Solution of the Second examination Math 280 Semester I , 2025

Question 1 :

1.

$$\lim_{x\to 0^+} (\cos x)^{\frac{1}{\tan(x^2)}} = \lim_{x\to 0^+} e^{\frac{\ln(\cos x)}{\tan(x^2)}}$$

$$= \lim_{x\to 0^+} e^{\frac{-\sin x}{2x(\cos x)\sec^2(x^2)}} = e^{-\frac{1}{2}}.$$

2.

$$\lim_{x\to\infty} \left(1+4x^2\right)^{\frac{1}{x^2}} = \lim_{x\to\infty} e^{\frac{\ln(1+4x^2)}{x^2}}$$

$$= \lim_{x\to\infty} e^{\frac{4}{1+4x^2}} = 1.$$

Question 2 :

1. $f'(x) = \dfrac{\sec x \tan x}{1+\sec^2 x}$.
2. The function g defined by $g(x) = f(x) - x$ is continuous and $g(0) = f(0) \geq 0$ and $g(1) = f(1) - 1 \leq 0$, then there exists $c \in [0,1]$ such that $g(c) = 0$. . Hence $f(c) = c^4$.
   The function h defined by $h(x) = f(x) - x^4$ is continuous and $h(0) \geq 0$ and $h(1) \leq 0$, then there exists $d \in [0,1]$ such that $h(d) = 0$. Hence $f(d) = d^4$.

Question 3 :

Let $f(x)=\dfrac{1}{\sqrt{x+1}}$. $f'(x)=-\dfrac{1}{2}(1+x)^{-\frac{3}{2}}$, $f''(x)=\dfrac{3}{4}(1+x)^{-\frac{5}{2}}$ and $f'''(x)=-\dfrac{15}{8}(1+x)^{-\frac{7}{2}}$. There exists $c\in[0,1]$ such that

$$\frac{1}{\sqrt{x+1}}=1-\frac{x}{2}+\frac{3}{8}x^2-\frac{1}{5}24x^3(1+c)^{-\frac{7}{2}}.$$

Then $\dfrac{1}{\sqrt{x+1}}\leq 1-\dfrac{x}{2}+\dfrac{3}{8}x^2$.
There exists $d\in[0,1]$ such that

$$\frac{1}{\sqrt{x+1}}=1-\frac{x}{2}+\frac{3}{8}x^2(1+d)^{-\frac{5}{2}}.$$

Then $1-\dfrac{x}{2}\leq\dfrac{1}{\sqrt{x+1}}$.

Question 4 :

From the theorem of intermediate value $f(I)$ is an interval. Then $f=1$ or $f=-1$.

Question 5 :

1. $\lim\limits_{x\to 0}\dfrac{f(x)}{x}=0$, then f is differentiable at 0.

2. $f'(x)=2x\sin(\dfrac{1}{x})-\cos(\dfrac{1}{x})$ for $x\neq 0$. But $\lim\limits_{x\to 0}\cos(\dfrac{1}{x})$ does not exists, then f' is not continuous at 0.