrable term-by-term

$$\int_0^x f(t) dt = \sum_{n=0}^{\infty} \int_0^x a_n t^n dt = \sum_{n=0}^{\infty} a_n \frac{x^{n+1}}{n+1}.$$

The resulting power series has the *same* radius R .

$\ln(1+x)$ **via integration.**

Start from the geometric series with $g(x) = -x$:

$$\frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n, \quad |x| < 1.$$

Integrate term-by-term from 0 to x (valid for $|x| < 1$):

$$\int_0^x \frac{1}{1+t} dt = \int_0^x \sum_{n=0}^{\infty} (-1)^n t^n dt = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}.$$

The left-hand side:

$$\int_0^x \frac{1}{1+t} dt = \ln(1+x) - \ln(1+0) = \ln(1+x).$$

Reindex ($m = n+1$) to the standard form:

$$\boxed{\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}}, \quad |x| < 1.$$

Endpoints: At $x = 1$ the series becomes the harmonic series (diverges). At $x = -1$ it is the alternating harmonic series (converges conditionally to

$$\ln 2 = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$$

.

Now start from the geometric series with $g(t) = -t^2$:

$$\frac{1}{1 - (-t^2)} = \sum_{n=0}^{\infty} (-t^2)^n = \sum_{n=0}^{\infty} (-1)^n t^{2n}, \quad |t| < 1.$$

Integrate term-by-term from 0 to x (valid for $|x| < 1$):

$$\int_0^x \frac{1}{1+t^2} dt = \int_0^x \sum_{n=0}^{\infty} (-1)^n t^{2n} dt = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}.$$

Recognize the left-hand side:

$$\int_0^x \frac{1}{1+t^2} dt = \arctan x - \arctan 0 = \arctan x.$$

Therefore

$$\boxed{\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}}, \quad |x| < 1.$$

Endpoints: At $x = \pm 1$ the series becomes the Leibniz series; it converges conditionally to $\pm\pi/4$

Exponential function e^x

Recall Maclaurin (Taylor at 0):

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \quad (\text{for } |x| \text{ within the radius}).$$

For $f(x) = e^x$, we have $f^{(n)}(x) = e^x$ for all n , hence $f^{(n)}(0) = 1$. Therefore

$$\boxed{e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}}.$$

Similarly,

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} \quad (R = \infty).$$

Functions $\sinh x$ and $\cosh x$.

Recall that, if $f(x) = \sum a_n x^n$, $g(x) = \sum b_n x^n$, then

$$(f(x) + g(x)) = \sum (a_n + b_n) x^n.$$

We know that

$$\sinh x := \frac{e^x - e^{-x}}{2}, \quad \cosh x := \frac{e^x + e^{-x}}{2}.$$

Substitute the power series of $e^{\pm x}$ and combine term-by-term *termwise addition/subtraction is justified* and preserves $R = \infty$:

$$\sinh x = \frac{1}{2} \left(\sum_{n=0}^{\infty} \frac{x^n}{n!} - \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} \right) = \sum_{n=0}^{\infty} \frac{1 - (-1)^n}{2} \frac{x^n}{n!}.$$

Note that $1 - (-1)^n$ is 0 for even n and 2 for odd n , so only *odd* powers remain:

$$\boxed{\sinh x = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}} \quad (R = \infty).$$

Similarly,

$$\cosh x = \frac{1}{2} \left(\sum_{n=0}^{\infty} \frac{x^n}{n!} + \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} \right) = \sum_{n=0}^{\infty} \frac{1 + (-1)^n}{2} \frac{x^n}{n!},$$

where $1 + (-1)^n$ is 2 for even n and 0 for odd n , so only *even* powers remain:

$$\boxed{\cosh x = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}} \quad (R = \infty).$$

Examples

- **Exponential function:**

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \text{for all } x \in \mathbb{R}.$$

- **Geometric series:**

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad \text{for } |x| < 1.$$

- **Sine and cosine:**

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \quad \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \quad \text{for all } x \in \mathbb{R}.$$

- **Natural logarithm:**

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}, \quad \text{for } |x| < 1.$$

- **Arctan:**

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \quad \text{for } |x| \leq 1.$$

Exercises

1. Show that

$$\lim_{x \rightarrow 0} \frac{\ln(1+x) - x + \frac{x^2}{2}}{x^3} = \frac{1}{3}.$$

Solution. Use the power series representation

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

Then

$$\ln(1+x) - x + \frac{x^2}{2} = \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \right) - x + \frac{x^2}{2} = \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

Divide by x^3 : $\frac{\ln(1+x) - x + \frac{x^2}{2}}{x^3} = \frac{1}{3} - \frac{x}{4} + \dots \rightarrow \boxed{\frac{1}{3}}.$

2. Show that

$$\lim_{x \rightarrow 0} \frac{1 - \cos x - \frac{x^2}{2}}{x^4} = \frac{1}{24}.$$

Solution. Use

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots.$$

Then

$$1 - \cos x - \frac{x^2}{2} = \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots\right) - 1 - \frac{x^2}{2} = \frac{x^4}{24} - \frac{x^6}{720} + \cdots.$$

Divide by x^4 and let $x \rightarrow 0$: $\rightarrow \boxed{\frac{1}{24}}$.

3. Evaluate

$$\sum_{n=0}^{\infty} (n+1) \left(\frac{1}{2}\right)^n.$$

Solution. Recall $\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1)x^n$ for $|x| < 1$. Set $x = \frac{1}{2}$:

$$\sum_{n=0}^{\infty} (n+1) \left(\frac{1}{2}\right)^n = \frac{1}{\left(1 - \frac{1}{2}\right)^2} = \frac{1}{\left(\frac{1}{2}\right)^2} = \boxed{4}.$$

4. Show that

$$\sum_{n=1}^{\infty} n^2 x^n = \frac{x(1+x)}{(1-x)^3}, \quad |x| < 1,$$

and then compute

$$\sum_{n=1}^{\infty} \frac{n^2}{3^n}.$$

Solution. Start from $S(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ ($|x| < 1$). Differentiate: $S'(x) = \sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}$. Multiply by x : $\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}$. Differentiate this identity:

$$\sum_{n=1}^{\infty} n^2 x^{n-1} = \frac{d}{dx} \left(\frac{x}{(1-x)^2} \right) = \frac{(1-x)^2 - x \cdot 2(1-x)(-1)}{(1-x)^4} = \frac{1+x}{(1-x)^3}.$$

Multiply by x : $\sum_{n=1}^{\infty} n^2 x^n = \frac{x(1+x)}{(1-x)^3}$, as claimed. Now plug $x = \frac{1}{3}$:

$$\sum_{n=1}^{\infty} \frac{n^2}{3^n} = \frac{\frac{1}{3}(1+\frac{1}{3})}{(1-\frac{1}{3})^3} = \frac{\frac{1}{3} \cdot \frac{4}{3}}{(\frac{2}{3})^3} = \frac{\frac{4}{9}}{\frac{8}{27}} = \frac{4}{9} \cdot \frac{27}{8} = \boxed{\frac{3}{2}}.$$

Fourier Series

Fourier series allow us to represent periodic functions as infinite sums of sines and cosines.

1. Periodic Functions

A function $f(t)$ is said to be **periodic** with period $L > 0$ if

$$f(t + L) = f(t) \quad \text{for all } t \in \mathbb{R}.$$

Example. Let $f(t) = \sin(t)$. Then f is periodic with period $L = 2\pi$, because:

$$f(t + 2\pi) = \sin(t + 2\pi) = \sin(t) = f(t).$$

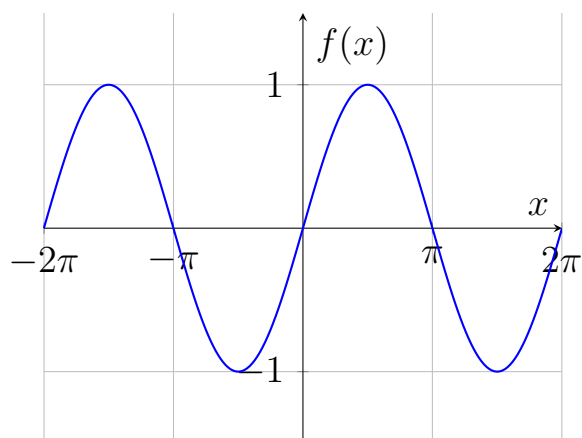
Note that $f(t) = \sin(t)$ also satisfies

$$f(t + 4\pi) = \sin(t + 4\pi) = \sin(t), \quad f(t + 6\pi) = \sin(t + 6\pi) = \sin(t), \quad \text{etc.}$$

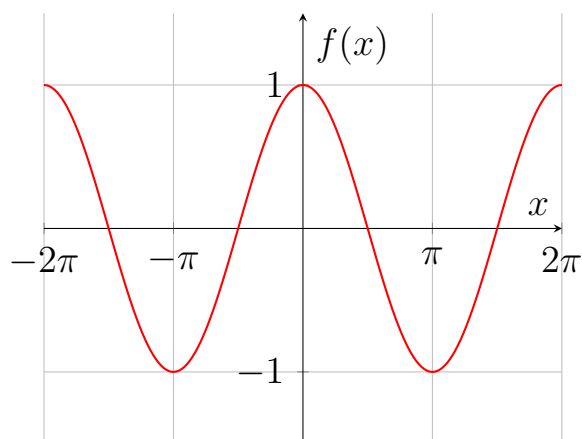
In fact, $\sin(t)$ has period $2n\pi$ for any positive integer $n = 1, 2, 3, \dots$. However, the **smallest positive period** is 2π , which is called the **period**.

