Chapter 5: Magnetostatics

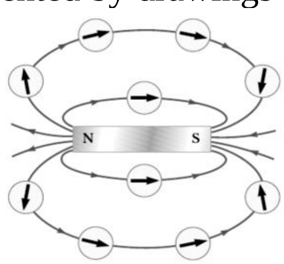
5.1 Magnetic Fields

As mentioned in chapter 2, an electrostatic field is produced by static (stationary) charges. If the charges are moving with a constant velocity, a static magnetic (or magnetostatic) field is produced.
This means a magnetostatic field is produced by a constant current flow.

There are two major laws governing magnetostatic fields:
(1) Biot–Savart's law
(2) Ampere's circuit law.

Like Coulomb's law, Biot–Savart's law is the general law of magnetostatics.
Also, as Gauss's law is a special case of Coulomb's law, Ampere's law is a special case of Biot–Savart's law and is easily applied in problems involving symmetrical current distribution.

The magnetic field can be represented by drawings with magnetic field lines.



The relation between the magnetic flux density B and the magnetic field intensity H is (in free space):

$$\mathbf{B} = \mu_o \mathbf{H}$$

The unit of B is weber/m^2 (Wb/m^2) or Tesla (T)

The unit of H is A/m

where μ_o is a constant called the permeability of free space

$$\mu_o = 4\pi \times 10^{-7} \quad T.m/A$$

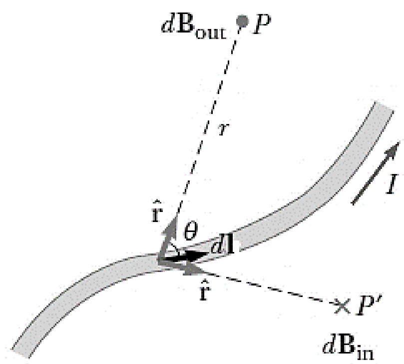
5.2 Biot-Savart Law

Biot–Savart's law states that the differential magnetic field intensity d**H** produced at a point P, as shown in the figure below, by the differential current element I dl is proportional to the product I dl and the sine of the angle θ between the element dl and the line joining P to the element and is inversely proportional to the square of the distance r between P and the element.

$$dH = \frac{1}{4\pi}\frac{Idl\,\sin\theta}{r^2}$$

For the magnetic flux density B:

$$dB = \frac{\mu_o}{4\pi}\frac{Idl\,\sin\theta}{r^2}$$



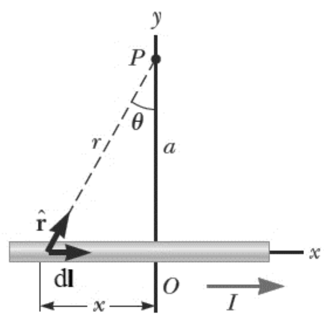
It can be written as:

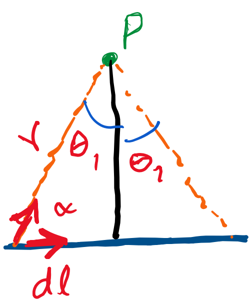
$$d\boldsymbol{B} = \frac{\mu_o}{4\pi}\frac{Id\boldsymbol{l}\times\boldsymbol{r}}{r^3}$$

$$\boldsymbol{B} = \frac{\mu_o I}{4\pi}\int\frac{d\boldsymbol{l}\times\boldsymbol{r}}{r^3}$$

Example 5.1

Consider a thin, straight wire of finite length carrying a constant current I and placed along the x axis as shown below. Determine the magnitude and direction of the magnetic flux density B at point P due to this steady current.





$$B = \frac{\mu_0 I}{4\pi} \int \frac{dl \sin\alpha}{r^2}$$

$$r = \frac{a}{\cos\theta}$$

$$dl = dx \qquad x = a\tan\theta \Rightarrow dx = a\sec^2\theta\, d\theta$$

$$dx = \frac{a}{\cos^2\theta} d\theta$$

$$\sin\alpha = \cos\theta$$

$$\Rightarrow B = \frac{\mu_0 I}{4\pi} \int_{\theta_1}^{\theta_2} \frac{\frac{\cancel{a}}{\cancel{\cos^2\theta}} d\theta}{\frac{a^{\cancel{2}}}{\cancel{\cos^2\theta}}} \cos\theta$$

