King Saud University
College of Sciences
Department of Mathematics

Second Mid Term Exam, S2 1442
M 380 – Stochastic Processes
Time: 90 minutes

Answer the following questions:

Q1: [4+4]

(a) For the Markov process $\{X_t\}$, t=0,1,2,...,n with states $i_0, i_1, i_2, \ldots, i_{n-1}, i_n$

Prove that: $\Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \ldots, X_n = i_n\} = p_{i_0} P_{i_0 i_1} P_{i_1 i_2} \ldots P_{i_{n-1} i_n}$ where $p_{i_0} = \mathrm{pr}\{X_0 = i_0\}$

(b) A Markov chain $X_0, X_1, X_2, \ldots$ has the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} 0.2 & 0.3 & 0.5 \\ 0.4 & 0.2 & 0.4 \\ 0.5 & 0.3 & 0.2 \end{array} \right\| \end{array}$$

and initial distribution p_0=0.5, p_1=0.2 and p_2=0.3 Determine the probabilities

$\mathrm{pr}\{X_0 = 1, X_1 = 1, X_2 = 0\}$ and $\mathrm{pr}\{X_1 = 1, X_2 = 1, X_3 = 0\}$

Q2: [4+4]

(a) Consider a spare parts inventory model in which either 0, 1, or 2 repair parts are demanded in any period, with $\Pr\{\xi_n = 0\} = 0.3$, $\Pr\{\xi_n = 1\} = 0.2$, $\Pr\{\xi_n = 2\} = 0.5$ and suppose s=0 and S=3. Determine the transition probability matrix for the Markov chain $\{X_n\}$, where X_n is defined to be the quantity on hand at the end of period n.

(b) For modelling weather phenomenon, let $\{X_n\}$ be a Markov chain with state space $S=\{1,2\}$ where 1 stands for rainy and 2 stands for dry. The transition probability matrix is given by

$$\mathbf{P} = \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \left\| \begin{array}{cc} 0.8 & 0.2 \\ 0.4 & 0.6 \end{array} \right\| \end{array}$$

Initially, assume that the probability of weather will be rainy on 1st June equals 3/8.

Find the probability for each of the following:

(i) The weather will be dry on 2nd June.

(ii) The weather will be dry on 3rd June.

(iii) The weather will be rainy on 5th June.

Q3: [4+5]

(a) Suppose that the social classes of successive generations in a family follow a Markov chain with transition probability matrix given by

		Son's class		
		Lower	Middle	Upper
Father's class	Lower	0.7	0.2	0.1
	Middle	0.2	0.6	0.2
	Upper	0.1	0.4	0.5

What fraction of families are middle class in the long run?

(b) Consider the Markov chain whose transition probability matrix is given by

$$\mathbf{P}= \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left\| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0.1 & 0.6 & 0.1 & 0.2 \\ 0.2 & 0.3 & 0.4 & 0.1 \\ 0 & 0 & 0 & 1 \end{array} \right\| \end{array}$$

(i) Starting in state 1, determine the probability that the Markov chain ends in state 0.

(ii) Determine the mean time to absorption.

(iii) Sketch, the Markov chain diagram, and determine whether it's an absorbing chain or not.

The Model Answer

Q1: [4+4]

(a)

$$\because \Pr\{X_0=i_0, X_1=i_1, X_2=i_2, \ldots, X_n=i_n\}$$
$$=\Pr\{X_0=i_0, X_1=i_1, X_2=i_2, \ldots, X_{n-1}=i_{n-1}\}.\Pr\{X_n=i_n \mid X_0=i_0, X_1=i_1, X_2=i_2, \ldots, X_{n-1}=i_{n-1}\}$$
$$=\Pr\{X_0=i_0, X_1=i_1, X_2=i_2, \ldots, X_{n-1}=i_{n-1}\}.P_{i_{n-1}i_n} \quad \text{Definition of Markov}$$

By repeating this argument $n-1$ times

$$\therefore \Pr\{X_0=i_0, X_1=i_1, X_2=i_2, \ldots, X_n=i_n\}$$
$$=p_{i_0}P_{i_0i_1}P_{i_1i_2}\ \ldots\ P_{i_{n-2}i_{n-1}}P_{i_{n-1}i_n}$$ where $p_{i_0}=\Pr\{X_0=i_0\}$ is obtained from the initial distribution of the process.