Graphical Interpretation. A function with period L has a graph that repeats itself every L units. That is, the graph remains unchanged when shifted left or right by L .

1. Sine Function $\sin x$



Cosine Function: $\cos x$



Example: Periodic Extension of x^2

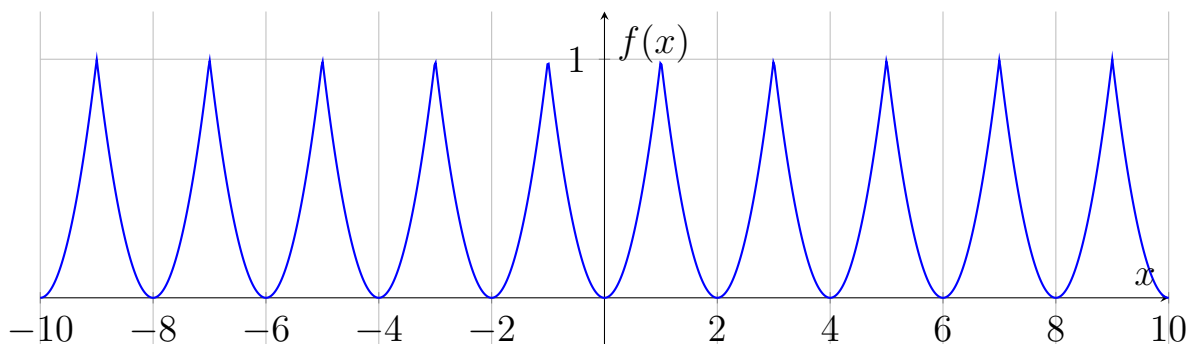
We define the function:

$$f(x) = x^2, \quad x \in (-1, 1),$$

and extend it to all real numbers by periodicity with period 2:

$$f(x + 2) = f(x), \quad \text{for all } x \in \mathbb{R}.$$

To repeat the shape of x^2 from $(-1, 1)$ across the real line, we shift the basic graph by multiples of 2. This means we “copy and paste” the curve x^2 into every interval of length 2.



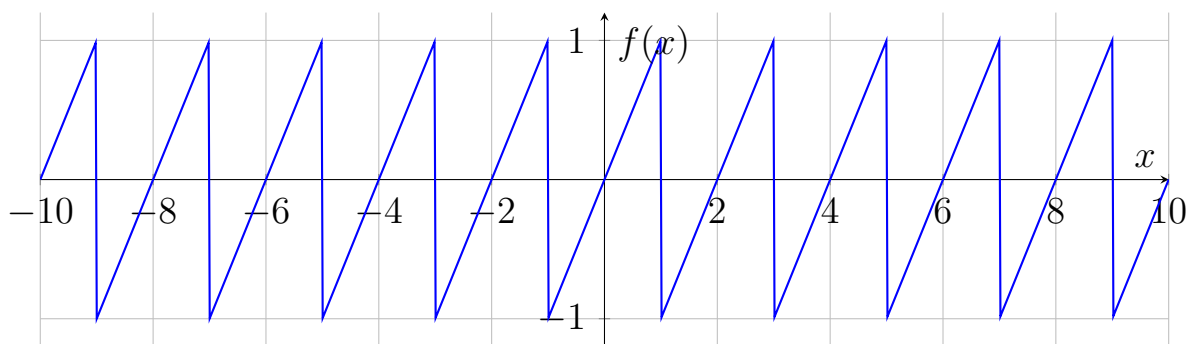
Example: Periodic Extension of x

We define the function:

$$f(x) = x, \quad x \in (-1, 1),$$

and extend it to all real numbers by periodicity with period 2:

$$f(x + 2) = f(x), \quad \text{for all } x \in \mathbb{R}.$$



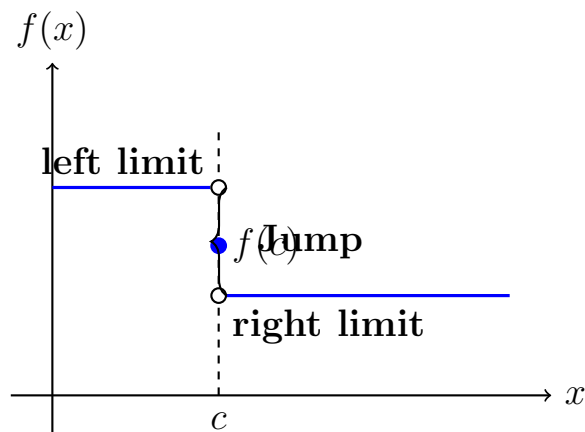
2. Piecewise Continuous Functions and Jump Discontinuities

A function f is called **piecewise continuous** on an interval $[a, b]$ if:

- It is continuous on parts of the interval, except at a finite number of **points of discontinuity**,
- At each point of discontinuity, the left-hand and right-hand limits exist and are finite.

When the two one-sided limits exist but are not equal, the function has a **jump discontinuity**.

Example: Jump Discontinuity



The function has a jump at $x = c$. Both one-sided limits exist but are not equal.

Orthogonality of Trigonometric Functions

The sine and cosine functions are orthogonal on $[-\pi, \pi]$:

$$\int_{-\pi}^{\pi} \cos(mx) \cos(nx) dx = \begin{cases} \pi & \text{if } m = n \neq 0, \\ 0 & \text{if } m \neq n, \end{cases}$$

$$\int_{-\pi}^{\pi} \sin(mx) \sin(nx) dx = \begin{cases} \pi & \text{if } m = n, \\ 0 & \text{if } m \neq n, \end{cases}$$

$$\int_{-\pi}^{\pi} \cos(mx) \sin(nx) dx = 0 \quad \text{for all } m, n.$$

These orthogonality relations are the foundation of Fourier coefficients.

3. Fourier Series of a Function

The Fourier series of a function $f(x)$ with period $2L$ is given by:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]$$

where the Fourier coefficients are:

$$a_0 = \frac{1}{L} \int_{-L}^L f(x) dx$$

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx, \quad b_n = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

provided that these integrals exist.

The symbol $\sim$ means that the Fourier series represents an approximation of the function $f(x)$.

In general, the Fourier series of a function f is not exactly equal to f . However, under suitable conditions on f , the series converges to $f(x)$ in various senses — such as:

- **Pointwise convergence** at each point (if f is piecewise smooth),
- **Uniform convergence** on an interval (if f is continuous and periodic),
- **Convergence in norm**, such as in the L^2 -sense (i.e., mean-square convergence).

4. Dirichlet's Theorem (Pointwise Convergence)

Let $f(x)$ be a function of period $2L$. Suppose that:

- f is piecewise continuous on $[-L, L]$,
- f' is piecewise continuous on $[-L, L]$.

Then the Fourier series of f converges at every point $x \in \mathbb{R}$ to the average of the left- and right-hand limits:

$$\frac{f(x^-) + f(x^+)}{2} = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]$$

where the **right-hand** and **left-hand** limits as:

$$f(x^+) = \lim_{\xi \rightarrow x^+} f(\xi), \quad f(x^-) = \lim_{\xi \rightarrow x^-} f(\xi)$$

Furthermore, if f is continuous at x , then the Fourier series converges to $f(x)$:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]$$

Summary

Let $S_N(x)$ denote the N -th partial sum of the Fourier series of a function f with period $2L$:

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^N \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]$$

This expression is known as a trigonometric polynomial of degree N .

- If f is **continuous** at a point x , then:

$$\lim_{N \rightarrow \infty} S_N(x) = f(x)$$

- If f has a **jump discontinuity** at x , then:

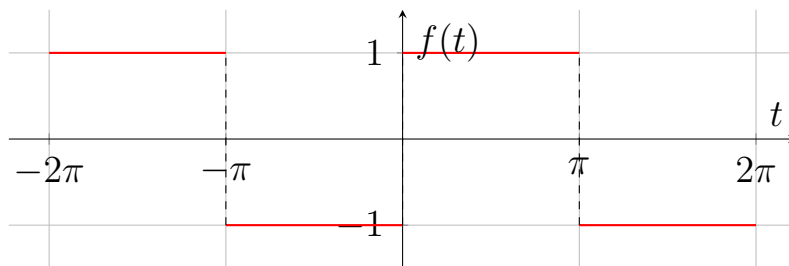
$$\lim_{N \rightarrow \infty} S_N(x) = \frac{1}{2} (f(x^-) + f(x^+))$$

That is, the Fourier series converges to the midpoint of the jump.