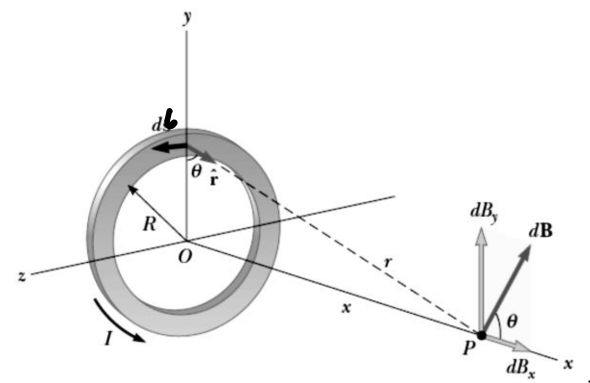
$$= \frac{\mu_0 I}{4\pi a} \int_{\theta_1}^{\theta_2} \cos\theta\, d\theta \Rightarrow \boxed{B = \frac{\mu_0 I}{4\pi a}\left[\sin\theta_2 - \sin\theta_1\right]}$$

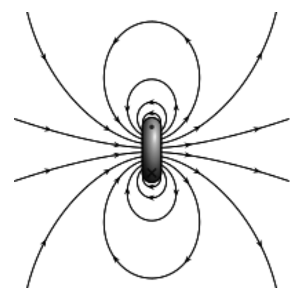
out of the page

$$\text{if } l \to \infty \Rightarrow \theta_2 = \frac{\pi}{2} \;\&\; \theta_1 = -\frac{\pi}{2} \Rightarrow \boxed{B = \frac{\mu_0 I}{2\pi a}}$$

Example 5.2

Consider a circular wire loop of radius R located in the yz plane and carrying a steady current I, as in the figure below. Calculate the magnetic flux density B at an axial point P a distance x from the center of the loop.
Then calculate B and H at the center of the coil if its average radius is 20 cm and it has 200 turns and a current of 3.5 A.





$$dB = \frac{\mu_0 I}{4\pi} \frac{dl}{r^2} \sin\alpha$$

$$\alpha = 90 \Rightarrow \sin\alpha = 1$$

$$r^2 = x^2 + R^2$$

due to symmetry the resultant of $B_y = 0$

$$\Rightarrow B = \oint B_x = \oint dB \cos\theta = \oint \frac{\mu_0 I}{4\pi} \frac{dl}{x^2+R^2} \cos\theta$$

$$\cos\theta = \frac{R}{r} = \frac{R}{(x^2+R^2)^{\frac{1}{2}}}$$

$$\Rightarrow B = \frac{\mu_0 I}{4\pi} \frac{R}{(x^2+R^2)^{\frac{3}{2}}} \oint dl$$

$$\oint dl = 2\pi R \quad \Rightarrow \quad \boxed{\vec{B} = \frac{\mu_0 I R^2}{2(x^2+R^2)^{\frac{3}{2}}} \hat{x}}$$

if the coil consists of N turns

$$\vec{B} = \frac{N\mu_0 I R^2}{2(x^2+R^2)^{3/2}} \hat{x}$$

at $x=0 \Rightarrow B = \frac{N\mu_0 I}{2R}$

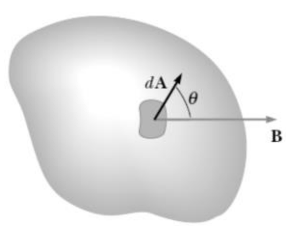
$$B(x=0) = \frac{4\pi \times 10^{-7} \times 200 \times 3.5}{2 \times 20 \times 10^{-2}} = 2.2 \times 10^{-3}\ T$$

$$H(x=0) = \frac{B}{\mu_0} = 1.75 \times 10^{3}\ A/m$$

5.3 Magnetic flux

The magnetic flux Φ_B through a surface A can be given by:

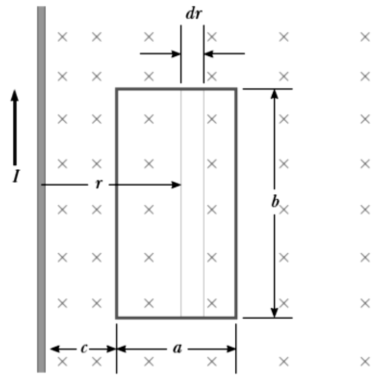
$$\Phi_B = \int \boldsymbol{B} \cdot d\boldsymbol{A}$$



where the magnetic flux Φ_B is in webers (Wb)

Example 5.3

A rectangular loop of width a and length b is located near a long wire carrying a current I. The distance between the wire and the closest side of the loop is c. The wire is parallel to the long side of the loop. Find the total magnetic flux through the loop due to the current in the wire.



$$\Phi_B = \int \vec{B} \cdot d\vec{a} = \int B\, da = \int \frac{\mu_0 I}{2\pi r}\, da$$

$$da = b\, dr$$

$$\Rightarrow \Phi_B = \int_c^{a+c} \frac{\mu_0 I}{2\pi r}\, b\, dr = \frac{\mu_0 I b}{2\pi} \int_c^{a+c} \frac{dr}{r}$$

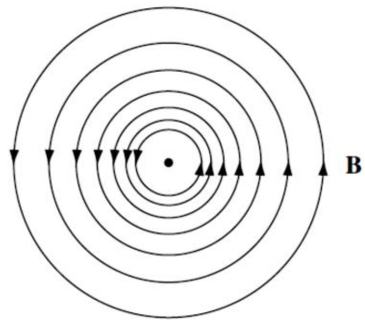
$$\Rightarrow \Phi_B = \frac{\mu_0 I b}{2\pi} \Big[\ln r\Big]_c^{a+c}$$

$$\Rightarrow \boxed{\Phi_B = \frac{\mu_0 I b}{2\pi} \ln\left(\frac{a+c}{c}\right)}$$

5.4 The curl of B

The magnetic flux density of an infinite straight wire discussed in Example 5.1 is shown below where the current is coming out of the page.

It is clear that this field has a nonzero curl (unlike the electrostatic field)



The integral of B around a circular path of radius S, centered at the wire, is:

$$\oint \mathbf{B} \cdot d\mathbf{l} = \frac{\mu_0 I}{2\pi} \oint \frac{1}{s} s \, d\phi = \frac{\mu_0 I}{2\pi} \int_0^{2\pi} d\phi = \mu_0 I.$$

- It is clear that the answer of the integral is independent of S, because B decreases at the same rate as the circumference increases.
- The close path (Amperian loop) does not have to be a circle, any loop that encloses the wire would give the same answer

If there is a group of straight wires passing through the loop:

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}}$$

<u>This is known as Ampere's law in integral form</u>.

It states that the line integral of B around a closed path is the same as $\mu_o I_{enc}$ where I_{enc} is the net current passing through any surface bounded by the closed path.

If the flow of charge is represented by a current density J, the enclosed current is:

$$I_{\text{enc}} = \int \mathbf{J} \cdot d\mathbf{a}$$

Applying Stokes' Theorem

$$\int (\nabla \times \mathbf{B}) \cdot d\mathbf{a} = \mu_0 \int \mathbf{J} \cdot d\mathbf{a},$$

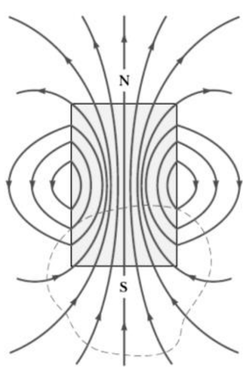
$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}.$$

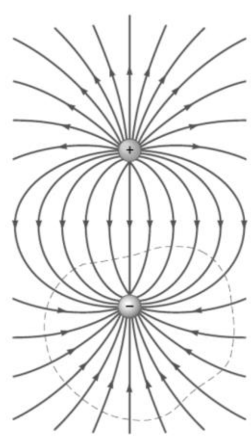
This equation is called Ampere's law (in differential form).

Again, unlike electrostatic fields, B has a nonzero curl.

