King Saud University
College of Sciences
Department of Mathematics

Second Mid Term Exam, S1 1443
M 380 – Stochastic Processes
Time: 90 minutes

Answer only three questions (including Q3) from the following questions:

Q1: [4+4]

(a) The probability of the thrower winning in the dice game is $p=0.5071$. Suppose player A is the thrower and begins the game with \$5, and player B, his opponent, begins with \$10. What is the probability that player A goes bankrupt before player B? Assume that the bet is \$1 per round.

(b) Let us model the daily stock price change as $Z=\xi_0+\xi_1+...+\xi_N$, where

$\xi_0,\xi_1,...,\xi_N$ are independent normally distributed random variables with common mean zero and variance 0.5, and N is the number of transactions during the day which has a Poisson distribution with mean 1.

(i) Determine the mean and variance of Z.

(ii) What is the distribution of Z?

Q2: [4+4]

(a) For the Markov process $\{X_t\}$, $t=0,1,2,...,n$ with states $i_0,i_1,i_2, \ldots ,i_{n-1},i_n$

Prove that: $\Pr\{X_0=i_0,X_1=i_1,X_2=i_2, \ldots ,X_n=i_n\}=p_{i_0}P_{i_0i_1}P_{i_1i_2}\ldots P_{i_{n-1}i_n}$ where $p_{i_0}=\text{pr}\{X_0=i_0\}$

(b) A Markov chain $X_0,X_1,X_2,\ldots$ has the transition probability matrix

$$\mathbf{P}=\begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\|\begin{array}{ccc} 0.2 & 0.3 & 0.5 \\ 0.4 & 0.2 & 0.4 \\ 0.5 & 0.3 & 0.2 \end{array}\right\| \end{array}$$

and initial distribution $p_0=0.3$, $p_1=0.5$ and $p_2=0.2$ Determine the probabilities

$\text{pr}\{X_0=1,X_1=1,X_2=0\}$ and $\text{pr}\{X_1=1,X_2=1,X_3=0\}$

Q3: [5+4]

(a) Consider the Markov chain whose transition probability matrix is given by

$$\mathbf{P}= \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left\| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0.1 & 0.6 & 0.1 & 0.2 \\ 0.2 & 0.3 & 0.4 & 0.1 \\ 0 & 0 & 0 & 1 \end{array} \right\| \end{array}$$

(i) Starting in state 2, determine the probability that the Markov chain ends in state 0.

(ii) Determine the mean time to absorption.

(iii) Sketch, the Markov chain diagram, and determine whether it's an absorbing chain or not.

(b) For modelling weather phenomenon, let $\{X_n\}$ be a Markov chain with state space $S=\{1,2\}$ where 1 stands for rainy and 2 stands for dry. The transition probability matrix is given by

$$\mathbf{P} = \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \left\| \begin{array}{cc} 0.8 & 0.2 \\ 0.4 & 0.6 \end{array} \right\| \end{array}$$

Initially, assume that the probability of weather will be rainy on 1^{st} June equals 3/8.

Find the probability for each of the following:

(i) The weather will be rainy on 2^{nd} June.

(ii) The weather will be rainy on 3^{rd} June.

(iii) The weather will be dry on 5^{th} June.

Q4: [4+4]

(a) Let $U_1, U_2, \ldots. U_n$ be independent random variables each uniformly distributed over the interval (0,1]. Show that $X_0 = 1$ and $X_n = 2^n\, U_1\, U_2 \ldots. U_n$, for n=1,2,… defines a martingale.

(b) Let $S_1, S_2, \ldots. S_n$ be independent random variables such that $E|S_i| < \infty$ for all i=1,2,…,n. Let $X_0 = 0$, $X_n = S_1 + S_2 + \ldots. + S_n$, $n \geq 1$.

Prove that: X_n is a martingale if and only if $E[S_n] = 0$ for all $n \geq 1$.