Example: Fourier Series of a Square Wave

Let $f(t)$ be the square wave of period 2π , defined over one period by:

$$f(t) = \text{sign}(t) = \begin{cases} -1, & -\pi < t < 0 \\ 1, & 0 < t < \pi \end{cases}$$



We now compute the Fourier series of $f(t)$. Since f is piecewise smooth on $[-\pi, \pi]$, the Fourier series converges (by Dirichlet's Theorem) to:

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nt) + b_n \sin(nt))$$

The Fourier coefficients are given by:

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) dt, \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt) dt, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(nt) dt$$

Step 1: Compute a_0

$$a_0 = \frac{1}{\pi} \left(\int_{-\pi}^0 (-1) dt + \int_0^{\pi} 1 dt \right) = \frac{1}{\pi} (-\pi + \pi) = 0$$

Step 2: Compute a_n

$$a_n = \frac{1}{\pi} \left(\int_{-\pi}^0 (-1) \cos(nt) dt + \int_0^{\pi} \cos(nt) dt \right)$$

Since cosine is even:

$$a_n = \frac{1}{\pi} \left(- \int_0^{\pi} \cos(nt) dt + \int_0^{\pi} \cos(nt) dt \right) = 0$$

Step 3: Compute b_n

$$b_n = \frac{1}{\pi} \left(\int_{-\pi}^0 (-1) \sin(nt) dt + \int_0^{\pi} \sin(nt) dt \right)$$

Since sine is odd:

$$b_n = \frac{1}{\pi} \left(- \int_0^{\pi} \sin(nt) dt + \int_0^{\pi} \sin(nt) dt \right) = \frac{2}{\pi} \int_0^{\pi} \sin(nt) dt$$

Now compute:

$$\int_0^{\pi} \sin(nt) dt = \begin{cases} 0, & \text{if } n \text{ is even} \\ \frac{2}{n}, & \text{if } n \text{ is odd} \end{cases}$$

Thus:

$$b_n = \begin{cases} 0, & \text{if } n \text{ is even} \\ \frac{4}{n\pi}, & \text{if } n \text{ is odd} \end{cases}$$

The Fourier series of $f(t)$ is:

$$\begin{aligned} f(t) &\sim \sum_{n=1, n \text{ odd}}^{\infty} \frac{4}{n\pi} \sin(nt) \\ &\sim \sum_{n=0}^{\infty} \frac{4}{(2n+1)\pi} \sin((2n+1)t) \end{aligned}$$

Only the sine terms with odd indices appear in the expansion.

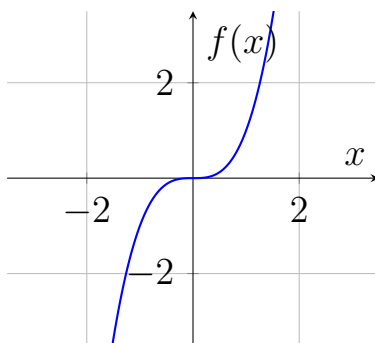
5. Symmetry : Even and Odd Functions

A function f is called **odd** if:

$$f(-x) = -f(x) \quad \text{for all } x \in \mathbb{R}.$$

Examples: $\text{sgn}(x)$, x , x^3 , $\sin(x)$.

Odd Function: $f(x) = x^3$



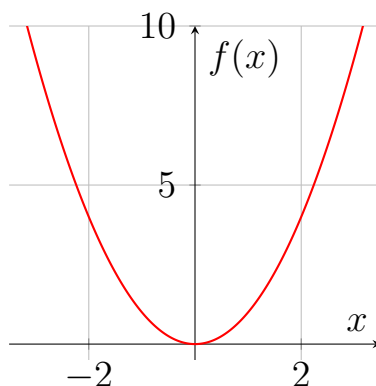
The graph is symmetric with respect to the origin.

A function f is called **even** if:

$$f(-x) = f(x) \quad \text{for all } x \in \mathbb{R}.$$

Examples: $|x|$, x^2 , $\cos(x)$.

Even Function: $f(x) = x^2$



The graph is symmetric with respect to the y -axis.

Even–Odd Decomposition of a Function

Every function $f(x)$ defined on a symmetric interval $[-L, L]$ can be uniquely written as:

$$f(x) = f_{\text{even}}(x) + f_{\text{odd}}(x)$$

With:

$$f_{\text{even}}(x) = \frac{f(x) + f(-x)}{2}, \quad f_{\text{odd}}(x) = \frac{f(x) - f(-x)}{2}$$

These satisfy:

- f_{even} is even,
- f_{odd} is odd,
- Their sum gives $f(x)$ exactly.

Example: Decompose $f(x) = e^x$

Let $f(x) = e^x$. Then:

$$f_{\text{even}}(x) = \frac{e^x + e^{-x}}{2} = \cosh(x) \quad (\text{even part})$$

$$f_{\text{odd}}(x) = \frac{e^x - e^{-x}}{2} = \sinh(x) \quad (\text{odd part})$$

So:

$$e^x = \cosh(x) + \sinh(x)$$

Even and Odd Decomposition in Fourier Series

According to Dirichlet's Theorem, any function $f(x)$ (under suitable conditions) can be written as the sum of an even and an odd function. This is reflected in the structure of the Fourier series:

$$f(x) \sim \underbrace{\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)}_{\text{even part}} + \underbrace{\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)}_{\text{odd part}}$$

- The **even part** consists of cosine terms and the constant term. Since cosine is even, this part is symmetric about the y -axis.
- The **odd part** consists of sine terms. Since sine is odd, this part is symmetric about the origin.

Consequence: Uniqueness of Even–Odd Decomposition

Since the Fourier series naturally separates into an even part (cosine terms) and an odd part (sine terms), we can make the following conclusions:

- If f is **even**, then its odd part must be identically zero. This happens if and only if all sine coefficients vanish:

$$b_n = 0 \quad \text{for all } n \geq 1.$$

Thus, the Fourier series contains only cosine terms (and possibly a_0).

- If f is **odd**, then its even part must be zero. This occurs only when all cosine coefficients vanish, including the constant term:

$$a_0 = 0, \quad a_n = 0 \quad \text{for all } n \geq 1.$$

Therefore, the Fourier series contains only sine terms.

Summary:

- If f is **even**, then all sine coefficients vanish: $b_n = 0$ for all $n \geq 1$.
- If f is **odd**, then all cosine coefficients vanish (including the constant term): $a_0 = 0$, $a_n = 0$ for all $n \geq 1$.

Fourier series on $[-L, L]$ with period $2L$

General form

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right],$$

$$a_0 = \frac{1}{L} \int_{-L}^L f(x) dx, \quad a_n = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx, \quad b_n = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

Even functions: $f(-x) = f(x)$ (**cosine series**)

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right),$$

$$a_0 = \frac{2}{L} \int_0^L f(x) dx, \quad a_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx, \quad n \geq 1,$$

$$b_n = 0 \quad \text{for all } n \geq 1.$$

Odd functions: $f(-x) = -f(x)$ (**sine series**)

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right),$$

$$a_0 = 0, \quad a_n = 0 \quad \text{for all } n \geq 1, \quad b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx, \quad n \geq 1.$$

Examples

1 (odd)

$$f(x) = \begin{cases} 1, & 0 < x < \pi, \\ -1, & -\pi < x < 0, \end{cases} \quad (f \text{ odd}).$$

Hence $a_0 = a_n = 0$ and

$$b_n = \frac{2}{\pi} \int_0^{\pi} 1 \cdot \sin(nx) dx = \frac{2}{\pi} \left[-\frac{\cos(nx)}{n} \right]_0^{\pi} = \frac{2}{\pi} \cdot \frac{1 - \cos(n\pi)}{n} = \frac{2}{\pi} \cdot \frac{1 - (-1)^n}{n}.$$

Thus $b_n = 0$ for n even, and $b_n = \frac{4}{\pi n}$ for n odd. Therefore

$$f(x) \sim \frac{4}{\pi} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{\sin(nx)}{n}$$

(Gibbs phenomenon appears at the jump points $x = 0, \pm\pi$; convergence is to the midpoint value there).

