Name: **ID:** **Section:** **Mark:**

King Saud University

College of Sciences, Department of Mathematics

1444/Semester-3/ MATH 380/ Quiz-2

Marks: 10 **Max. Time: 35 Minutes**

Answer the following questions.

Q1: [3]

The number of accidents occurring in a factory in a week is a Poisson random variable with mean 3. The number of individuals injured in different accidents is independently distributed, each with mean 2 and variance 4. Determine the mean and variance of the number of individuals injured in a week.

Q2: [1+3]

(a) Define a martingale.

(b) Let $\zeta_1, \zeta_2, \zeta_3, \ldots..$ be independent Bernoulli random variables with parameter p, $0<p<1$. Show that $X_0 = 1$ and $X_n = p^{-n}\,\zeta_1\,\zeta_2 \ldots \zeta_n$, $n = 1,2,\ldots$, defines a nonnegative martingale.

Q3: [3]

Consider a sequence of items from a production process, with each item being graded as good or defective. Suppose that a good item is followed by another good item with probability α and is followed by a defective item with probability $1-\alpha$. If the first item is good, what is the probability that the first defective item to appear is the fifth item ?

The Model Answer

Q1: [3]

Let $Z = \xi_1 + \xi_2 + ... + \xi_N$, where
N is # of accidents in a week and
ξ_k is # of individuals injured for kth accident.
$N \sim \text{Poisson}(3)$, consequently $E(N) = 3$, $Var(N) = 3$.
and $\because E(\xi_k) = 2$, $Var(\xi_k) = 4$
$\therefore E(\text{Z}) = \mu\upsilon = 2(3) = 6$
and $Var(Z) = \upsilon\sigma^2 + \mu^2\tau^2$
$= 3(4) + 4(3) = 24$

Q2: [1+3]

(a)
A stochastic process $\{X_n;\ n = 0,1,2,...\}$ is a martingale if for $n = 0,1,2,...$
(i) $E\left[|X_n|\right] < \infty$,
(ii) $E\left[X_{n+1}|X_0,...,X_n\right] = X_n$.
(b)
(1) $E[|X_n|] = E[X_n] = E[p^{-n}\zeta_1\zeta_2 \dots \zeta_n]$, and as $\zeta_{i's}$ are independent,
$= p^{-n}\, E[\zeta_1] \dots . E[\zeta_n]$
$= p^{-n}\, p^n = 1$, as $E[\zeta_\text{k}] = p$,
$\therefore\ E[|X_n|] = E[X_n] = 1 < \infty$.
(2) $E[X_{n+1}|X_0, \dots\dots, X_n] = E[p^{-(n+1)}\zeta_1\zeta_2 \dots \zeta_{n+1}|X_0, \dots\dots, X_n]$
$= E[p^{-n}\zeta_1\zeta_2 \dots \zeta_n\ p^{-1}\zeta_{n+1}|X_0, \dots\dots, X_n]$
$= p^{-n}\zeta_1\zeta_2 \dots \zeta_n\ E[p^{-1}\zeta_{n+1}|X_0, \dots\dots, X_n]$,
as $\zeta_1\zeta_2 \dots \zeta_n$ are determined by $X_0, \dots\dots, X_n$
$= p^{-n}\zeta_1\zeta_2 \dots \zeta_n\ p^{-1}E[\zeta_{n+1}|X_0, \dots\dots, X_n]$
$E[X_{n+1}|X_0, \dots\dots, X_n] = p^{-n}\zeta_1\zeta_2 \dots \zeta_n\ p^{-1}E[\zeta_{n+1}]$, as ζ_{n+1} is independent of $X_{i's}$,
$= p^{-n}\zeta_1\zeta_2 \dots \zeta_n\ p^{-1}\ p$
$\therefore\ E[X_{n+1}|X_0, \dots\dots, X_n] = p^{-n}\zeta_1\zeta_2 \dots \zeta_n = X_n$.
We have proved from (1) and (2) that X_n defines a nonnegative martingale.

Q3: [3]

$$\Pr\left\{X_2 = G,\ X_3 = G, X_4 = G, X_5 = D \middle| X_1 = G\right\}$$
$$= \Pr\left\{X_5 = D,\ X_4 = G, X_3 = G, X_2 = G \middle| X_1 = G\right\}$$
$$= \Pr\left\{X_5 = D \middle| X_4 = G\right\} \cdot \Pr\left\{X_4 = G \middle| X_3 = G\right\} \cdot \Pr\left\{X_3 = G \middle| X_2 = G\right\} \cdot \Pr\left\{X_2 = G \middle| X_1 = G\right\}$$
$$= p_{GD} p_{GG}^3$$
$$= (1-\alpha)\alpha^3$$
$$= \alpha^3(1-\alpha)$$

Also, you can solve it as follows.

$$p_1 p_{12} p_{23} p_{34} p_{45},\ p_1 = \Pr(X_1 = G) = 1$$
$$= p_G p_{GG}^3 p_{GD},\ p_G = 1$$
$$= \alpha^3(1-\alpha)$$