(b)

i) $\text{pr}\{X_0=1, X_1=1, X_2=0\}=p_1P_{11}P_{10}$, $p_1=\text{pr}\{X_0=1\}$

$=0.2(0.2)(0.4)$

$=0.016$

ii) $\text{pr}\{X_1=1, X_2=1, X_3=0\}=p_1P_{11}P_{10}$, $p_1=\text{pr}\{X_1=1\}$

$$\text{pr}\{X_1=1\}=\Pr(X_1=1 \mid X_0=0)\Pr(X_0=0)+\Pr(X_1=1 \mid X_0=1)\Pr(X_0=1)+\Pr(X_1=1 \mid X_0=2)\Pr(X_0=2)$$
$$=P_{01}p_0+P_{11}p_1+P_{21}p_2$$
$$=0.3(0.5)+0.2(0.2)+0.3(0.3)=0.28$$

$$\therefore \text{pr}\{X_1=1, X_2=1, X_3=0\}=0.28(0.2)(0.4)=0.0224$$

Q2: [4+4]

(a)

$$\begin{array}{c|ccccc|}
 & -1 & 0 & 1 & 2 & 3 \\
-1 & 0 & 0 & 0.5 & 0.2 & 0.3 \\
0 & 0 & 0 & 0.5 & 0.2 & 0.3 \\
1 & 0.5 & 0.2 & 0.3 & 0 & 0 \\
2 & 0 & 0.5 & 0.2 & 0.3 & 0 \\
3 & 0 & 0 & 0.5 & 0.2 & 0.3
\end{array}$$

P_{ij} =Pr($\xi_{n+1} = S - j$) , $i \le s$ for replenishment

$P_{-1,-1}$=Pr($\xi_{n+1} = 4) = 0$, P_{01}=Pr($\xi_{n+1} = 2) = 0.5$

P_{ij} =Pr($\xi_{n+1} = i - j$) , $s < i \le S$ for non-replenishment

$P_{1,-1}$=Pr($\xi_{n+1} = 2) = 0.5$, P_{11}=Pr($\xi_{n+1} = 0) = 0.3$, P_{21}=Pr($\xi_{n+1} = 1) = 0.2$

(b)

The Markov chain $X_0, X_1, X_2, \ldots$ represents the day's weather

$$\because \; pr(X_0 = 1) = p_1 = 3/8$$
$$\therefore \; pr(X_0 = 2) = p_2 = 5/8$$

$\Rightarrow$ The initial probability distribution is $[3/8 \quad 5/8]$

$\text{pr}(X_n = k) = \sum_{j=1}^{\infty} p_j P_{jk}^n$ is the probability of the process being in state k at time n.

(i) The prob. of weather will be dry on 2nd June is

$$\begin{aligned} \text{pr}(X_1 = 2) &= p_1 P_{12} + p_2 P_{22} \\ &= 3/8(0.2) + 5/8(0.6) \\ &= 0.45 \end{aligned}$$

(ii) The prob. of weather will be dry on 3rd June is

$$pr(X_2 = 2) = p_1 P_{12}^2 + p_2 P_{22}^2$$

$$\because \; \mathbf{P}^2 = \begin{bmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{bmatrix} \begin{bmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{bmatrix} = \begin{bmatrix} 0.72 & 0.28 \\ 0.56 & 0.44 \end{bmatrix}$$

$$\begin{aligned} \therefore \; pr(X_2 = 2) &= p_1 P_{12}^2 + p_2 P_{22}^2 \\ &= 3/8(0.28) + 5/8(0.44) \\ &= 0.38 \end{aligned}$$