2) (odd)

$$f(x) = x \quad \text{on } [-\pi, \pi] \quad (f \text{ odd}).$$

Again $a_0 = a_n = 0$ and

$$b_n = \frac{2}{\pi} \int_0^{\pi} x \sin(nx) dx.$$

Integrate by parts with $u = x$, $dv = \sin(nx) dx$ so $du = dx$, $v = -\frac{\cos(nx)}{n}$:

$$\int_0^{\pi} x \sin(nx) dx = \left[-\frac{x \cos(nx)}{n} \right]_0^{\pi} + \frac{1}{n} \int_0^{\pi} \cos(nx) dx = -\frac{\pi \cos(n\pi)}{n} + \frac{1}{n} \left[\frac{\sin(nx)}{n} \right]_0^{\pi} = -\frac{\pi \cos(n\pi)}{n}$$

Hence

$$b_n = \frac{2}{\pi} \left(-\frac{\pi(-1)^n}{n} \right) = -\frac{2(-1)^n}{n},$$

and the Fourier series is

$$f(x) \sim -2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin(nx)$$

3) Absolute Value (even)

$$f(x) = |x| \quad \text{on } [-\pi, \pi] \quad (f \text{ even}).$$

Thus $b_n = 0$. Compute

$$a_0 = \frac{2}{\pi} \int_0^{\pi} x dx = \frac{2}{\pi} \cdot \frac{\pi^2}{2} = \pi \quad \Rightarrow \quad \frac{a_0}{2} = \frac{\pi}{2}.$$

For $n \geq 1$,

$$a_n = \frac{2}{\pi} \int_0^\pi x \cos(nx) dx.$$

Integrate by parts with $u = x$, $dv = \cos(nx) dx$ so $du = dx$, $v = \frac{\sin(nx)}{n}$:

$$\int_0^\pi x \cos(nx) dx = \left[\frac{x \sin(nx)}{n} \right]_0^\pi - \frac{1}{n} \int_0^\pi \sin(nx) dx = \frac{\pi \sin(n\pi)}{n} - \frac{1}{n} \left[-\frac{\cos(nx)}{n} \right]_0^\pi.$$

Since $\sin(n\pi) = 0$,

$$\int_0^\pi x \cos(nx) dx = -\frac{1}{n} \cdot \frac{-\cos(n\pi) + \cos(0)}{n} = -\frac{1}{n^2} (1 - (-1)^n).$$

Therefore

$$a_n = \frac{2}{\pi} \cdot \left(-\frac{1}{n^2} (1 - (-1)^n) \right).$$

Hence $a_n = 0$ for n even, and for n odd $a_n = -\frac{4}{\pi n^2}$. The series is

$$|x| \sim \frac{\pi}{2} - \frac{4}{\pi} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{\cos(nx)}{n^2}.$$

At $x = 0$, we have $\cos(0) = 1$, so

$$|x| \sim \frac{\pi}{2} - \frac{4}{\pi} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{\cos(nx)}{n^2} \implies 0 = \frac{\pi}{2} - \frac{4}{\pi} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{n^2}.$$

Hence

$$\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{8}.$$

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}.$$

4. (even)

$$f(x) = x^2 \quad \text{on } [-\pi, \pi] \quad (f \text{ even}).$$

Thus $b_n = 0$. First,

$$a_0 = \frac{2}{\pi} \int_0^\pi x^2 dx = \frac{2}{\pi} \cdot \frac{\pi^3}{3} = \frac{2\pi^2}{3} \quad \Rightarrow \quad \frac{a_0}{2} = \frac{\pi^2}{3}.$$

For $n \geq 1$,

$$a_n = \frac{2}{\pi} \int_0^\pi x^2 \cos(nx) dx.$$

Let $I = \int_0^\pi x^2 \cos(nx) dx$. Integrate by parts twice.

(1) First parts: $u = x^2$, $dv = \cos(nx) dx$ so $du = 2x dx$, $v = \frac{\sin(nx)}{n}$:

$$I = \left[\frac{x^2 \sin(nx)}{n} \right]_0^\pi - \frac{2}{n} \int_0^\pi x \sin(nx) dx = 0 - \frac{2}{n} J,$$

since $\sin(n\pi) = 0$, where $J = \int_0^\pi x \sin(nx) dx$.

(2) Second parts for J : $u = x$, $dv = \sin(nx) dx$ so $du = dx$, $v = -\frac{\cos(nx)}{n}$:

$$J = \left[-\frac{x \cos(nx)}{n} \right]_0^\pi + \frac{1}{n} \int_0^\pi \cos(nx) dx = -\frac{\pi \cos(n\pi)}{n} + \frac{1}{n} \left[\frac{\sin(nx)}{n} \right]_0^\pi = -\frac{\pi(-1)^n}{n}.$$

Thus

$$I = -\frac{2}{n} \left(-\frac{\pi(-1)^n}{n} \right) = \frac{2\pi(-1)^n}{n^2}, \quad a_n = \frac{2}{\pi} I = \frac{2}{\pi} \cdot \frac{2\pi(-1)^n}{n^2} = \frac{4(-1)^n}{n^2}.$$

Therefore

$$x^2 \sim \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos(nx).$$

At $x = 0$, we have $\cos(0) = 1$, so the series becomes

$$x^2 \sim \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos(nx) \quad \Rightarrow \quad 0 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}.$$

Dividing both sides by 4, we obtain

$$\boxed{\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}.$$

Exercise Find the Fourier series on $[-\pi, \pi]$ (with 2π -periodic extension) of

$$f(x) = \frac{\pi}{2} - |x|.$$

Then evaluate your series at $x = 0$ to deduce the value of $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$.

Solution Since f is even, its Fourier series is a cosine series

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx), \quad a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx, \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx.$$

Compute a_0 . Using evenness,

$$a_0 = \frac{2}{\pi} \int_0^{\pi} \left(\frac{\pi}{2} - x \right) dx = \frac{2}{\pi} \left[\frac{\pi x}{2} - \frac{x^2}{2} \right]_0^{\pi} = 0.$$

Compute a_n . Again by evenness,

$$a_n = \frac{2}{\pi} \int_0^{\pi} \left(\frac{\pi}{2} - x \right) \cos(nx) dx = \frac{2}{\pi} \left[\frac{\pi}{2} \int_0^{\pi} \cos(nx) dx - \int_0^{\pi} x \cos(nx) dx \right].$$

The first integral is 0 since $\int_0^{\pi} \cos(nx) dx = \frac{\sin(n\pi)}{n} = 0$. For the second, integration by parts gives

$$\int_0^{\pi} x \cos(nx) dx = \frac{\pi \sin(n\pi)}{n} + \frac{\cos(n\pi) - 1}{n^2} = \frac{(-1)^n - 1}{n^2}.$$

Hence

$$a_n = -\frac{2}{\pi} \cdot \frac{(-1)^n - 1}{n^2} = \frac{2}{\pi} \cdot \frac{1 - (-1)^n}{n^2} = \begin{cases} \frac{4}{\pi n^2}, & n \text{ odd}, \\ 0, & n \text{ even}. \end{cases}$$

Therefore

$$\boxed{\frac{\pi}{2} - |x| \sim \frac{4}{\pi} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{\cos(nx)}{n^2} \quad (-\pi < x < \pi).$$

At $x = 0$. Since $f(0) = \frac{\pi}{2}$ and $\cos(0) = 1$,

$$\frac{\pi}{2} = \frac{4}{\pi} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{n^2} \quad \Longrightarrow \quad \boxed{\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8}}.$$

Fourier Integrals

Motivation

While Fourier series represent periodic functions as sums of sines and cosines, many functions in practice are defined on the entire real line and are not periodic. In such cases, we use the **Fourier integral** to represent non-periodic functions using integrals rather than infinite sums.

Fourier Integral Representation

Let $f(x)$ be a real-valued function defined on $(-\infty, \infty)$. Suppose that f satisfies the following conditions:

- (i) f and f' are piecewise continuous on every bounded interval of $\mathbb{R}$,
- (ii) f is **absolutely integrable** on $\mathbb{R}$, i.e.,

$$\int_{-\infty}^{\infty} |f(x)| dx \quad \text{converges} \quad (\text{as an improper integral}).$$

Then the **Fourier integral representation** of f is given by:

$$f(x) = \frac{1}{\pi} \int_0^{\infty} [A(\lambda) \cos(\lambda x) + B(\lambda) \sin(\lambda x)] d\lambda, \quad (1)$$

where the Fourier cosine and sine transforms are defined as:

$$A(\lambda) = \int_{-\infty}^{\infty} f(t) \cos(\lambda t) dt, \quad B(\lambda) = \int_{-\infty}^{\infty} f(t) \sin(\lambda t) dt.$$

The identity (1) holds at every point $x \in \mathbb{R}$ where f is continuous. If x is a point of discontinuity, the Fourier integral converges to the average of the

left- and right-hand limits:

$$\frac{1}{2} (f(x^-) + f(x^+)) = \frac{1}{\pi} \int_0^\infty [A(\lambda) \cos(\lambda x) + B(\lambda) \sin(\lambda x)] d\lambda.$$

Symmetry

- If f is **even**, i.e. $f(-x) = f(x)$, then the sine terms vanish ($B(\lambda) = 0$) and we obtain the **Fourier cosine integral**:

$$f(x) = \frac{1}{\pi} \int_0^\infty A(\lambda) \cos(\lambda x) d\lambda,$$

where

$$A(\lambda) = 2 \int_0^\infty f(t) \cos(\lambda t) dt.$$

- If f is **odd**, i.e. $f(-x) = -f(x)$, then the cosine terms vanish ($A(\lambda) = 0$) and we obtain the **Fourier sine integral**:

$$f(x) = \frac{1}{\pi} \int_0^\infty B(\lambda) \sin(\lambda x) d\lambda,$$

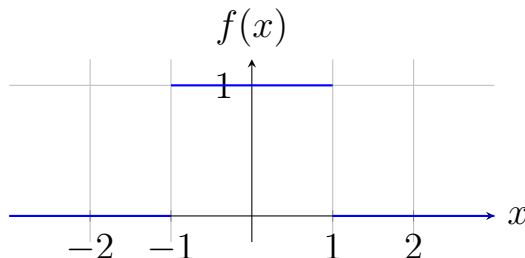
where

$$B(\lambda) = 2 \int_0^\infty f(t) \sin(\lambda t) dt.$$

Example 1.