5.5 The divergence of B

In chapter 2 we found from Gauss's law that the electric flux through a closed surface surrounding a net charge is proportional to that charge. However, the magnetic fields have no sources or sinks and their lines are always continuous. Therefore, for any closed surface, such as the one outlined in the figure below (left), the number of lines entering the surface equals the number leaving the surface; thus, the net magnetic flux is zero. In contrast, for a closed surface surrounding one charge of an electric dipole (the figure on the right), the net electric flux is not zero.





Thus, the total flux of B through a closed surface in a magnetic field must be zero

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$$

This means an isolated magnetic poles (monopoles) does not exist

By applying the divergence theorem:

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = \int_v \nabla \cdot \mathbf{B}\, dv = 0$$

$$\boxed{\nabla \cdot \mathbf{B} = 0}$$

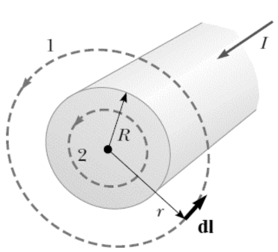
This equation is one of Maxwell’s equations

5.6 Applications of ampere’s law

Example 5.4

A long, straight wire of radius R carries a steady current I that is uniformly distributed through the cross section of the wire. Calculate the magnetic flux density B a distance r from the center of the wire in the regions:

1) $r \geq R$
2) $r < R$



1) $\oint B \cdot dl = \mu_0 I$

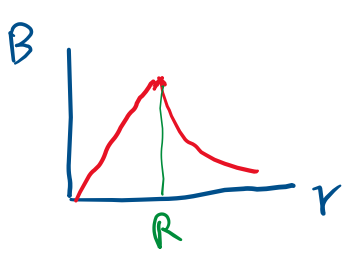
$B \oint dl = \mu_0 I \Rightarrow B(2\pi r) = \mu_0 I$

$\Rightarrow \boxed{B = \dfrac{\mu_0 I}{2\pi r}}$ $r \geq R$

2) $\dfrac{I'}{I} = \dfrac{\pi r^2}{\pi R^2} \Rightarrow I' = \dfrac{r^2}{R^2} I$

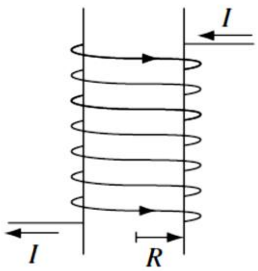
$\oint B \cdot dl = \mu_0 \dfrac{r^2}{R^2} I \Rightarrow B(2\pi r) = \mu_0 \dfrac{r^2}{R^2} I$

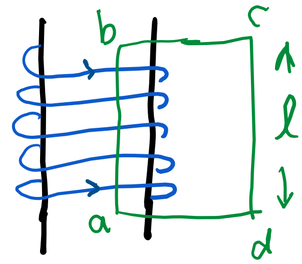
$\Rightarrow \boxed{B = \dfrac{\mu_0 I}{2\pi R^2} r}$ $r < R$



Example 5.4

Find the magnetic flux density B of a very long solenoid, consisting of n closely wound turns per unit length on a cylinder of radius R and each turn carries a steady current *I*.





(note that as the length of the solenoid increases, the interior field becomes more uniform and the exterior field becomes weaker)

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc} \Rightarrow \int_a^b B\cdot dl + \cancel{\int_b^c B\cdot dl} + \int_c^d B\cdot dl + \cancel{\int_d^a B\cdot dl} = \mu_0 N I$$

$\theta = 90^\circ$ $\quad \approx 0$ $\quad \theta = 90^\circ$

B very weak

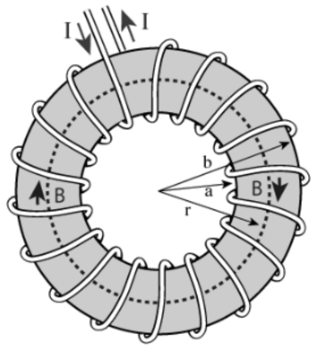
$$\Rightarrow B\int_a^b dl = \mu_0 N I \Rightarrow B l = \mu_0 N I$$

$$\boxed{B = \frac{\mu_0 N I}{l} = \mu_0 n I}$$

inside the solenoid

Example 5.5

A toroid whose dimensions are shown in the figure below has N turns and carries current I. Determine B inside and outside the toroid.



