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King Saud University

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1444/Semester-3/ MATH 380/ Quiz-2

Marks: 10 **Max. Time: 35 Minutes**

Answer the following questions.

Q1: [3]

An observation is made of a Poisson random variable N with parameter λ. Then N independent Bernoulli trials are performed, each with probability p of success. Let Z be the total number of successes observed in the N trials. Formulate Z as a random sum and determine its mean and variance. What is the distribution of Z?

Q2: [1+3]

(a) Define a martingale.

(b) Let $U_1, U_2, \ldots$ be independent identically random variables each uniformly distributed over the interval (0,1]. Show that $X_0 = 1$ and $X_n = 2^n U_1 \ldots U_n$ for $n = 1, 2, \ldots$ defines a martingale.

Q3: [3]

Consider a sequence of items from a production process, with each item being graded as good or defective. Suppose that a defective item is followed by another defective item with probability β and is followed by a good item with probability $1-\beta$. If the first item is defective, what is the probability that the first good item to appear is the fifth item ?

The Model Answer

Q1: [3]

Let $Z=\xi_1+\xi_2+...+\xi_N,\ N>0$ Then

$E(\xi_k)=\mu=p,\ \text{Var}(\xi_k)=\sigma^2=p(1-p)$

$E(N)=v=\lambda,\ \ \text{Var}(N)=\tau^2=\lambda$

$\because E(Z)=\mu v$

$\therefore E(Z)=\lambda p$

$\because\ \text{Var(Z)}=v\sigma^2+\mu^2\tau^2$

$\therefore\ \text{Var(Z)}=\lambda p(1-p)+p^2\lambda$

$\qquad\qquad=\lambda p$

Consequently, $Z\sim$ Poisson(λp).

Q2: [1+3]

(a)

A stochastic process $\{X_n;\ n=0,1,2,...\}$ is a martingale if for $n=0,1,2,...$

(i) $E\left[|X_n|\right]<\infty,$

(ii) $E\left[X_{n+1}|X_0,...,X_n\right]=X_n.$

(b)

(1) To show that $E\left[|\text{X}_n|\right]<\infty,$

$\because E\left[|X_n|\right]=E\left[X_n\right]$

$\qquad=E\left[2^n U_1\ ...\ U_n\right]$

$\qquad=2^n E\left[U_1\right]\left[U_2\right]\ ...\ \left[U_n\right]$ as $U_{i's}$ are indep. $r.v_s$

$\therefore E\left[|X_n|\right]=2^n.\frac{1}{2}.\frac{1}{2}\ ...\ \frac{1}{2}=\frac{2^n}{2^n}=1<\infty$

(2) To show that $E\left[X_{n+1}|X_0,...,X_n\right]=X_n,$

$\because E\left[X_{n+1}|X_0,...,X_n\right]=E\left[2^{n+1}U_1\ ...\ U_n U_{n+1}|X_0,...,X_n\right]$

$\qquad=2^n U_1\ ...\ U_n E\left[2U_{n+1}|X_0,...,X_n\right]$, as $U_1\ ...\ U_n$ is determined by $X_{i's}$

$\qquad=2^n U_1\ ...\ U_n.2E\left[U_{n+1}\right]$, as U_{n+1} is indep. of $X_{i's}$

$\qquad=\ 2^n U_1\ ...\ U_n.2.\frac{1}{2}$ where $E\left[U_i\right]=\frac{1}{2},\ i=1,2,...$

$\therefore E\left[X_{n+1}|X_0,...,X_n\right]=X_n$

That is from (1) and (2), we have proved that $X_n,\ n=0,1,2,...$ where $X_0=1$ is a martingale.

Q3: [3]

$$\Pr\left\{X_2 = D,\ X_3 = D, X_4 = D, X_5 = G \middle| X_1 = D\right\}$$
$$= \Pr\left\{X_5 = G,\ X_4 = D, X_3 = D, X_2 = D \middle| X_1 = D\right\}$$
$$= \Pr\left\{X_5 = G \middle| X_4 = D\right\} \cdot \Pr\left\{X_4 = D \middle| X_3 = D\right\} \cdot \Pr\left\{X_3 = D \middle| X_2 = D\right\} \cdot \Pr\left\{X_2 = D \middle| X_1 = D\right\}$$
$$= p_{DG} p_{DD}^3$$
$$= (1-\beta)\beta^3$$
$$= \beta^3(1-\beta)$$

Also, you can solve it as follows.

$$p_1 p_{12} p_{23} p_{34} p_{45},\ p_1 = \Pr(X_1 = D) = 1$$
$$= p_D p_{DD}^3 p_{DG},\ p_D = 1$$
$$= \beta^3(1-\beta)$$