Define:

$$f(x) = \begin{cases} 1, & |x| \leq 1, \\ 0, & |x| > 1. \end{cases}$$



Plot of $f(x) = 1$ for $|x| \leq 1$ and $f(x) = 0$ otherwise.

Since f is even, we have $B(\lambda) = 0$, and the Fourier cosine transform is:

$$A(\lambda) = 2 \int_0^{\infty} f(x) \cos(\lambda x) dx = 2 \int_0^1 \cos(\lambda x) dx = \frac{2 \sin(\lambda)}{\lambda}.$$

Hence, the Fourier integral representation of $f(x)$ is:

$$f(x) = \frac{1}{\pi} \int_0^{\infty} A(\lambda) \cos(\lambda x) d\lambda = \frac{2}{\pi} \int_0^{\infty} \frac{\sin(\lambda)}{\lambda} \cos(\lambda x) d\lambda.$$

At $x = 0$:

$$f(0) = 1 = \frac{2}{\pi} \int_0^{\infty} \frac{\sin \lambda}{\lambda} d\lambda.$$

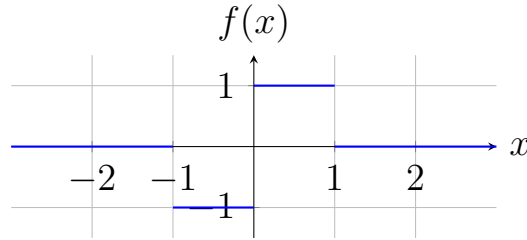
Then

$$\int_0^{\infty} \frac{\sin \lambda}{\lambda} d\lambda = \frac{\pi}{2}.$$

Example 2.

Define:

$$f(x) = \begin{cases} -1, & -1 \leq x \leq 0, \\ 1, & 0 \leq x \leq 1, \\ 0, & |x| > 1. \end{cases}$$



Since f is odd, we have $A(\lambda) = 0$, and the Fourier sine transform is:

$$B(\lambda) = 2 \int_0^{\infty} f(x) \sin(\lambda x) dx.$$

Substituting $f(x)$ gives:

$$B(\lambda) = 2 \int_0^1 \sin(\lambda x) dx = \frac{2(1 - \cos(\lambda))}{\lambda}.$$

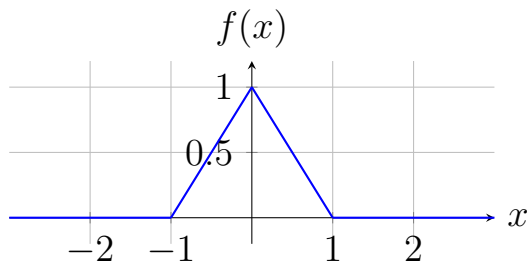
Hence, the Fourier integral representation of $f(x)$ reduces to:

$$f(x) = \frac{1}{\pi} \int_0^{\infty} B(\lambda) \sin(\lambda x) d\lambda = \frac{2}{\pi} \int_0^{\infty} \frac{1 - \cos(\lambda)}{\lambda} \sin(\lambda x) d\lambda.$$

Example 3.

Let

$$f(x) = \begin{cases} 1 - |x|, & |x| \leq 1, \\ 0, & |x| > 1. \end{cases}$$



Since f is even, in the Fourier–integral representation

$$f(x) = \frac{1}{\pi} \int_0^\infty [A(\lambda) \cos(\lambda x) + B(\lambda) \sin(\lambda x)] d\lambda,$$

we have $B(\lambda) \equiv 0$ and only $A(\lambda)$ is needed:

$$A(\lambda) = \int_{-\infty}^\infty f(t) \cos(\lambda t) dt = 2 \int_0^1 (1 - t) \cos(\lambda t) dt.$$

Compute:

$$A(\lambda) = 2 \left[\int_0^1 \cos(\lambda t) dt - \int_0^1 t \cos(\lambda t) dt \right].$$

Now,

$$\int_0^1 \cos(\lambda t) dt = \frac{\sin \lambda}{\lambda}, \quad \int_0^1 t \cos(\lambda t) dt = \frac{\sin \lambda}{\lambda} + \frac{\cos \lambda - 1}{\lambda^2}.$$

Hence,

$$A(\lambda) = 2 \left[\frac{\sin \lambda}{\lambda} - \frac{\sin \lambda}{\lambda} - \frac{\cos \lambda - 1}{\lambda^2} \right] = \frac{2[1 - \cos \lambda]}{\lambda^2}.$$

Since $B(\lambda) = 0$,

$$f(x) = \frac{1}{\pi} \int_0^\infty \frac{2[1 - \cos \lambda]}{\lambda^2} \cos(\lambda x) d\lambda.$$

At $x = 0$, we have $f(0) = 1$, so

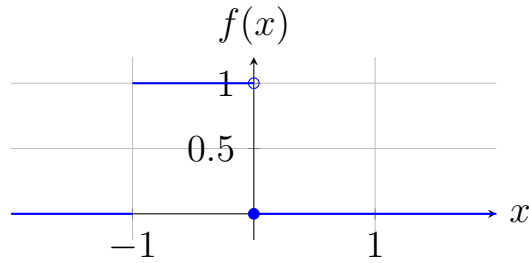
$$1 = f(0) = \frac{2}{\pi} \int_0^\infty \frac{1 - \cos \lambda}{\lambda^2} d\lambda.$$

Thus,

$$\boxed{\int_0^\infty \frac{1 - \cos \lambda}{\lambda^2} d\lambda = \frac{\pi}{2}.}$$

Example 4.

$$f(x) = \begin{cases} 1, & -1 \leq x < 0, \\ 0, & 0 \leq x \leq 1, \\ 0, & |x| > 1. \end{cases}$$



Recall that, if $f(x)$ and $f'(x)$ are piecewise continuous on every finite interval and $f(x)$ is absolutely integrable on $(-\infty, \infty)$, then

$$f(x) = \frac{1}{\pi} \int_0^\infty [A(\lambda) \cos(\lambda x) + B(\lambda) \sin(\lambda x)] d\lambda,$$

where

$$A(\lambda) = \int_{-\infty}^\infty f(t) \cos(\lambda t) dt, \quad B(\lambda) = \int_{-\infty}^\infty f(t) \sin(\lambda t) dt.$$

At a point of discontinuity x_0 ,

$$\boxed{\frac{1}{2} [f(x_0^-) + f(x_0^+)] = \frac{1}{\pi} \int_0^\infty [A(\lambda) \cos(\lambda x_0) + B(\lambda) \sin(\lambda x_0)] d\lambda.}$$

Here $f(x)$ is supported on $[-1, 1]$, so

$$A(\lambda) = \int_{-1}^0 \cos(\lambda t) dt = \frac{\sin(\lambda)}{\lambda}, \quad B(\lambda) = \int_0^1 \sin(\lambda t) dt = \frac{1 - \cos(\lambda)}{\lambda}.$$

Hence

$$f(x) = \frac{1}{\pi} \int_0^\infty \left[\frac{\sin(\lambda)}{\lambda} \cos(\lambda x) + \frac{1 - \cos(\lambda)}{\lambda} \sin(\lambda x) \right] d\lambda.$$

At $x = 0$, $\sin(\lambda x) = 0$ and $\cos(\lambda x) = 1$, giving

$$f(0) = \frac{1}{\pi} \int_0^\infty \frac{\sin(\lambda)}{\lambda} d\lambda = \frac{1}{\pi} \cdot \frac{\pi}{2} = \frac{1}{2}.$$

By the Fourier Integral Theorem,

$$\frac{1}{2}[f(0^-) + f(0^+)] = \frac{1}{2}(1 + 0) = \frac{1}{2},$$

which agrees with the computed value.

$$f(0) = \frac{1}{2}.$$

Exercise. Show that

$$\int_0^\infty \frac{\sin(\pi\lambda)}{1 - \lambda^2} \sin(\lambda x) d\lambda = \begin{cases} \frac{\pi}{2} \sin x, & 0 \leq x \leq \pi, \\ 0, & x > \pi, \end{cases}$$

(and the value is 0 also for $x < 0$). *Hint:* Use the Fourier *sine* transform.