The Model Answer

Q1: [4+4]

(a)

The fortune for player A is $i = \$5$ and the total amount is $N = \$5 + \$10 = \$15$

p=0.5071 $\Rightarrow$ q=0.4929

$$u_i = pr\left\{X_n \text{ reaches state 0 before state } N \middle| X_0 = i\right\}$$

$$u_i = \frac{(q/p)^i - (q/p)^N}{1-(q/p)^N}, \; p \neq q$$

$$\therefore u_i = \frac{(0.4929/0.5071)^5 - (0.4929/0.5071)^{15}}{1-(0.4929/0.5071)^{15}}$$

$$u_i = 0.61837$$

(b)

$$E(\xi_k) = \mu = 0, \; \text{Var}(\xi_k) = \sigma^2 = 0.5$$

$$E(N) = v = 1, \; \text{Var}(N) = \tau^2 = 1$$

$$\because Z = \xi_0 + \xi_1 + ... + \xi_N$$

$$\therefore E(Z) = \mu(v+1) = 0(2) = 0 \text{ and}$$

$$\text{Var}(Z) = (v+1)\sigma^2 + \mu^2\tau^2 = 2(0.5) = 1$$

$$\Rightarrow \; Z \sim N(0,1)$$

$\therefore$ Z has the standard normal distribution.

Q2: [4+4]

(a)

$\because \Pr\{X_0=i_0, X_1=i_1, X_2=i_2, \ldots, X_n=i_n\}$

$=\Pr\{X_0=i_0, X_1=i_1, X_2=i_2, \ldots, X_{n-1}=i_{n-1}\}.\Pr\{X_n=i_n | X_0=i_0, X_1=i_1, X_2=i_2, \ldots, X_{n-1}=i_{n-1}\}$

$=\Pr\{X_0=i_0, X_1=i_1, X_2=i_2, \ldots, X_{n-1}=i_{n-1}\}.P_{i_{n-1}i_n}$ Definition of Markov

By repeating this argument $n-1$ times

$\therefore \Pr\{X_0=i_0, X_1=i_1, X_2=i_2, \ldots, X_n=i_n\}$

$=p_{i_0}P_{i_0i_1}P_{i_1i_2}\ldots P_{i_{n-2}i_{n-1}}P_{i_{n-1}i_n}$ where $p_{i_0}=\Pr\{X_0=i_0\}$ is obtained from the initial distribution of the process.

(b)

i) $\text{pr}\{X_0=1, X_1=1, X_2=0\}=p_1P_{11}P_{10}$, $p_1=\text{pr}\{X_0=1\}$

$=0.5(0.2)(0.4)$

$=0.04$

ii) $\text{pr}\{X_1=1, X_2=1, X_3=0\}=p_1P_{11}P_{10}$, $p_1=\text{pr}\{X_1=1\}$

$\text{pr}\{X_1=1\}=\Pr(X_1=1|X_0=0)\Pr(X_0=0)+\Pr(X_1=1|X_0=1)\Pr(X_0=1)+\Pr(X_1=1|X_0=2)\Pr(X_0=2)$

$=P_{01}p_0+P_{11}p_1+P_{21}p_2$

$=0.3(0.3)+0.2(0.5)+0.3(0.2)=0.25$

$\therefore \text{pr}\{X_1=1, X_2=1, X_3=0\}=0.25(0.2)(0.4)=0.02$

Q3: [5+4]

(a)

$$\mathbf{P}= \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left\|\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0.1 & 0.6 & 0.1 & 0.2 \\ 0.2 & 0.3 & 0.4 & 0.1 \\ 0 & 0 & 0 & 1 \end{array}\right\| \end{array}$$

$u_i=pr\{X_T=0|X_0=i\}$ for i=1,2,

$and\ \ v_i=E[T|X_0=i]$ for i=1,2.

