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King Saud University

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1444/Semester-2/ MATH 380/ Quiz-2

Marks: 10 **Max. Time: 35 Minutes**

Answer the following questions.

Q1: [2+1]

An observation is made of a Poisson random variable N with parameter λ. Then N independent Bernoulli trials are performed, each with probability p of success. Let Z be the total number of successes observed in the N trials. Formulate Z as a random sum and determine its mean and variance. What is the distribution of Z?

Q2: [1+3]

(a) Define a martingale.

(b) Suppose $X_1, X_2, X_3, \ldots$ are identically independent distributed random variables where $\Pr\{X_k = 1\} = \Pr\{X_k = -1\} = \frac{1}{2}$ and $S_n = \sum_{k=1}^{n} X_k$. Show that S_n is a martingale.

Q3: [3]

Consider a sequence of items from a production process, with each item being graded as good or defective. Suppose that a defective item is followed by another defective item with probability β and is followed by a good item with probability $1-\beta$. If the first item is defective, what is the probability that the first good item to appear is the fifth item ?

The Model Answer

Q1: [2+1]

Let $Z = \xi_1 + \xi_2 + ... + \xi_N$, $N > 0$ Then

$E(\xi_k) = \mu = p$, $\text{Var}(\xi_k) = \sigma^2 = p(1-p)$

$E(N) = v = \lambda$, $\text{Var}(N) = \tau^2 = \lambda$

$\because E(Z) = \mu v$

$\therefore E(Z) = \lambda p$

$\because \text{Var(Z)} = v\sigma^2 + \mu^2\tau^2$

$\therefore \text{Var(Z)} = \lambda p(1-p) + p^2\lambda$

$= \lambda p$

Consequently, $Z \sim \text{Poisson}(\lambda p)$.

Q2: [1+3]

(a)

A stochastic process $\{X_n; n = 0,1,2,...\}$ is a martingale if

(i) $E\left[|X_n|\right] < \infty$,

(ii) $E\left[X_{n+1}|X_0,...,X_n\right] = X_n$.

(b)

(1) To show that $E\left[|S_n|\right] < \infty$,

$|S_n| = |X_1 + ... + X_n| \leq |X_1| + ... + |X_n|$

$\leq 1 + ... + 1 = n$

$E\left[|S_n|\right] \leq E[n] = n < \infty$.

(2) To show that $E\left[S_{n+1}|X_1,...,X_n\right] = S_n$,

$E\left[S_{n+1}|X_1,...,X_n\right] = E\left[S_n + X_{n+1}|X_1,...,X_n\right]$

$= E\left[S_n|X_1,...,X_n\right] + E\left[X_{n+1}|X_1,...,X_n\right]$

$= S_n + E[X_{n+1}]$,

where S_n is determined by $X_1,...,X_n$ and X_{n+1} is independent of $X_{i's}$,

and $\because E[X_{n+1}] = (1).\Pr\{X_{n+1} = 1\} + (-1).\Pr\{X_{n+1} = -1\}$

$= (1)(1/2) + (-1)(1/2) = 0$

$\therefore E\left[S_{n+1}|X_1,...,X_n\right] = S_n$

That is from (1) and (2), we have proved that S_n is a martingale.

Q3: [3]

$$\Pr\left\{X_2 = D,\ X_3 = D, X_4 = D, X_5 = G \middle| X_1 = D\right\}$$

$$= \Pr\left\{X_5 = G,\ X_4 = D, X_3 = D, X_2 = D \middle| X_1 = D\right\}$$

$$= \Pr\left\{X_5 = G \middle| X_4 = D\right\} \cdot \Pr\left\{X_4 = D \middle| X_3 = D\right\} \cdot \Pr\left\{X_3 = D \middle| X_2 = D\right\} \cdot \Pr\left\{X_2 = D \middle| X_1 = D\right\}$$

$$= p_{DG} p_{DD}^3$$

$$= (1-\beta)\beta^3$$

$$= \beta^3(1-\beta)$$

Also, you can solve it as follows.

$$p_1 p_{12} p_{23} p_{34} p_{45},\ p_1 = \Pr(X_1 = D) = 1$$

$$= p_D p_{DD}^3 p_{DG},\ p_D = 1$$

$$= \beta^3(1-\beta)$$
