Fist Midterm Exam Solutions for math 203 (2nd semester 1445)

1. (5pts) Determine whether the sequence $\left\{\left(\frac{n^2-2}{n^2+3}\right)^n\right\}$ converges or diverges and if converges find its imit.

Solution:

Let $y = f(x) = \left(\frac{x^2-2}{x^2+3}\right)^x$ $then\ f(n) = \left(\frac{n^2-2}{n^2+3}\right)^n$ and

$$\lim_{x\to\infty} Ln\, y = \underset{x\to\infty}{Lim}\ Ln\left(\frac{x^2-2}{x^2+3}\right)^x = \underset{x\to\infty}{Lim}\ x[Ln(x^2-2) - Ln(x^2+3)] =$$

$$\underset{x\to\infty}{Lim}\ \frac{1}{\frac{1}{x}}[Ln(x^2-2) - Ln(x^2+3)] \overset{L'Hopital}{=} \underset{x\to\infty}{Lim}\frac{\frac{2x}{x^2-2} - \frac{2x}{x^2+3}}{\frac{-1}{x^2}}$$

$$= \underset{x\to\infty}{Lim}\ \frac{-10x^3}{(x^2-2)(x^2+3)} = 0$$

Hence $\underset{x\to\infty}{Lim}\ y = \underset{x\to\infty}{Lim}\ e^{Lny} = e^0 = 1$

So the sequence $\left\{\left(\frac{n^2-2}{n^2+3}\right)^n\right\}$ converges and $\lim_{n\to\infty}\left(\frac{n^2-2}{n^2+3}\right)^n = 1$.

2.(3 pts) Find the sum of the series:

$$\sum_{n=1}^{\infty}\left[\cos\left(\frac{1}{n}\right) - cos\left(\frac{1}{n+3}\right)\right]$$

Solution:

$$\sum_{n=1}^{\infty}\left[\cos\left(\frac{1}{n}\right) - cos\left(\frac{1}{n+3}\right)\right] = \sum_{n=1}^{\infty} a_n$$

$$a_1 = cos(1) - cos\left(\frac{1}{4}\right)$$

$$a_2 = cos\left(\frac{1}{2}\right) - cos\left(\frac{1}{5}\right)$$

$$a_3 = cos\left(\frac{1}{3}\right) - cos\left(\frac{1}{6}\right)$$

$$a_4 = cos\left(\frac{1}{4}\right) - cos\left(\frac{1}{7}\right)$$

.

.

.

$$a_{n-3} = cos\left(\frac{1}{n-3}\right) - cos\left(\frac{1}{n}\right)$$

$$a_{n-2} = cos\left(\frac{1}{n-2}\right) - cos\left(\frac{1}{n+1}\right)$$

$$a_{n-1} = cos\left(\frac{1}{n-1}\right) - cos\left(\frac{1}{n+2}\right)$$

$$a_n = cos\left(\frac{1}{n}\right) - cos\left(\frac{1}{n+3}\right).$$

$$S_n = a_1 + a_2 + \cdots + a_n = cos(1) + cos\left(\frac{1}{2}\right) + cos\left(\frac{1}{3}\right)$$

$$-cos\left(\frac{1}{n+1}\right) - cos\left(\frac{1}{n+2}\right) - cos\left(\frac{1}{n+3}\right)$$

and $\lim_{n\to\infty} S_n = \cos(1) + \cos\left(\frac{1}{2}\right) + \cos\left(\frac{1}{3}\right) - 3$.

Therefore

$$\sum_{n=1}^{\infty}\left[\cos\left(\frac{1}{n}\right) - cos\left(\frac{1}{n+3}\right)\right] = cos(1) + cos\left(\frac{1}{2}\right) + cos\left(\frac{1}{3}\right) - 3.$$

3.**(5pts)** Test the convergence of the series $\sum_{n=1}^{\infty} \frac{3+\cos(n)}{e^n}$.

Solution:

Let $\sum\limits_{n=1}^{\infty} \frac{3+\cos(n)}{e^n} = \sum\limits_{n=1}^{\infty} a_n$ and taking $\sum\limits_{n=1}^{\infty} b_n = \sum\limits_{n=1}^{\infty} \frac{4}{e^n} = 4\sum\limits_{n=1}^{\infty} \left(\frac{1}{e}\right)^n$

The series $\sum\limits_{n=1}^{\infty} \left(\frac{1}{e}\right)^n$ is a geometric series with $|r| = \frac{1}{e} < 1$ and hence

converges $\Longrightarrow$ $\sum\limits_{n=1}^{\infty} b_n = \sum\limits_{n=1}^{\infty} \frac{4}{e^n}$ converges , but

$$\frac{3+\cos(n)}{e^n} \le \frac{4}{e^n} \qquad \forall n$$

and by the basic comparison test the series $\sum\limits_{n=1}^{\infty} \frac{3+\cos(n)}{e^n}$ converges.

4.(6pts) Find the radius and the interval of convergence for the power series $\sum_{n=1}^{\infty} \frac{(x+2)^n}{n\,(-5)^n}$

Solution:

Taking $u_n = \frac{(x+2)^n}{n\ (-5)^n} = (-1)^n \left(\frac{1}{5}\right)^n \frac{(x+2)^n}{n}$ then

$$\underset{n\to\infty}{Lim} \left|\frac{u_{n+1}}{u_n}\right| = \frac{1}{5}\,|x+3| \lim_{n\to\infty} \frac{n}{n+1} = \frac{1}{5}|x+2|.$$

By the ratio test, the series converges if $\frac{1}{5}|x+2| < 1$ or $|x+2| < 5$

That is $-5 < x+2 < 5 \Rightarrow -7 < x < 3$.

If $x=3$ we have the series $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ which converges by the alternating series test, and If $x=-7$ we get the series $\sum_{n=1}^{\infty}\frac{1}{n}$ which diverges by the integral test. **So**

The interval of convergence is $I=(-7\ ,\ 3]$ and the radius of convergence is $r\ =\ 5$.

5.(**6pts**) Find the power series representation for function $f(x)=\frac{x}{(1+x)^2}$

and use its first three nonzero terms to approximate the integral

$$\int_0^1 \frac{x^2}{\left(1+x^2\right)^2}\ dx$$

Solution: We know that $\sum_{n=0}^{\infty} u^n=\frac{1}{1-u} \quad \Leftrightarrow |u|<1$ taking $u=-x$ we get

$$\frac{1}{1+x}=\sum_{n=0}^{\infty}(-1)^n x^n \quad \Leftrightarrow \quad |x|<1$$ differentiating both sides, we get

$$\frac{1}{(1+x)^2}=\sum_{n=1}^{\infty}(-1)^{n+1} n x^{n-1} \qquad |x|<1$$ hence

$$f(x) = \frac{x}{(1+x)^2} = \sum_{n=1}^{\infty} (-1)^{n+1} n x^n \qquad |x| < 1$$

Replacing x by x^2 we get

$$\frac{x^2}{(1+x^2)^2} = \sum_{n=1}^{\infty} (-1)^{n+1} n x^{2n} \qquad |x| < 1 \text{ and}$$

$$\int_0^1 \frac{x^2}{(1+x^2)} dx = \sum_{n=1}^{\infty} (-1)^{n-1} n \int_0^1 x^{2n} dx \approx \int_0^1 x^2 dx - 2\int_0^1 x^4 dx + 3\int_0^1 x^6 dx$$

$$= \frac{1}{3} - \frac{2}{5} + \frac{3}{7} = \frac{38}{105}$$