(i)

$$u_1=p_{10}+p_{11}u_1+p_{12}u_2$$
$$u_2=p_{20}+p_{21}u_1+p_{22}u_2$$

$\Rightarrow$

$$u_1 = 0.1 + 0.6u_1 + 0.1u_2$$
$$u_2 = 0.2 + 0.3u_1 + 0.4u_2$$

$\Rightarrow$

$$4u_1 - u_2 = 1 \qquad (1)$$
$$3u_1 - 6u_2 = -2 \qquad (2)$$

Solving (1) and (2), we get

$$u_1 = \tfrac{8}{21} \text{ and } u_2 = \tfrac{11}{21}$$

Starting in state 2, the probability that the Markov chain ends in state 0 is

$$u_2 = u_{20} = \tfrac{11}{21}$$
$$\approx 0.52$$

(ii) Also, the mean time to absorption can be found as follows

$$v_1 = 1 + p_{11}v_1 + p_{12}v_2$$
$$v_2 = 1 + p_{21}v_1 + p_{22}v_2$$

$\Rightarrow$

$$v_1 = 1 + 0.6v_1 + 0.1v_2$$
$$v_2 = 1 + 0.3v_1 + 0.4v_2$$

$\Rightarrow$

$$4v_1 - v_2 = 10 \qquad (1)$$
$$3v_1 - 6v_2 = -10 \qquad (2)$$

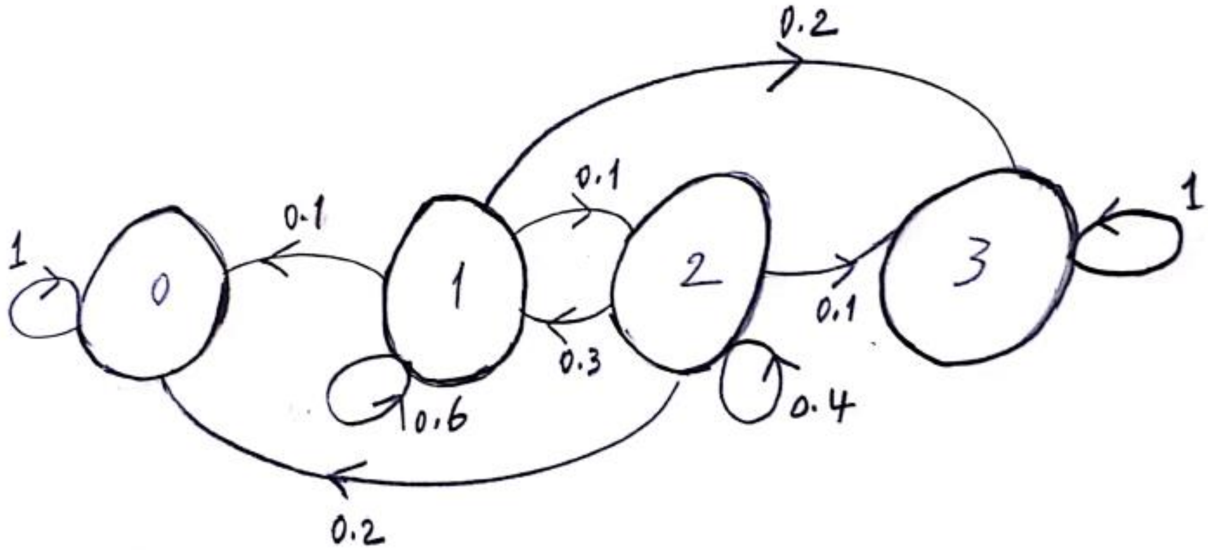
Solving (1) and (2), we get

$\therefore$ The mean time to absorption is

$$v_1 = v_2 = \tfrac{10}{3}$$

$$\therefore \; v_2 = v_{20} = \tfrac{10}{3}$$
$$\approx 3.3$$

(iii) It's an absorbing Markov Chain.



Markov Chain Diagram

(b)

The Markov chain $X_0, X_1, X_2, \ldots$ represents the day's weather

$\because \ pr(X_0 = 1) = p_1 \ = 3/8$
$\therefore \ pr(X_0 = 2) = p_2 \ = 5/8$

$\Rightarrow$ The initial probability distribution is $[3/8 \quad 5/8]$

(i) To get the prob. of weather will be rainy on 2nd June

$$\because \ \mathbf{P} = \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \left\| \begin{array}{cc} 0.8 & 0.2 \\ 0.4 & 0.6 \end{array} \right\| \end{array}$$

$$\begin{aligned} pr(X_1 = 1) &= \Pr(X_1 = 1 | X_0 = 1)\Pr(X_0 = 1) + \Pr(X_1 = 1 | X_0 = 2)\Pr(X_0 = 2) \\ &= P_{11}p_1 + P_{21}p_2 \\ &= (0.8)(\tfrac{3}{8}) + (0.4)(\tfrac{5}{8}) \end{aligned}$$

