King Saud University
College of Sciences
Department of Mathematics

Second Mid Term Exam, S1 1441/1442
M 380 – Stochastic Processes
Time: 90 minutes

Answer the following questions:

Q1: [4+4]

(a) For the Markov process $\{X_t\}$, t=0,1,2,...,n with states $i_0, i_1, i_2, \ldots, i_{n-1}, i_n$

Prove that: $\Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \ldots, X_n = i_n\} = p_{i_0} P_{i_0 i_1} P_{i_1 i_2} \ldots P_{i_{n-1} i_n}$ where $p_{i_0} = \mathrm{pr}\{X_0 = i_0\}$

(b) A Markov chain $X_0, X_1, X_2, \ldots$ has the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} 0.2 & 0.3 & 0.5 \\ 0.4 & 0.2 & 0.4 \\ 0.5 & 0.3 & 0.2 \end{array} \right\| \end{array}$$

and initial distribution p_0=0.3, p_1=0.5 and p_2=0.2 Determine the probabilities

$\mathrm{pr}\{X_0 = 1, X_1 = 1, X_2 = 0\}$ and $\mathrm{pr}\{X_1 = 1, X_2 = 1, X_3 = 0\}$

Q2: [4+4]

(a) Consider a spare parts inventory model in which either 0, 1, or 2 repair parts are demanded in any period, with $\Pr\{\xi_n = 0\} = 0.5$, $\Pr\{\xi_n = 1\} = 0.4$, $\Pr\{\xi_n = 2\} = 0.1$ and suppose s=0 and S=2 Determine the transition probability matrix for the Markov chain $\{X_n\}$, where X_n is defined to be the quantity on hand at the end of period n.

(b) Let X_n denote the quality of the nth item that produced in a certain factory with $X_n = 0$ meaning "good" and $X_n = 1$ meaning "defective". Suppose that $\{X_n\}$ be a Markov chain whose transition matrix is

$$P = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \left\| \begin{array}{cc} 0.99 & 0.01 \\ 0.12 & 0.88 \end{array} \right\| \end{array}$$

i) What is the probability that the fourth item is defective given that the first item is good?

ii) In the long run, what is the probability that an item produced by this system is defective?

Q3: [5+4]

(a) Suppose that the social classes of successive generations in a family follow a Markov chain with transition probability matrix given by

		Son's class		
		Lower	Middle	Upper
Father's class	Lower	0.7	0.2	0.1
	Middle	0.2	0.6	0.2
	Upper	0.1	0.4	0.5

What fraction of families are upper class in the long run?

(b) Consider the Markov chain whose transition probability matrix is given by

$$\mathbf{P}=\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left\|\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0.4 & 0.4 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.6 & 0.2 \\ 0 & 0 & 0 & 1 \end{array}\right\| \end{array}$$

(i) Starting in state 1, determine the probability that the Markov chain ends in state 0.

(ii) Determine the mean time to absorption.

The Model Answer

Q1: [4+4]

(a)

$\because \Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \dots, X_n = i_n\}$

$= \Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \dots, X_{n-1} = i_{n-1}\}.\Pr\{X_n = i_n | X_0 = i_0, X_1 = i_1, X_2 = i_2, \dots, X_{n-1} = i_{n-1}\}$

$= \Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \dots, X_{n-1} = i_{n-1}\}.P_{i_{n-1}i_n}$ Definition of Markov

By repeating this argument $n-1$ times

$\therefore \Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \dots, X_n = i_n\}$

$= p_{i_0} P_{i_0 i_1} P_{i_1 i_2} \dots P_{i_{n-2} i_{n-1}} P_{i_{n-1} i_n}$ where $p_{i_0} = \Pr\{X_0 = i_0\}$ is obtained from the initial distribution of the process.

(b)

i) $\text{pr}\{X_0 = 1, X_1 = 1, X_2 = 0\} = p_1 P_{11} P_{10}$, $p_1 = \text{pr}\{X_0 = 1\}$

$=0.5(0.2)(0.4)$

$=0.04$

ii) $\text{pr}\{X_1 = 1, X_2 = 1, X_3 = 0\} = p_1 P_{11} P_{10}$, $p_1 = \text{pr}\{X_1 = 1\}$

$\text{pr}\{X_1 = 1\} = \Pr(X_1 = 1 | X_0 = 0)\Pr(X_0 = 0) + \Pr(X_1 = 1 | X_0 = 1)\Pr(X_0 = 1) + \Pr(X_1 = 1 | X_0 = 2)\Pr(X_0 = 2)$

