[Solution Key]

KING SAUD UNIVERSITY
COLLEGE OF SCIENCES
DEPARTMENT OF MATHEMATICS

Semester 452 / MATH-244 (Linear Algebra) / Mid-term Exam 2

Max. Marks: 25 **Max. Time: $1\frac{1}{2}$ hr**

Question 1: [Marks: (1+2+2) + (1+2)]

a) **Determine** whether the following statements are true. **Justify** your answers.

i) $\{(a, b, c) \mid a, b, c$ are non-negative real numbers$\}$ is a subspace of $\mathbb{R}^3$.

Solution: This statement is not true because the set $\{(a, b, c) \mid a, b, c$ are non-negative real numbers$\}$ is not closed under scalar multiplication; for example, take the scaler k = −1. [Mark 1]

ii) For any fixed matrix $Y \in M_n(\mathbb{R})$, $\{A \in M_n(\mathbb{R}) \mid AY = YA\}$ is a subspace of the vector space $M_n(\mathbb{R})$ of all real matrices of type $n \times n$

Solution: This is a true statement because $OY = O = YO$ and $(\alpha A + B)Y = \alpha(AY) + BY = Y(\alpha A) + YB = Y(\alpha A + B)$ for all $A.B \in \{A \in M_n(\mathbb{R}) \mid AY = YA\}, \alpha \in \mathbb{R}$. [Marks 0.5+1.5]

iii) Any **set of five** 2×2 matrices must be linearly dependent.

Solution: This statement is true because vector space $M_2(\mathbb{R})$ of all real matrices of type 2×2 is of dimension 4 and so any **set of five** 2×2 matrices must be linearly dependent. [Marks 1 +1]

b) **Consider** the vector subspace $W = \{p(x) \in P_2 \mid p(1) = p(2)\}$ of P_2, where P_2 denotes the vector space of all real polynomials in x with degree at most 2 under the usual addition and scalar multiplication. Then:

i) **Show** that $P_2 - W \neq \emptyset$.

Solution: The polynomial $q(x) = x$ being of degree 1 is in P_2 but $q(1) \neq q(2)$. Hence, $q \in P_2 - W$. [Mark 1]

ii) **Show** that $\{1, (x-1)(x-2)\}$ is a linearly independent subset and **explain why** it must be a basis for W.

Solution: For any $\alpha, \beta \in \mathbb{R}$, $\alpha 1 + \beta(x-1)(x-2) = 0 \Longrightarrow \alpha = 0\ (with\ x = 1)$ and $\beta = 0\ (with\ x = 0)$. Hence, the subset $\{1, (x-1)(x-2)\}$ of W is linearly independent. But, $dimW \leq dimP_2 = 3$ by Part b)/i). Therefore, $\{1, (x-1)(x-2)\}$ must be a basis for W. [Marks 1+ 0.5+0.5]

Question 2: [Marks: 3 + 2 + 2 + 2]

Let $B := \{u_1 = (2,1),\ u_2 = (5,2)\}$ and $C := \{v_1 = (1,-2), v_2 = (-3,7)\}$. Then:

a) **Show** that both B and C are bases for the vector space $\mathbb{R}^2$.

Solution: If we construct the matrices $\acute{B} = \begin{bmatrix} 2 & 5 \\ 1 & 2 \end{bmatrix}$ and $\acute{C} = \begin{bmatrix} 1 & -3 \\ -2 & 7 \end{bmatrix}$ then det($\acute{B}$) = −1 and det($\acute{C}$) = 1; so, both the matrices $\acute{B}$ and $\acute{C}$ are invertible.; which means that the sets B and C are linearly independent in the 2-dimensional vector space. $\mathbb{R}^2$. Hence, both B and C are bases for $\mathbb{R}^2$. [Marks 2x0.5+2x 0.5+2x0.5]

b) **Construct** the transition matrix ${}_C P_B$ from the basis B to the basis C.

Solution: ${}_C P_B = [[u_1]_C \quad [u_2]_C] = \begin{bmatrix} 17 & 41 \\ 5 & 12 \end{bmatrix}$ [Marks 1+ 2 x1]

c) **Use** the matrix ${}_C P_B$ to **find** the transition matrix ${}_B P_C$.

Solution: ${}_B P_C = {}_C P_B^{-1} = \begin{bmatrix} 17 & 41 \\ 5 & 12 \end{bmatrix}^{-1} = \frac{1}{-1}\begin{bmatrix} 12 & -41 \\ -5 & 17 \end{bmatrix} = \begin{bmatrix} -12 & 41 \\ 5 & -17 \end{bmatrix}$. [Marks 1+ 1]

d) **Find** the coordinate vectors $[u_1]_C$ and $[v_2]_B$ by **using** the matrices ${}_C P_B$ and ${}_B P_C$, respectively.

Solution: $[u_1]_C = {}_C P_B [u_1]_B = \begin{bmatrix} 17 & 41 \\ 5 & 12 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 17 \\ 5 \end{bmatrix}$ and $[v_2]_B = {}_B P_C [v_2]_C = \begin{bmatrix} -12 & 41 \\ 5 & -17 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 41 \\ -17 \end{bmatrix}$. [Marks 1+ 1]

Question 3: [Marks: 2 + 3 + 3]

Let $RREF(A) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $RREF(A^T) = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ be the reduced row-echelon forms of a matrix A and its transpose A^T, respectively. Then, **find**:

i) $rank(A)$ and $nullity(A)$.

Solution: $rank(A) = number\ of\ nonzero\ rows\ in\ RREF\ (A) = 2$. Since $rank(A) + nullity(A) = number\ of\ columns\ in\ A = 3$, we have $nullity(A) = 3 - 2 = 1$. [Marks 1+ 1]

ii) A basis for each of the vector spaces: $col(A)$, $row(A)$, $N(A)$.

Solution: The set of non-zero rows in $RREF(A^T) = \{(1,0,1,2), (0,1,1,1\}$ is a basis for $col(A)$. [Mark 1]
The set of non-zero rows in $RREF(A) = \{(1,0,1), (0,1,1\}$ is a basis for $row(A)$. [Mark 1]
Since $(0,0,0) \neq (1,1,-1) \in N(A)$ and $dim\,N(A) = 1$, $\{(1,1,-1)\}$ is a basis for $N(A)$. [Mark 1]

iii) Three vectors u, v and w such that $u \in \mathbb{R}^3 \backslash row(A)$, $v \in \mathbb{R}^4 \backslash col(A)$ and $w \in \mathbb{R}^3 \backslash N(A)$.

Solution: $u := (0,0,1) \in \mathbb{R}^3 \backslash row(A)$, $v := (0,0,1,0) \in \mathbb{R}^4 \backslash col(A)$ and $w: (1,0,0) \in \mathbb{R}^3 \backslash N(A)$. [Mark 1+1+1]

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