$\therefore pr(X_1 = 1) = 0.55$

(ii) To get the prob. of weather will be rainy on 3rd June

$$\because \mathbf{P}^2 = \begin{bmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{bmatrix}\begin{bmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{bmatrix}$$

$$= \begin{bmatrix} 0.72 & 0.28 \\ 0.56 & 0.44 \end{bmatrix}$$

$$pr(X_2 = 1) = \Pr(X_2 = 1 | X_0 = 1)\Pr(X_0 = 1) + \Pr(X_2 = 1 | X_0 = 2)\Pr(X_0 = 2)$$

$$= P_{11}{}^2 p_1 + P_{21}{}^2 p_2$$

$$= (0.72)(3/8) + (0.56)(5/8)$$

$$\therefore pr(X_2 = 1) = 0.62$$

(iii) To get the prob. of weather will be dry on 5th June

$$\because \mathbf{P}^4 = \begin{bmatrix} 0.72 & 0.28 \\ 0.56 & 0.44 \end{bmatrix}\begin{bmatrix} 0.72 & 0.28 \\ 0.56 & 0.44 \end{bmatrix}$$

$$= \begin{bmatrix} 0.6752 & 0.3248 \\ 0.6496 & 0.3504 \end{bmatrix}$$

$$pr(X_4 = 2) = \Pr(X_4 = 2 | X_0 = 1)\Pr(X_0 = 1) + \Pr(X_4 = 2 | X_0 = 2)\Pr(X_0 = 2)$$

$$= P_{12}^4 p_1 + P_{22}^4 p_2$$

$$= (0.3248)(3/8) + (0.3504)(5/8)$$

$$\therefore pr(X_4 = 2) = 0.3408$$

Q4: [4+4]

(a)

1- As X_n is a non-negative random variable,

$$E[|X_n|] = E[X_n] = E[2^n\, U_1\, U_2 \ldots . U_n] = 2^n\, E[U_1] \ldots . E[U_n]$$

$$= 2^n \cdot \left(\tfrac{1}{2}\right)^n = 1 < \infty.$$

This is because $U_i's$ are independent, also, since $U_i \sim uniform(0,1]$, then $E[U_i] = \frac{1}{2}$.

2- $E[X_{n+1}|X_0, \ldots, X_n] = E[2^{n+1}\, U_1\, U_2 \ldots . U_n . U_{n+1}|X_0, \ldots, X_n]$

$$= 2^n\, U_1\, U_2 \ldots . U_n\, E[2 . U_{n+1}|X_0, \ldots, X_n]$$

$$= 2^n\, U_1\, U_2 \ldots . U_n\;\; 2\, E[U_{n+1}|X_0, \ldots, X_n]$$

$$= 2^n\, U_1\, U_2 \ldots . U_n\;\; 2\, E[U_{n+1}]$$

$$= 2^n \, U_1 \, U_2 \, \ldots . \, U_n \;\; 2 . \frac{1}{2} = 2^n \, U_1 \, U_2 \, \ldots . \, U_n = X_n.$$

From 1 and 2, we proved that X_n is a martingale.

(b)

For $n \geq 1$,

1- $E|X_n| = E|S_1 + S_2 + \ldots . + S_n| \leq E|S_1| + E|S_2| + \ldots .. + E|S_n| < \infty$, since $E|S_i| < \infty$ for all i=1,2,…,n.

2- $E[X_{n+1}|X_0, \ldots ., X_n] = E[(S_1 + S_2 + \ldots . + S_n + S_{n+1})|X_0, \ldots ., X_n]$
$=(S_1 + S_2 + \ldots . + S_n) + E[S_{n+1}|X_0, \ldots ., X_n]$
$= X_n + E[S_{n+1}]$
$= X_n$ if and only if $E[S_{n+1}] = 0$.

Therefore, X_n is a martingale if and only if $E[S_n] = 0$ for all $n \geq 1$.