(iii) The prob. of weather will be rainy on 5th June is

$$pr(X_4 = 1) = p_1 P_{11}^4 + p_2 P_{21}^4$$

$$\because\ \mathbf{P}^4 = \begin{bmatrix} 0.72 & 0.28 \\ 0.56 & 0.44 \end{bmatrix}\begin{bmatrix} 0.72 & 0.28 \\ 0.56 & 0.44 \end{bmatrix}$$
$$= \begin{bmatrix} 0.6752 & 0.3248 \\ 0.6496 & 0.3504 \end{bmatrix}$$

$$\begin{aligned} \therefore\ pr(X_4 = 1) &= p_1 P_{11}^4 + p_2 P_{21}^4 \\ &= 3/8(0.6752) + 5/8(0.6496) \\ &= 0.6592 \end{aligned}$$

Another Solution:

(i) The prob. of weather will be dry on 2nd June is

$$\begin{aligned} pr(X_1 = 2) &= \Pr(X_1 = 2 | X_0 = 1)\Pr(X_0 = 1) + \Pr(X_1 = 2 | X_0 = 2)\Pr(X_0 = 2) \\ &= P_{12}p_1 + P_{22}p_2 \\ &= (0.2)(\tfrac{3}{8}) + (0.6)(\tfrac{5}{8}) \\ &= 0.45 \end{aligned}$$
$$\therefore pr(X_1 = 1) = 0.55$$

(ii)The prob. of weather will be dry on 3rd June is

$$\begin{aligned} pr(X_2 = 2) &= \Pr(X_2 = 2 | X_1 = 1)\Pr(X_1 = 1) + \Pr(X_2 = 2 | X_1 = 2)\Pr(X_1 = 2) \\ &= P_{12}p_1 + P_{22}p_2 \\ &= (0.2)(0.55) + (0.6)(0.45) \\ &= 0.38 \end{aligned}$$
$$\therefore pr(X_2 = 1) = 0.62$$

(iii)The prob. of weather will be rainy on 5th June is

$$\begin{aligned} pr(X_4 = 1) &= \Pr(X_4 = 1 | X_2 = 1)\Pr(X_2 = 1) + \Pr(X_4 = 1 | X_2 = 2)\Pr(X_2 = 2) \\ &= P_{11}^2 p_1 + P_{21}^2 p_2 \\ &= (0.72)(0.62) + (0.56)(0.38) \\ &= 0.6592 \end{aligned}$$

Q3: [4+5]

(a)

Let $\pi = (\pi_0, \pi_1, \pi_2)$ be the limiting distribution

$\Rightarrow$

$$\pi_0 = 0.7\pi_0 + 0.2\pi_1 + 0.1\pi_2$$

$\pi_1 = 0.2\pi_0 + 0.6\pi_1 + 0.4\pi_2$

$\pi_2 = 0.1\pi_0 + 0.2\pi_1 + 0.5\pi_2$

$\pi_0 + \pi_1 + \pi_2 = 1$

Solving the following equations

$$3\pi_0 - 2\pi_1 - \pi_2 = 0 \qquad (1)$$
$$\pi_0 + 2\pi_1 - 5\pi_2 = 0 \qquad (2)$$
$$\pi_0 + \pi_1 + \pi_2 = 1 \qquad (3)$$

By solving equations using Cramer's rule, we get

$$\Delta = \begin{vmatrix} 3 & -2 & -1 \\ 1 & 2 & -5 \\ 1 & 1 & 1 \end{vmatrix} = 34,\ \Delta_0 = \begin{vmatrix} 0 & -2 & -1 \\ 0 & 2 & -5 \\ 1 & 1 & 1 \end{vmatrix} = 12$$

$$\Delta_1 = \begin{vmatrix} 3 & 0 & -1 \\ 1 & 0 & -5 \\ 1 & 1 & 1 \end{vmatrix} = 14,\ \Delta_2 = \begin{vmatrix} 3 & -2 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 8$$

$\therefore\ \pi_0 = \frac{\Delta_0}{\Delta} = \frac{6}{17},\ \pi_1 = \frac{\Delta_1}{\Delta} = \frac{7}{17},\ \pi_2 = \frac{\Delta_2}{\Delta} = \frac{4}{17}$

$\therefore$ The limitting distribution is $\pi = (\pi_0, \pi_1, \pi_2) = (6/17, 7/17, 4/17)$

$\therefore$ In the long run, approximately 41.2% of families are middle class.

