Name: **ID:**

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1444/Semester-1/ MATH 380/ Quiz-1

Marks: 10 **Max. Time: 35 Minutes**

Answer the following questions.

Q1: [2+2]

(a) The lifetime T of a certain component has an exponential distribution with parameter $\lambda = 0.02$. Find $\Pr(T \le 120 \mid T > 100)$

(b) Suppose that X is a Poisson distributed random variable with mean $\lambda = 2$. Determine $\Pr\{X \le \lambda\}$.

Q2: [2+1]

An observation is made of a Poisson random variable N with parameter λ. Then N independent Bernoulli trials are performed, each with probability p of success. Let Z be the total number of successes observed in the N trials. Formulate Z as a random sum and determine its mean and variance. What is the distribution of Z?

Q3: [1+2]

(a) Define a martingale.

(b) Suppose $X_1, X_2, X_3, \ldots$ are identically independent distributed random variables where $\Pr\{X_k = 1\} = \Pr\{X_k = -1\} = \frac{1}{2}$ and $S_n = \sum_{k=1}^{n} X_k$. Show that S_n is a martingale.

Answer:

The Model Answer

Q1: [2+2]

(a)

$$\Pr(T \le 120 | T > 100) = 1 - pr(T > 120 | T > 100)$$
$$= 1 - pr(T > 20)$$
$$= 1 - e^{-0.02(20)}$$
$$\therefore \Pr(T \le 120 | T > 100) = 1 - e^{-0.4} \simeq 0.33$$

(b)

$$\because \Pr\{X \le 2\} = \sum_{x=0}^{2} \frac{e^{-2} 2^x}{x!}$$
$$= e^{-2}\left[1 + \frac{2}{1!} + \frac{2^2}{2!}\right]$$
$$\therefore \Pr\{X \le 2\} = 5e^{-2} \simeq 0.6767$$

Q2: [2+1]

Let $Z = \xi_1 + \xi_2 + ... + \xi_N,\ N > 0$ Then

$$E(\xi_k) = \mu = p,\ \text{Var}(\xi_k) = \sigma^2 = p(1-p)$$
$$E(N) = v = \lambda,\ \ \text{Var}(N) = \tau^2 = \lambda$$
$$\because E(Z) = \mu v$$
$$\therefore E(Z) = \lambda p$$
$$\because\ \text{Var(Z)} = v\sigma^2 + \mu^2 \tau^2$$
$$\therefore\ \text{Var(Z)} = \lambda p(1-p) + p^2 \lambda$$
$$= \lambda p$$

Consequently, $Z \sim$ Poisson(λp).

Q3: [1+2]

(a)

A stochastic process $\{X_n; n = 0, 1, 2, ...\}$ is a martingale if

(i) $E\left[|X_n|\right] < \infty$,

(ii) $E\left[X_{n+1} | X_0, ..., X_n\right] = X_n$.

(b)

(1) To show that $E\left[|S_n|\right] < \infty$,

$$|S_n| = |X_1 + ... + X_n| \le |X_1| + ... + |X_n|$$
$$\le 1 + ... + 1 = n$$

$$E\left[|S_n|\right] \le E[n] = n < \infty.$$

(2) To show that $E\left[S_{n+1} | X_1, ..., X_n\right] = S_n$,

$$E\left[S_{n+1} | X_1, ..., X_n\right] = E\left[S_n + X_{n+1} | X_1, ..., X_n\right]$$
$$= E\left[S_n | X_1, ..., X_n\right] + E\left[X_{n+1} | X_1, ..., X_n\right]$$
$$= S_n + E[X_{n+1}],$$

where S_n is determined by $X_1, ..., X_n$ and X_{n+1} is independent of $X_{i's}$,

and $\because E[X_{n+1}] = (1).\Pr\{X_{n+1} = 1\} + (-1).\Pr\{X_{n+1} = -1\}$

$$= (1)(1/2) + (-1)(1/2) = 0$$

$$\therefore E\left[S_{n+1} | X_1, ..., X_n\right] = S_n$$

That is from (1) and (2), we have proved that S_n is a martingale.