Solution. Consider the function

$$f(x) = \begin{cases} \sin x, & 0 \leq x \leq \pi, \\ 0, & x > \pi, \end{cases} \quad (\text{and } f(x) = 0 \text{ for } x < 0).$$

Its (one-sided) Fourier *sine* transform is

$$B(\lambda) = \int_0^\infty f(t) \sin(\lambda t) dt = \int_0^\pi \sin t \sin(\lambda t) dt.$$

Using the product-to-sum identity

$$\sin t \sin(\lambda t) = \frac{1}{2} \left(\cos((\lambda - 1)t) - \cos((\lambda + 1)t) \right),$$

we get

$$\begin{aligned} B(\lambda) &= \frac{1}{2} \left[\int_0^\pi \cos((\lambda - 1)t) dt - \int_0^\pi \cos((\lambda + 1)t) dt \right] \\ &= \frac{1}{2} \left[\frac{\sin((\lambda - 1)\pi)}{\lambda - 1} - \frac{\sin((\lambda + 1)\pi)}{\lambda + 1} \right]. \end{aligned}$$

Since $\sin((\lambda \pm 1)\pi) = \sin(\pi\lambda) \cos \pi \pm \cos(\pi\lambda) \sin \pi = -\sin(\pi\lambda)$, this simplifies to

$$B(\lambda) = \frac{1}{2} \left[-\frac{\sin(\pi\lambda)}{\lambda - 1} + \frac{\sin(\pi\lambda)}{\lambda + 1} \right] = \sin(\pi\lambda) \cdot \frac{-1}{\lambda^2 - 1} = \frac{\sin(\pi\lambda)}{1 - \lambda^2}.$$

For suitable piecewise smooth, absolutely integrable f , the sine-transform inversion reads

$$f(x) = \frac{2}{\pi} \int_0^\infty B(\lambda) \sin(\lambda x) d\lambda \quad \text{for } x > 0,$$

with the usual midpoint value at discontinuities. Since f is continuous on $[0, \pi]$ and vanishes for $x > \pi$, we obtain

$$\int_0^\infty \frac{\sin(\pi\lambda)}{1 - \lambda^2} \sin(\lambda x) d\lambda = \frac{\pi}{2} f(x) = \begin{cases} \frac{\pi}{2} \sin x, & 0 \leq x \leq \pi, \\ 0, & x > \pi, \end{cases}$$

and the integral is 0 also for $x < 0$ (since $f(x) = 0$ there).

First Order Differential Equations

Introduction

In this chapter, we study methods for solving **first-order differential equations**. The most general form is:

$$\frac{dy}{dt} = f(y, t)$$

There is no general formula for solving this equation. Instead, we explore several special cases.

Topics Covered

- Linear equations
- Bernoulli equations
- Separable equations
- Homogenizes equations
- Exact equations
- Non-exact and integrating factors.

Linear First-Order Differential Equations

A linear first-order differential equation has the form:

$$\frac{dy}{dt} + p(t)y = g(t)$$

where $p(t)$ and $g(t)$ are continuous.

Solution Method

1. Compute the **integrating factor**:

$$\mu(t) = e^{\int p(t) dt}$$

2. Multiply the entire equation by $\mu(t)$:

$$\mu(t) \frac{dy}{dt} + \mu(t)p(t)y = \mu(t)g(t)$$

3. Recognize the left side as the derivative of a product:

$$\frac{d}{dt}[\mu(t)y(t)] = \mu(t)g(t)$$

4. Integrate both sides:

$$\mu(t)y(t) = \int \mu(t)g(t) dt + C$$

5. Solve for $y(t)$:

$$y(t) = \frac{1}{\mu(t)} \left(\int \mu(t)g(t) dt + C \right)$$

Summary

The general solution is:

$$y(t) = \frac{1}{e^{\int p(t) dt}} \left(\int e^{\int p(t) dt} g(t) dt + C \right)$$

However, it's often easier to use the process above rather than memorizing the formula.

Example 1.

Solve the IVP:

$$ty' + 2y = t^2 - t + 1, \quad y(1) = \frac{1}{2}$$

Step 1: Standard form.

Divide by t (assume $t > 0$):

$$y' + \frac{2}{t}y = t - 1 + \frac{1}{t}$$

Step 2: Integrating factor.

$$\mu(t) = e^{\int \frac{2}{t} dt} = e^{2 \ln |t|} = t^2$$

Step 3: Multiply through by $\mu(t) = t^2$:

$$t^2 y' + 2ty = t^3 - t^2 + t \quad \Rightarrow \quad \frac{d}{dt}(t^2 y) = t^3 - t^2 + t$$

Step 4: Integrate both sides.

$$t^2 y = \int (t^3 - t^2 + t) dt = \frac{1}{4}t^4 - \frac{1}{3}t^3 + \frac{1}{2}t^2 + C$$

Step 5: Solve for $y(t)$.

$$y(t) = \frac{1}{4}t^2 - \frac{1}{3}t + \frac{1}{2} + \frac{C}{t^2}$$

Step 6: Apply initial condition.

$$y(1) = \frac{1}{4} - \frac{1}{3} + \frac{1}{2} + C = \frac{1}{2} \Rightarrow C = \frac{1}{12}$$

Final Solution:

$$y(t) = \frac{1}{4}t^2 - \frac{1}{3}t + \frac{1}{2} + \frac{1}{12t^2}$$

Example 2

Solve the initial value problem:

$$\cos(x)y' + \sin(x)y = 2\cos^3(x)\sin(x) - 1, \quad y\left(\frac{\pi}{4}\right) = 3\sqrt{2}, \quad 0 \leq x < \frac{\pi}{2}$$

Dividing both sides by $\cos(x)$, we obtain:

$$y' + \tan(x)y = 2\cos^2(x)\sin(x) - \sec(x)$$

The integrating factor is:

$$\mu(x) = e^{\int \tan(x) dx} = \sec(x)$$

Multiplying the equation by the integrating factor:

$$\sec(x)y' + \sec(x)\tan(x)y = 2\sec(x)\cos^2(x)\sin(x) - \sec^2(x)$$

$$\frac{d}{dx}[\sec(x)y] = 2\cos(x)\sin(x) - \sec^2(x)$$

Integrating both sides:

$$\sec(x)y = \int (2\cos(x)\sin(x) - \sec^2(x)) dx = \int \sin(2x) dx - \int \sec^2(x) dx$$

$$\sec(x)y = -\frac{1}{2}\cos(2x) - \tan(x) + C$$

Solving for $y(x)$:

$$y(x) = \cos(x) \left(-\frac{1}{2}\cos(2x) - \tan(x) + C \right)$$

$$y(x) = -\frac{1}{2}\cos(x)\cos(2x) - \cos(x)\tan(x) + C\cos(x)$$

$$y(x) = -\frac{1}{2}\cos(x)\cos(2x) - \sin(x) + C\cos(x)$$

Using the initial condition $y\left(\frac{\pi}{4}\right) = 3\sqrt{2}$, we have:

$$3\sqrt{2} = -\frac{1}{2} \cdot \frac{\sqrt{2}}{2} \cdot 0 - \frac{\sqrt{2}}{2} + C \cdot \frac{\sqrt{2}}{2} \Rightarrow 3\sqrt{2} = -\frac{\sqrt{2}}{2} + C \cdot \frac{\sqrt{2}}{2} \Rightarrow C = 7$$

Thus, the solution is:

$$y(x) = -\frac{1}{2}\cos(x)\cos(2x) - \sin(x) + 7\cos(x)$$

Bernoulli Differential Equations

A **Bernoulli equation** is a first-order nonlinear differential equation of the form:

$$\frac{dy}{dx} + P(x)y = Q(x)y^n, \quad n \neq 0, 1.$$

Method of Solution

1. Divide both sides by y^n (assuming $y \neq 0$):

$$y^{-n} \frac{dy}{dx} + P(x)y^{1-n} = Q(x).$$

2. Make the substitution:

$$v = y^{1-n} \quad \Rightarrow \quad \frac{dv}{dx} = (1-n)y^{-n} \frac{dy}{dx}.$$

3. Replace $y^{-n} \frac{dy}{dx}$ in the equation to get a linear equation in v :

$$\frac{dv}{dx} + (1-n)P(x)v = (1-n)Q(x).$$

4. Solve this **linear first-order ODE** using the integrating factor method:

$$\mu(x) = e^{\int (1-n)P(x) dx}.$$

5. After finding $v(x)$, use the back-substitution $y = v^{\frac{1}{1-n}}$.

Example.

Solve

$$\frac{dy}{dx} + y = xy^2.$$

This is a Bernoulli equation with $P(x) = 1$, $Q(x) = x$, and $n = 2$. Dividing both sides by y^2 gives

$$y^{-2} \frac{dy}{dx} + y^{-1} = x.$$

Let $v = y^{-1}$, so that $\frac{dv}{dx} = -y^{-2}\frac{dy}{dx}$. Substituting into the equation yields

$$-\frac{dv}{dx} + v = x, \quad \text{or equivalently} \quad \frac{dv}{dx} - v = -x.$$

This is a linear first-order equation in v . The integrating factor is

$$\mu(x) = e^{-\int 1 dx} = e^{-x}.$$

Multiplying through by e^{-x} ,

$$e^{-x}\frac{dv}{dx} - e^{-x}v = -xe^{-x}.$$

The left-hand side is the derivative of ve^{-x} , so

$$\frac{d}{dx}(ve^{-x}) = -xe^{-x}.$$

Integrating both sides gives

$$ve^{-x} = \int -xe^{-x} dx = (x+1)e^{-x} + C.$$

Hence,

$$v = x + 1 + Ce^x.$$

Substituting back $v = y^{-1}$, we obtain the general solution

$$\boxed{y = \frac{1}{x+1+Ce^x}}.$$

At $x = 0$, if $C = 0$, then $y(0) = \frac{1}{1+0} = 1$. This can serve as an initial condition if specified.

Example.