$$\oint \vec{B} \cdot d\vec{l} = \mu_o I_{enc} \Rightarrow \oint B\, dl = \mu_o N I \Rightarrow B \oint dl = \mu_o N I$$

$$B(2\pi r) = \mu_o N I \Rightarrow \boxed{B = \frac{\mu_o N I}{2\pi r}} \quad a < r < b$$

5.7 Magnetic Forces

5.7.1 Force on a Charged Particle

A magnetic field can exert force only on a moving charge. From experiments, it is found that the magnetic force on a charge Q, moving with velocity v in a magnetic field with flux density B, is

$$\mathbf{F}_{\text{mag}} = Q(\mathbf{v} \times \mathbf{B})$$

or

$$F_{\text{mag}} = QvB \sin\theta$$

This clearly shows that F_{mag} is perpendicular to both v and B.

In the presence of both electric and magnetic fields, the net force on Q would be

$$\mathbf{F} = Q[\mathbf{E} + (\mathbf{v} \times \mathbf{B})]$$

This is known as the Lorentz force law

Example 5.6

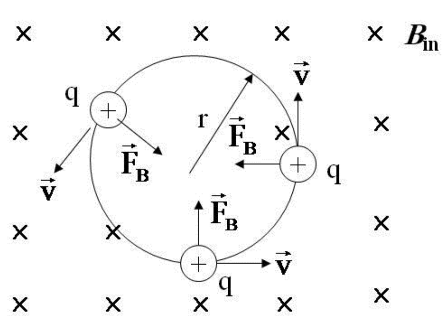
A charged particle of mass 2 kg and 1C charge, starts at the origin with velocity 3 ŷ m/s and travels in a region of uniform magnetic field B = 10 ẑ T. At t = 4 s, find the magnitude of the magnetic force acting on the particle and its acceleration.

$$\vec{F}_{mag} = Q\vec{v}\times\vec{B} \Rightarrow |\vec{F}_{mag}| = QvB\sin(90) = (1)(3)(10) = 30\ N$$

$$m\vec{a} = Q\vec{v}\times\vec{B} \Rightarrow |\vec{a}| = \frac{|\vec{F}_{mag}|}{m} = \frac{30}{2} = 15\ m/s^2$$

5.7.1.1 Motion of charges in a uniform magnetic field

When a charged particle moves with a velocity being perpendicular to a uniform magnetic field, the particle moves in a circular path in a plane perpendicular to B. as shown in the figure below for a positively charges particle.



From Newton's second law with centripetal acceleration

$$F_B = qvB = \frac{mv^2}{r}$$

The angular speed of the particle:

$$\omega = \frac{v}{r} = \frac{qB}{m}$$

The period of the motion:

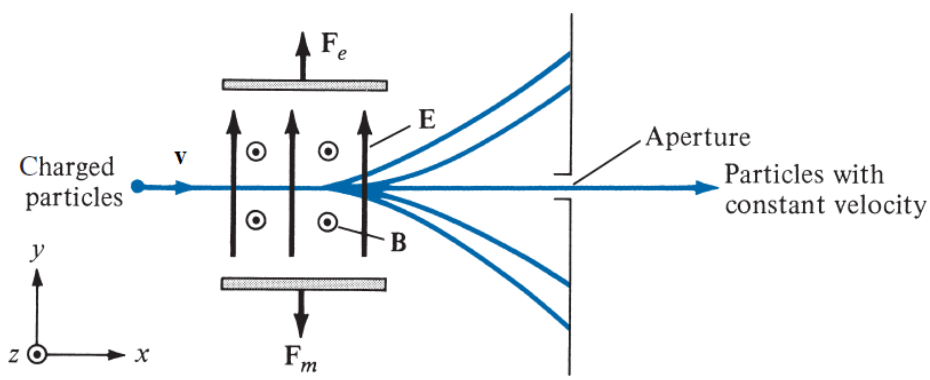
$$T = \frac{2\pi r}{v} = \frac{2\pi}{\omega} = \frac{2\pi m}{qB}$$

What if a charged particle moves in a uniform magnetic field, but the angle between its velocity and B is not 90^{o} ?