$= P_{01} p_0 + P_{11} p_1 + P_{21} p_2$

$=0.3(0.3)+0.2(0.5)+0.3(0.2)=0.25$

$\therefore \text{pr}\{X_1 = 1, X_2 = 1, X_3 = 0\} = 0.25(0.2)(0.4) = 0.02$

Q2: [4+4]

(a)

$$\begin{array}{c} \\ -1 \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{cccc} -1 & 0 & 1 & 2 \\ \left\| \begin{array}{cccc} 0 & 0.1 & 0.4 & 0.5 \\ 0 & 0.1 & 0.4 & 0.5 \\ 0.1 & 0.4 & 0.5 & 0 \\ 0 & 0.1 & 0.4 & 0.5 \end{array} \right\| \end{array}$$

where

$$P_{ij}=\begin{cases} pr=(\xi_n=2-j), \ \mathrm{i}\le 0 & \text{replenishment} \\ pr=(\xi_n=i-j), \ 0<\mathrm{i}\le 2 & \text{without replenishment} \end{cases}$$

(b)

i)

$$P^3=\begin{bmatrix} 0.9737 & 0.0263 \\ 0.3152 & 0.6848 \end{bmatrix}$$

$$pr\{X_3=1|X_0=0\}=p_{01}^3=0.0263$$

ii)

In the long run, the probability that an item produced by this system is defective is given by:

$$a/(a+b)=\frac{0.01}{0.01+0.12}$$

$$=\frac{1}{13}=7.69\ \%\ ,$$

where $\lim_{n\to\infty} P^n=\begin{bmatrix} \frac{b}{a+b} & \frac{a}{a+b} \\ \frac{b}{a+b} & \frac{a}{a+b} \end{bmatrix}$

Q3: [5+4]

(a)

Let $\pi=(\pi_0,\pi_1,\pi_2)$ be the limiting distribution

$\Rightarrow$

$$\pi_0=0.7\pi_0+0.2\pi_1+0.1\pi_2$$

$$\pi_1=0.2\pi_0+0.6\pi_1+0.4\pi_2$$

$$\pi_2=0.1\pi_0+0.2\pi_1+0.5\pi_2$$

$$\pi_0+\pi_1+\pi_2=1$$

Solving the following equations

$$3\pi_0 - 2\pi_1 - \pi_2 = 0 \qquad (1)$$
$$\pi_0 + 2\pi_1 - 5\pi_2 = 0 \qquad (2)$$
$$\pi_0 + \pi_1 + \pi_2 = 1 \qquad (3)$$

We get $\pi_0 = \frac{6}{17},\ \pi_1 = \frac{7}{17},\ \pi_2 = \frac{4}{17}$

$\therefore$ In the long run, approximately 23.5% of families are upper class.

(b)

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left\| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0.4 & 0.4 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.6 & 0.2 \\ 0 & 0 & 0 & 1 \end{array} \right\| \end{array}$$

$u_i = pr\{X_T = 0 | X_0 = i\}$ for i=1,2,

$and\ \ v_i = E[T | X_0 = i]$ for i=1,2.

(i)

$$u_1 = p_{10} + p_{11}u_1 + p_{12}u_2$$
$$u_2 = p_{20} + p_{21}u_1 + p_{22}u_2$$

$\Rightarrow$

$$u_1 = 0.4 + 0.4u_1 + 0.1u_2$$
$$u_2 = 0.1 + 0.1u_1 + 0.6u_2$$

$\Rightarrow$

$$6u_1 - u_2 = 4 \qquad (1)$$
$$u_1 - 4u_2 = -1 \qquad (2)$$

Solving (1) and (2), we get

$$u_1 = \frac{17}{23} \text{ and } u_2 = \frac{10}{23}$$

Starting in state 1, the probability that the Markov chain ends in state 0 is

$$u_1 = u_{10} = \frac{17}{23}$$
$$\approx 0.74$$

(ii) Also, the mean time to absorption can be found as follows

$$v_1 = 1 + p_{11}v_1 + p_{12}v_2$$
$$v_2 = 1 + p_{21}v_1 + p_{22}v_2$$

$\Rightarrow$

$$v_1 = 1 + 0.4v_1 + 0.1v_2$$
$$v_2 = 1 + 0.1v_1 + 0.6v_2$$

$\Rightarrow$

$$6v_1 - v_2 = 10 \qquad (1)$$
$$v_1 - 4v_2 = -10 \qquad (2)$$

Solving (1) and (2), we get

$v_1 = \frac{50}{23}$ and $v_2 = \frac{70}{23}$

$$v_1 = v_{10} = \frac{50}{23}$$
$$\approx 2.17$$