Solve the initial value problem

$$\frac{dy}{dx} - \frac{2y}{x} = x^2y^3, \quad y(1) = 1, \quad x > 0.$$

This is a Bernoulli equation with $P(x) = -\frac{2}{x}$, $Q(x) = x^2$, $n = 3$. Dividing both sides by y^3 (assuming $y \neq 0$) gives

$$y^{-3} \frac{dy}{dx} - \frac{2y^{-2}}{x} = x^2.$$

Let $v = y^{-2}$, so that $\frac{dv}{dx} = -2y^{-3} \frac{dy}{dx}$, or equivalently

$$y^{-3} \frac{dy}{dx} = -\frac{1}{2} \frac{dv}{dx}.$$

Substituting in the equation yields

$$-\frac{1}{2} \frac{dv}{dx} - \frac{2v}{x} = x^2,$$

which simplifies to

$$\frac{dv}{dx} + \frac{4v}{x} = -2x^2.$$

This is a first-order linear differential equation in v . The integrating factor is

$$\mu(x) = e^{\int \frac{4}{x} dx} = e^{4 \ln x} = x^4.$$

Multiplying through by x^4 ,

$$x^4 \frac{dv}{dx} + 4x^3 v = -2x^6,$$

which gives

$$\frac{d}{dx}(x^4 v) = -2x^6.$$

Integrating both sides:

$$x^4 v = -\frac{2x^7}{7} + C.$$

Hence,

$$v = -\frac{2x^3}{7} + \frac{C}{x^4}.$$

Recalling $v = y^{-2}$,

$$y^{-2} = -\frac{2x^3}{7} + \frac{C}{x^4}, \quad \text{or} \quad y = \frac{1}{\sqrt{-\frac{2x^3}{7} + \frac{C}{x^4}}}.$$

Applying the initial condition $y(1) = 1$:

$$1^{-2} = -\frac{2(1)^3}{7} + \frac{C}{(1)^4} \Rightarrow 1 = -\frac{2}{7} + C \Rightarrow C = \frac{9}{7}.$$

Thus the particular solution is

$$y(x) = \frac{1}{\sqrt{-\frac{2x^3}{7} + \frac{9}{7x^4}}} = \frac{\sqrt{7}}{\sqrt{-2x^3 + \frac{9}{x^4}}}.$$

Interval of Convergence. For $y(x)$ to be real and defined, the denominator must be positive:

$$-\frac{2x^3}{7} + \frac{9}{7x^4} > 0 \implies -2x^7 + 9 > 0 \implies x^7 < \frac{9}{2}.$$

Hence,

$$0 < x < \sqrt[7]{\frac{9}{2}} \approx 1.48.$$

Therefore, the solution is valid for $x > 0$ and $x < \sqrt[7]{9/2}$.

$$y(x) = \frac{1}{\sqrt{-\frac{2x^3}{7} + \frac{9}{7x^4}}}, \quad 0 < x < \sqrt[7]{\frac{9}{2}}.$$

Separable Equations

We now consider nonlinear first-order differential equations, starting with separable equations. A differential equation is separable if it can be written as:

$$N(y) \frac{dy}{dx} = M(x)$$

This means all y -terms are with dy and all x -terms are with dx . Rearranging gives:

$$N(y) dy = M(x) dx$$

We solve by integrating both sides:

$$\int N(y) dy = \int M(x) dx$$

This gives an *implicit solution*, which may or may not be solvable for an *explicit solution* $y = y(x)$.

Be aware of the **interval of validity**: the solution is only valid where it is defined (no division by zero, negative logs, etc.).

Most separable equations can be solved using this straightforward technique. We begin with a simple example to illustrate the method.

Example 1

Solve the differential equation:

$$\frac{dy}{dx} = 6y^2x, \quad y(1) = \frac{1}{25}$$

This is a separable equation:

$$y^{-2}dy = 6x dx$$

Integrating both sides:

$$\int y^{-2}dy = \int 6x dx \quad \Rightarrow \quad -\frac{1}{y} = 3x^2 + C$$

Apply the initial condition $y(1) = \frac{1}{25}$:

$$-\frac{1}{1/25} = 3(1)^2 + C \quad \Rightarrow \quad -25 = 3 + C \Rightarrow C = -28$$

So the solution is:

$$-\frac{1}{y} = 3x^2 - 28 \quad \Rightarrow \quad y(x) = \frac{1}{28 - 3x^2}$$

Interval of Validity:

To avoid division by zero, we require:

$$28 - 3x^2 \neq 0 \quad \Rightarrow \quad x \neq \pm\sqrt{\frac{28}{3}} \approx \pm 3.055$$

The valid intervals are:

$$(-\infty, -\sqrt{28/3}), \quad (-\sqrt{28/3}, \sqrt{28/3}), \quad (\sqrt{28/3}, \infty)$$

Since the initial condition is at $x = 1$, the correct interval of validity is:

$$\boxed{-\sqrt{28/3} < x < \sqrt{28/3}}$$

Note: With different initial conditions, the solution remains the same, but the interval of validity changes:

- If $y(-4) = -\frac{1}{20}$, then the interval is $(-\infty, -\sqrt{28/3})$
- If $y(6) = -\frac{1}{80}$, then the interval is $(\sqrt{28/3}, \infty)$

Example 2

Solve the IVP:

$$\frac{dy}{dx} = \frac{3x^2 + 4x - 4}{2y - 4}, \quad y(1) = 3$$

This is a separable equation. Rearranging and integrating:

$$(2y - 4)dy = (3x^2 + 4x - 4)dx$$

$$\int (2y - 4) dy = \int (3x^2 + 4x - 4) dx \Rightarrow y^2 - 4y = x^3 + 2x^2 - 4x + C$$

Apply the initial condition $y(1) = 3$:

$$3^2 - 4(3) = 1^3 + 2(1)^2 - 4(1) + C \Rightarrow 9 - 12 = 1 + 2 - 4 + C \Rightarrow -3 = -1 + C \Rightarrow C = -2$$

So the implicit solution is:

$$y^2 - 4y = x^3 + 2x^2 - 4x - 2$$

Solve explicitly using the quadratic formula:

$$y = \frac{4 \pm \sqrt{16 + 4(x^3 + 2x^2 - 4x - 2)}}{2} = 2 \pm \sqrt{x^3 + 2x^2 - 4x + 2}$$

Use the initial condition to determine the correct sign:

$$y(1) = 2 \pm \sqrt{1 + 2 - 4 + 2} = 2 \pm \sqrt{1} = 2 \pm 1 \Rightarrow y(1) = 3 \Rightarrow \text{use } +$$

Thus, the explicit solution is:

$$y(x) = 2 + \sqrt{x^3 + 2x^2 - 4x + 2}$$

Interval of Validity:

We require the argument of the square root to be non-negative:

$$x^3 + 2x^2 - 4x + 2 \geq 0$$

Graphing, the real root is approximately $x \approx -3.36523$, so the interval of validity is:

$$x \geq -3.36523$$

which contains $x = 1$, satisfying the initial condition.

Homogeneous Differential Equations

1. Definition.

A first-order differential equation of the form

$$\frac{dy}{dx} = F(x, y)$$

is said to be **homogeneous** if the function $F(x, y)$ satisfies

$$F(tx, ty) = F(x, y) \quad \text{for all } t > 0.$$

That is, $F(x, y)$ is homogeneous of degree 0.

Equivalently, the differential equation

$$M(x, y) dx + N(x, y) dy = 0$$

is homogeneous if both M and N are homogeneous functions of the **same degree**.

Recall: a function $f(x, y)$ is homogeneous of degree n if

$$f(tx, ty) = t^n f(x, y).$$

2. Standard Form and Substitution.

If

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right),$$

then the equation is homogeneous. We use the substitution:

$$y = vx \quad \Rightarrow \quad \frac{dy}{dx} = v + x \frac{dv}{dx}.$$

Substitute these into the original equation. This converts the given homogeneous equation into a ****separable equation**** in v and x .

3. Method of Solution.

1. Express the given ODE as $\frac{dy}{dx} = F(y/x)$.
2. Substitute $y = vx$, so that $\frac{dy}{dx} = v + x \frac{dv}{dx}$.
3. Replace y/x by v everywhere and simplify.
4. Rearrange to get a separable equation in v and x :

$$\frac{dv}{dx} = g(v, x) \quad \Rightarrow \quad \frac{dv}{f(v)} = \frac{dx}{x}.$$

5. Integrate both sides and then back-substitute $v = y/x$ to get the final relation between x and y .
-

4. Example.

Solve

$$\frac{dy}{dx} = \frac{x + y}{x - y}.$$

Solution. The right-hand side depends only on y/x , so the equation is homogeneous.