Its path will be a helix

5.7.1.2 Some related applications

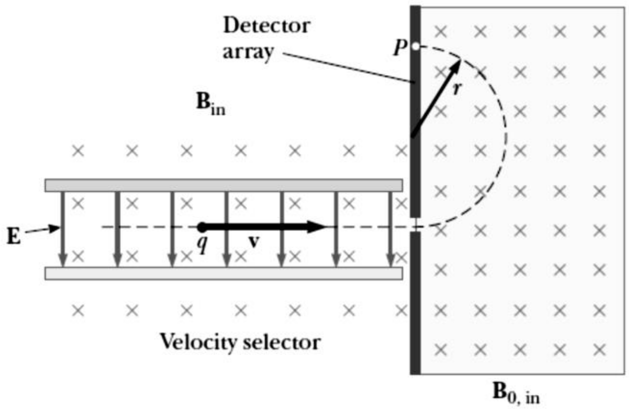
- **Velocity filter (selector) for charged particles**



$$v = \frac{E}{B}$$

Can you prove this?

- ## The Mass Spectrometer

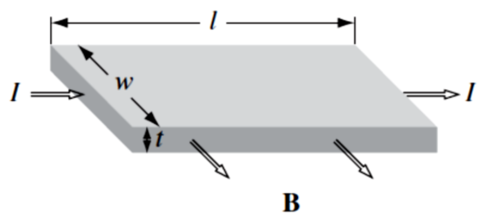


$$\frac{m}{q} = \frac{rB_0}{v}$$

Can you prove this?

- ## The Hall Effect

When a conducting sample carrying current *I* is placed in a magnetic field with magnetic flux density B, a potential difference (electric field) is generated in a direction perpendicular to both *I* and B.



From the directions of v and B shown in the figure above the direction of the magnetic force will be downward and this will produce a buildup of positive charge on the lower side of the sample and leaves an excess of negative charge on upper side, thus the direction of the generated electric field (E_H) is opposite the direction of the magnetic force.

In equilibrium,

$$qv_dB = qE_{\mathrm{H}}$$

$$E_{\mathrm{H}} = v_dB$$

If t is the thickness of the conductor, the Hall voltage is

$$\Delta V_H = E_H t = v_d B t$$

We can obtain the charge-carrier density n by measuring the current in the sample

$$\boldsymbol{v_d} = \frac{\boldsymbol{I}}{\boldsymbol{nqA}}$$

$$\Delta \boldsymbol{V_H} = \frac{\boldsymbol{IBt}}{\boldsymbol{nqA}}$$

$$\Delta \boldsymbol{V_H} = \frac{\boldsymbol{IB}}{\boldsymbol{nqw}}$$

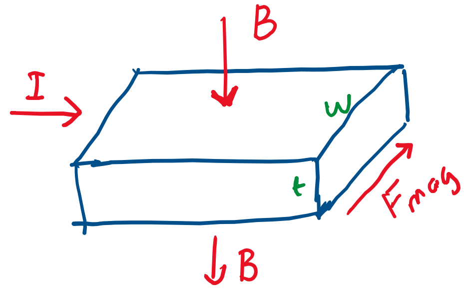
where w is the width of the conductor

and $\mathbf{R_H = 1/nq}$ is called the Hall coefficient

Also this relationship shows that a conductor with a known Hall coefficient and properly calibrated can be used to measure the magnitude of an unknown magnetic field.

Example 5.7

A rectangular copper strip 1.5 cm wide and 0.1 cm thick carries a current of 5 A. Find the Hall voltage for a 1.2 T magnetic field applied in a direction perpendicular to the strip. The density of copper is 8.92 g/cm^3



As discussed in example 4.1 (chapter 4)

$$n = \frac{N_a}{V} \quad \& \quad V = \frac{M}{\rho} \quad \Rightarrow n = \frac{N_a \rho}{M}$$