(b)

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left\Vert \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0.1 & 0.6 & 0.1 & 0.2 \\ 0.2 & 0.3 & 0.4 & 0.1 \\ 0 & 0 & 0 & 1 \end{array} \right\Vert \end{array}$$

$u_i = pr\{X_T = 0 | X_0 = i\}$ for i=1,2,

$and\ \ v_i$ =E[$T | X_0 = i$] for i=1,2.

(i)

$$u_1 = p_{10} + p_{11}u_1 + p_{12}u_2$$
$$u_2 = p_{20} + p_{21}u_1 + p_{22}u_2$$

$\Rightarrow$

$$u_1 = 0.1 + 0.6u_1 + 0.1u_2$$
$$u_2 = 0.2 + 0.3u_1 + 0.4u_2$$

$\Rightarrow$

$$4u_1 - u_2 = 1 \qquad (1)$$
$$3u_1 - 6u_2 = -2 \qquad (2)$$

Solving (1) and (2), we get

$$u_1 = \tfrac{8}{21} \text{ and } u_2 = \tfrac{11}{21}$$

Starting in state 1, the probability that the Markov chain ends in state 0 is

$$u_1 = u_{10} = \tfrac{8}{21}$$
$$\approx 0.38$$

(ii) Also, the mean time to absorption can be found as follows

$$v_1 = 1 + p_{11}v_1 + p_{12}v_2$$
$$v_2 = 1 + p_{21}v_1 + p_{22}v_2$$

$\Rightarrow$

$$v_1 = 1 + 0.6v_1 + 0.1v_2$$
$$v_2 = 1 + 0.3v_1 + 0.4v_2$$

$\Rightarrow$

$$4v_1 - v_2 = 10 \qquad (1)$$
$$3v_1 - 6v_2 = -10 \qquad (2)$$

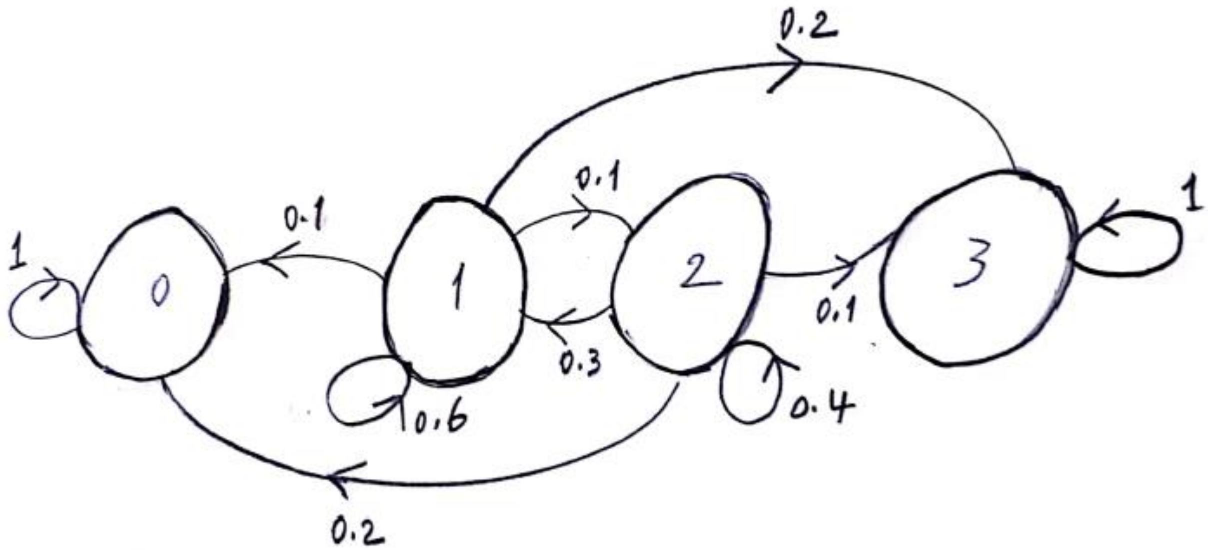
Solving (1) and (2), we get

$$v_1 = v_2 = \tfrac{10}{3}$$

$\therefore\ v_1 = v_{10} = \frac{10}{3}$

≈ 3.3

(ii) It's an absorbing Markov Chain.



Markov Chain Diagram