Let $y = vx$. Then

$$\frac{dy}{dx} = v + x \frac{dv}{dx}.$$

Substituting in:

$$v + x \frac{dv}{dx} = \frac{x + vx}{x - vx} = \frac{1 + v}{1 - v}.$$

Simplify:

$$x \frac{dv}{dx} = \frac{1 + v}{1 - v} - v = \frac{1 + v - v(1 - v)}{1 - v} = \frac{1 + v^2}{1 - v}.$$

Hence:

$$\frac{1 - v}{1 + v^2} dv = \frac{dx}{x}.$$

Integrate both sides:

$$\int \frac{1 - v}{1 + v^2} dv = \int \frac{dx}{x}.$$

Split the integral on the left:

$$\int \frac{dv}{1 + v^2} - \int \frac{v dv}{1 + v^2} = \ln |x| + C.$$

The first term is $\tan^{-1} v$, and the second is $\frac{1}{2} \ln(1 + v^2)$. Thus:

$$\tan^{-1} v - \frac{1}{2} \ln(1 + v^2) = \ln |x| + C.$$

Substituting back $v = \frac{y}{x}$, we get

$$\boxed{\tan^{-1}\left(\frac{y}{x}\right) - \frac{1}{2} \ln\left(1 + \frac{y^2}{x^2}\right) = \ln |x| + C.}$$

Simplifying the logarithmic term:

$$\tan^{-1}\left(\frac{y}{x}\right) - \frac{1}{2} \ln(x^2 + y^2) = C'.$$

Hence, the general solution is:

$$\boxed{\tan^{-1}\left(\frac{y}{x}\right) - \frac{1}{2} \ln(x^2 + y^2) = C.}$$

5. Remarks.

- The key idea in solving homogeneous equations is the substitution $y = vx$.
- This converts the equation into a separable form.
- After integration, always replace v by y/x to get the final result.

Exact Equations

We now explore a new class of first-order differential equations: **exact equations**. Before diving into the solution method, we demonstrate the concept using an example.

Example 1

$$2xy - 9x^2 + (2y + x^2 + 1)\frac{dy}{dx} = 0$$

Assume a function $\Psi(x, y)$ exists such that:

$$\Psi(x, y) = y^2 + (x^2 + 1)y - 3x^3$$

Then,

$$\frac{\partial \Psi}{\partial x} = 2xy - 9x^2, \quad \frac{\partial \Psi}{\partial y} = 2y + x^2 + 1$$

So the equation becomes:

$$\frac{d}{dx}[\Psi(x, y(x))] = 0 \Rightarrow \Psi(x, y) = C$$

Implicit solution:

$$y^2 + (x^2 + 1)y - 3x^3 = C$$

General Form

A differential equation is exact if:

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0$$

and there exists a function $\Psi(x, y)$ such that:

$$\frac{\partial \Psi}{\partial x} = M, \quad \frac{\partial \Psi}{\partial y} = N$$

Then the solution is:

$$\Psi(x, y) = C$$

To test for exactness:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Example 2:

$$2xy - 9x^2 + (2y + x^2 + 1) \frac{dy}{dx} = 0, \quad y(0) = -3$$

$$M = 2xy - 9x^2 \Rightarrow M_y = 2x, \quad N = 2y + x^2 + 1 \Rightarrow N_x = 2x$$

Exact equation.

Integrate M with respect to x :

$$\Psi(x, y) = \int (2xy - 9x^2) dx = x^2y - 3x^3 + h(y)$$

Differentiate with respect to y :

$$\frac{\partial \Psi}{\partial y} = x^2 + h'(y) = 2y + x^2 + 1 \Rightarrow h'(y) = 2y + 1 \Rightarrow h(y) = y^2 + y$$

So:

$$\Psi(x, y) = x^2y - 3x^3 + y^2 + y$$

Implicit solution:

$$x^2y - 3x^3 + y^2 + y = C$$

Apply initial condition $y(0) = -3$:

$$0 + 9 - 3 = C \Rightarrow C = 6$$

$$y^2 + (x^2 + 1)y - 3x^3 = 6$$

Solve using quadratic formula:

$$y = -(x^2 + 1) \pm \frac{\sqrt{(x^2 + 1)^2 + 12x^3 + 24}}{2}$$

Choose $-$ sign based on initial condition:

$$y(x) = -(x^2 + 1) - \frac{\sqrt{x^4 + 12x^3 + 2x^2 + 25}}{2}$$

Interval of validity:

$$x^4 + 12x^3 + 2x^2 + 25 \geq 0 \Rightarrow x \geq -1.396911133$$

Example 3

$$2xy^2 + 4 = 2(3 - x^2y)y', \quad y(-1) = 8$$

Rewriting:

$$2xy^2 + 4 + 2(x^2y - 3)y' = 0$$

$$M = 2xy^2 + 4, \quad N = 2x^2y - 6$$

$$M_y = 4xy, \quad N_x = 4xy \Rightarrow \text{Exact}$$

Integrate N w.r.t. y :

$$\Psi(x, y) = \int (2x^2y - 6)dy = x^2y^2 - 6y + h(x)$$

Differentiate w.r.t. x :

$$\frac{\partial \Psi}{\partial x} = 2xy^2 + h'(x) = 2xy^2 + 4 \Rightarrow h'(x) = 4 \Rightarrow h(x) = 4x$$

$$\Psi(x, y) = x^2y^2 - 6y + 4x$$

Implicit solution:

$$x^2y^2 - 6y + 4x = C$$

Apply initial condition:

$$64 - 48 - 4 = C \Rightarrow C = 12$$

$$x^2y^2 - 6y + 4x - 12 = 0$$

Solve for y :

$$y = \frac{6 \pm \sqrt{36 + 48x^2 - 16x^3}}{2x^2} = 3 \pm \frac{\sqrt{9 + 12x^2 - 4x^3}}{x^2}$$

Use initial condition to choose “+”:

$$y(x) = 3 + \frac{\sqrt{9 + 12x^2 - 4x^3}}{x^2}$$

Interval of validity:

$$x \neq 0, \quad 9 + 12x^2 - 4x^3 > 0 \Rightarrow x \in (-\infty, 0) \text{ since } x = -1 \text{ is valid}$$

$$\boxed{(-\infty, 0)}$$

Non-Exact Differential Equations

1. Definition.

A first-order differential equation of the form

$$M(x, y) dx + N(x, y) dy = 0$$

is called **exact** if there exists a function $F(x, y)$ such that

$$\frac{\partial F}{\partial x} = M(x, y), \quad \frac{\partial F}{\partial y} = N(x, y).$$

That is, the equation represents the total differential $dF = 0$.

If this condition is *not* satisfied—i.e.

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x},$$

then the equation is said to be **non-exact**.

2. The Idea.

Even when the given equation is not exact, it may be possible to multiply it by a function $\mu(x, y)$, called an **integrating factor**, which makes it exact.

Thus we look for $\mu(x, y)$ such that

$$\mu(x, y) M(x, y) dx + \mu(x, y) N(x, y) dy = 0$$

becomes exact.

3. Condition for Exactness.

For an equation $M dx + N dy = 0$, exactness requires

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

If this does not hold, one may find an integrating factor μ satisfying:

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}.$$

—

4. Common Cases for Integrating Factors.

1. If

$$\frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right)$$

is a function of x only, then an integrating factor depending on x exists and is given by

$$\mu(x) = e^{\int \frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) dx}.$$

2. If

$$\frac{1}{M} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right)$$

is a function of y only, then an integrating factor depending on y exists and is given by

$$\mu(y) = e^{\int \frac{1}{M} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dy}.$$

—

5. Example.

Solve

$$(2xy + y^2) dx + (x^2 + 2xy) dy = 0.$$

Here,

$$M = 2xy + y^2, \quad N = x^2 + 2xy.$$

Compute the partial derivatives:

$$\frac{\partial M}{\partial y} = 2x + 2y, \quad \frac{\partial N}{\partial x} = 2x + 2y.$$

Thus, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, so in fact the equation *is exact*. For demonstration, let us modify slightly to make it non-exact.

Consider instead:

$$(2xy + y^2) dx + (x^2 + 3xy) dy = 0.$$

Now,

$$\frac{\partial M}{\partial y} = 2x + 2y, \quad \frac{\partial N}{\partial x} = 2x + 3y,$$

which are not equal; hence it is non-exact.

We test for an integrating factor depending on x :

$$\frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) = \frac{1}{x^2 + 3xy} [(2x + 2y) - (2x + 3y)] = \frac{-y}{x^2 + 3xy},$$

which is not a function of x alone.

Next, test for an integrating factor depending on y :

$$\frac{1}{M} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) = \frac{1}{2xy + y^2} [(2x + 3y) - (2x + 2y)] = \frac{y}{2xy + y^2} = \frac{1}{y(2x/y + 1)}.$$

This expression simplifies to a function of y only when x/y is constant (i.e. along straight lines), so no simple $\mu(y)$ exists in this case.

However, note that the equation can be divided by y^2 to yield a homogeneous form:

$$\left(2\frac{x}{y} + 1 \right) dx + \left(\left(\frac{x}{y} \right)^2 + 3\frac{x}{y} \right) dy = 0,$$

which may be solved by the substitution $v = x/y$.

Hence, when an equation is not exact, one either looks for an integrating factor or converts it into a homogeneous or separable form.

6. Summary.

- An exact equation satisfies $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.
- A non-exact equation can sometimes be made exact using an integrating factor $\mu(x)$ or $\mu(y)$.
- If no simple integrating factor exists, attempt other transformations (e.g. homogeneous substitution).