N_a is Avogadro's number $= 6.02 \times 10^{23}\ mol^{-1}$

M is molar mass of copper $= 63.5\ g/mol$

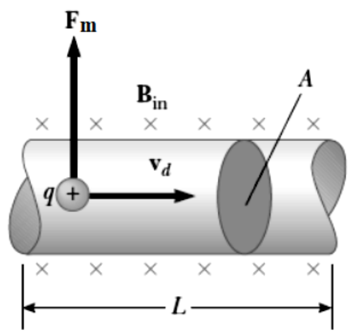
$$n = \frac{6.02 \times 10^{23} \times 8.92 \times 10^{6}}{63.5} = 8.46 \times 10^{28}\ m^{-3}$$

$$V_H = \frac{IB}{nqt} = \frac{5 \times 1.2}{8.46 \times 10^{28} \times 1.6 \times 10^{-19} \times 0.1 \times 10^{-2}} = 0.44\ \mu V$$

why we used (t) not (w) here ??

5.7.2 Force on a Current Element

The following figure shows a piece of a wire carrying current *I* in a magnetic field.



We found previously for a charge Q moving with constant velocity v:

$$\mathbf{F}_{\text{mag}} = Q(\mathbf{v} \times \mathbf{B})$$

The moving charge dQ in the wire is:

$$dQ = I\,dt$$

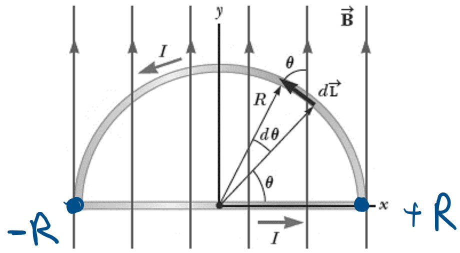
thus,

$$dF_{mag} = Idt(v \times B)$$

$$\boldsymbol{F_{mag}} = \int \boldsymbol{IdL} \times \boldsymbol{B}$$

Example 5.8

A wire bent into a semicircle of radius R and forms a closed circuit. The wire carries a current *I* and lies in the xy plane, and a uniform magnetic field is directed along the positive y axis as in the figure below. Find the magnitude and direction of the magnetic force acting on the straight portion of the wire and on the curved portion.



Solution

The magnetic force on the straight portion of the loop is directed out of the page. Whereas, the magnetic force on the curved portion is directed into the page.

<u>For the straight portion</u>

$$\vec{F}_1 = \int I\, d\vec{L} \times \vec{B} = I \int_{-R}^{R} B\, dx\, \hat{z} = I B [x]_{-R}^{R} \hat{z} = 2IBR\hat{z}$$

$(\hat{x} \times \hat{y} = \hat{z})$ & $dL = dx$

<u>For the curved portion</u>

$$\vec{F}_2 = \int I\, d\vec{L} \times \vec{B} = I \int_0^{\pi} -BR\sin\theta\, d\theta \;\hat{z}$$

$(-\hat{x} \times \hat{y} = -\hat{z})$ & $dL = ds = R\,d\theta$

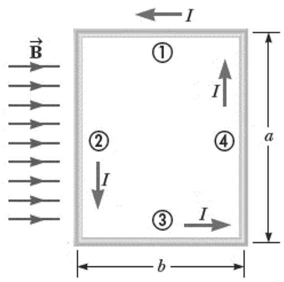
$$\Rightarrow \vec{F}_2 = -I BR [-\cos\theta]_0^{\pi} \hat{z} = IBR[-1-1]\hat{z} = -2IBR\hat{z}$$

$$\vec{F}_1 + \vec{F}_2 = 0$$

The net magnetic force acting on any closed current loop in a **uniform** magnetic field is zero.

5.8 Magnetic Torque and Moment

Consider a rectangular loop carrying a current I in the presence of a uniform magnetic field directed parallel to the plane of the loop as shown in Figure

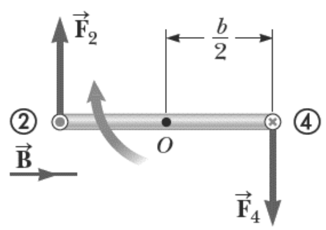


No magnetic forces act on sides 1 and 3 because these wires are parallel to the field. Magnetic forces do, however, act on sides 2 and 4.
The **magnitude** of these forces is:

$F_2 = F_4 = IaB$

and they point in opposite directions (The direction of F_2 is out of the page whereas that of F_4 is into the page)

If the loop is allowed to rotate about point O, these two forces produce a torque about O that rotates the loop clockwise.



