Name: **ID:** **Section:** **Mark:**

King Saud University

College of Sciences, Department of Mathematics

1444/Semester-3/ MATH 380/ Quiz-1

Marks: 10 **Max. Time: 35 Minutes**

Answer the following questions.

Q1: [2.5+2.5]

(a) Prove that if $T \sim \exp(\lambda)$ then $\Pr(T > t + s | T > s) = \Pr(T > t)\ \forall t, s \geq 0$

(b) The lifetime T of a certain component has an exponential distribution with parameter $\lambda = 0.02$. Find $\Pr(T \leq 120 | T > 100)$.

Q2: [2.5+2.5]

(a) A fraction $p = 0.05$ of the items coming off a production process are defective. The output of the process is sampled, one by one, in a random manner.

(i) What is the probability that the first defective item found is the tenth item sampled?

(ii) Determine the mean and variance.

(b) Let X and Y two random variables have the joint normal distribution. What value of α that minimizes the variance of $Z = \alpha X + (1 - \alpha) Y$? Simplify your result when X and Y are independent.

The Model Answer

Q1: [2.5+2.5]

(a)

$$\Pr(T > t+s \mid T > s) = \frac{\Pr(T > t+s, T > s)}{\Pr(T > s)}$$
$$= \frac{\Pr(T > t+s)}{\Pr(T > s)}$$

$\because T \sim \exp(\lambda)$

$$\therefore \Pr(T > t+s \mid T > s) = \frac{e^{-\lambda(t+s)}}{e^{-\lambda s}} = e^{-\lambda t}$$
$$= R(t) = \Pr(T > t)$$

(b)

$$\Pr(T \le 120 \mid T > 100) = 1 - pr(T > 120 \mid T > 100)$$
$$= 1 - pr(T > 20)$$
$$= 1 - \mathrm{e}^{-0.02(20)}$$
$$\therefore \Pr(T \le 120 \mid T > 100) = 1 - \mathrm{e}^{-0.4} \simeq 0.33$$

Q2: [2.5+2.5]

(a)

Let X counts the number of failures prior to the first success

$\mathrm{X} \sim \text{Geom}\ (p),\ p = 0.05$

$$\text{pr(X=10)} = p(1-p)^{k-1}, k = 1, 2, \ldots$$
$$= 0.05(1-0.05)^9$$
$$= 0.0315$$

The mean is $\mathrm{E(X)} = 1/p = 20$ and variance is $\mathrm{Var(X)} = (1-p)/p^2 = 380$

(b)

$$\mathrm{Z} = \alpha \mathrm{X} + (1-\alpha)\mathrm{Y}$$
$$\mathrm{Var(Z)} = \alpha^2 \sigma_X^2 + 2\alpha(1-\alpha)\rho\sigma_X\sigma_Y + (1-\alpha)^2\sigma_Y^2$$

To get α^* that minimizes Var(Z) let $\dfrac{\partial V}{\partial \alpha} = 0$

$$\therefore\ 2\alpha\sigma_X^2 + (2-4\alpha)\rho\sigma_X\sigma_Y + (-2+2\alpha)\sigma_Y^2 = 0$$

$\Rightarrow$

$$\alpha^* = \frac{\sigma_Y^2 - \rho\sigma_X\sigma_Y}{\sigma_X^2 - 2\rho\sigma_X\sigma_Y + \sigma_Y^2}, \quad -1 < \rho < 1$$

If X and Y are independent random variables, then $\rho = 0$

Consequently $\alpha^* = \dfrac{\sigma_Y^2}{\sigma_X^2 + \sigma_Y^2}$.