The magnitude of this torque t_{max} is:

$$\tau_{\max} = F_2\frac{b}{2} + F_4\frac{b}{2} = (IaB)\frac{b}{2} + (IaB)\frac{b}{2} = IabB$$

$$\tau_{\max} = IAB$$

In general:

$$\boldsymbol{\tau} = I\mathbf{A}\times\mathbf{B}$$

The product of current and area IA is defined to be the **magnetic dipole moment** "**m**" (simply called the "magnetic moment") of the loop and its direction is normal to the loop.

$$\mathbf{m} = \mathrm{I}\mathbf{A}$$

so

$$\boldsymbol{\tau} = \mathbf{m}\times\mathbf{B}$$

Example 5.9

A rectangular coil of dimensions 5.4 cm × 8.5 cm consists of 25 turns of wire and carries a current of 15 mA. A magnetic field with B = 0.35T is applied parallel to the plane of the coil.

a) Calculate the magnitude of the magnetic dipole moment of the coil.
b) What is the magnitude of the torque acting on the loop?

a) $m = NIA = (25)(5.4\times10^{-2}\times8.5\times10^{-2})(15\times10^{-3}) = 1.72\times10^{-3}\ A.m^2$

b) $\vec{\tau} = \vec{m}\times\vec{B} \Rightarrow \tau_{max} = mB = (1.72\times10^{-3})(0.35) = 6.02\ N.m$

5.9 Magnetization in materials

A very small magnet of microscopic (subatomic) dimensions is equivalent to a flow of a small electric charge around a loop (such as electrons circulating around nuclei) that create magnetic dipoles.

Our discussion here will be similar to that on polarization of materials in an electric field. We will assume that the atomic model is an electron orbiting about a positive nucleus or about their own axes (spin). Thus, an internal magnetic field is produced by these electrons similar to the magnetic field produced by a current loop.

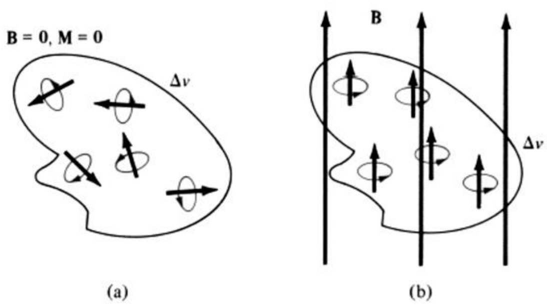
The equivalent current loop has a magnetic dipole moment of $m = I_b A$, where A is the area of the loop and I_b is the bound current that is bound to the atom.

The **magnetization M** is the magnetic dipole moment per unit volume.

$$\mathbf{M} = \lim_{\Delta\nu \to 0} \frac{\sum_{k=1}^{N} \mathbf{m}_k}{\Delta\nu}$$

It is in amperes per meter and it plays a role analogous to the polarization **P** in electrostatics.

The following figures show the magnetic dipole moment in a volume Δv before B is applied (a) and after B is applied (b).



In linear materials, magnetization M depends linearly on H:

$$\mathbf{M} = \chi_m \mathbf{H}$$

where χ_m (ratio of M to H) called magnetic susceptibility of the medium. It is a dimensionless quantity and it shows how susceptible (or sensitive) the material is to a magnetic field.

The relation between B and H in general is:

$$\mathbf{B} = \mu_o \mu_r \mathbf{H}$$

$$\mathbf{B} = \mu_o (1 + \chi_m) \mathbf{H} = \mu \mathbf{H}$$

where,

$$\mu_r = 1 + \chi_m = \frac{\mu}{\mu_o}$$

The quantity $\mu = \mu_o\mu_r$ is called the permeability of the material and μ_r is the ratio of the permeability of a given material to that of free space and is known as the relative permeability of the material and it is dimensionless.

Materials can be grouped into three major classes:
Diamagnetic materials (which have very small negative χ_m)
Paramagnetic materials (which have very small positive χ_m)
Ferromagnetic materials (which have very large positive χ_m)

Ferromagnetism occurs in materials whose atoms have relatively large permanent magnetic moment. Iron, cobalt and nickel are examples of ferromagnetic materials.