King Saud University
College of Sciences
Department of Mathematics

Second Mid Term, S1 1441
M 380 - Stochastic Processes
Time: 90 minutes - Female Sec.

Answer the following questions:

Q1: [4+5]

a) For the Markov process $\{X_t\}$, t=0,1,2,...,n with states $i_0, i_1, i_2, \dots, i_{n-1}, i_n$

Prove that: $\Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \dots, X_n = i_n\} = p_{i_0} P_{i_0 i_1} P_{i_1 i_2} \dots P_{i_{n-1} i_n}$ where $p_{i_0} = \mathrm{pr}\{X_0 = i_0\}$

b) Consider a spare parts inventory model in which either 0, 1, or 2 repair parts are demanded in any period, with $\Pr\{\xi_n = 0\} = 0.5$, $\Pr\{\xi_n = 1\} = 0.4$, $\Pr\{\xi_n = 2\} = 0.1$ and suppose s=0 and S=2 Determine the transition probability matrix for the Markov chain $\{X_n\}$, where X_n is defined to be the quantity on hand at the end of period n.

Q2: [3+6]

a) A particle moves among the states 0, 1, 2 according to a Markov process whose transition probability matrix is

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{array} \right\| \end{array}$$

Let X_n denote the position of the particle at the nth move. Calculate $\Pr\{X_n = 0 | X_0 = 0\}$ for $n = 0, 1, 2$

b) Consider the Markov chain whose transition probability matrix is given by

$$\mathbf{P}= \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left\| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0.1 & 0.4 & 0.1 & 0.4 \\ 0.2 & 0.1 & 0.6 & 0.1 \\ 0 & 0 & 0 & 1 \end{array} \right\| \end{array}$$

(i) Starting in state 2, determine the probability that the Markov chain ends in state 0.

(ii) Determine the mean time to absorption.

Q3: [7]

Suppose that the summands $\xi_1, \xi_2, \ldots$ are continuous random variables having a probability density function $f(z) = \begin{cases} \lambda e^{-\lambda z} & \text{for } z \geq 0 \\ 0 & \text{for } z < 0 \end{cases}$

and $P_N(n) = \beta(1-\beta)^{n-1}$ for $n = 1, 2, \ldots$

Find the probability density function for $X = \xi_1 + \xi_2 + \ldots + \xi_N$

The Model Answer

Q1: [4+5]

a)

$\because \Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \ldots, X_n = i_n\}$

$= \Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \ldots, X_{n-1} = i_{n-1}\}.\Pr\{X_n = i_n \mid X_0 = i_0, X_1 = i_1, X_2 = i_2, \ldots, X_{n-1} = i_{n-1}\}$

$= \Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \ldots, X_{n-1} = i_{n-1}\}.P_{i_{n-1}i_n}$ Definition of Markov

By repeating this argument $n-1$ times

$\therefore \Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \ldots, X_n = i_n\}$

$= p_{i_0} P_{i_0 i_1} P_{i_1 i_2} \ldots P_{i_{n-2} i_{n-1}} P_{i_{n-1} i_n}$ where $p_{i_0} = \Pr\{X_0 = i_0\}$ is obtained from the initial distribution of the process.

b)

	−1	0	1	2
−1	0	0.1	0.4	0.5
0	0	0.1	0.4	0.5
1	0.1	0.4	0.5	0
2	0	0.1	0.4	0.5

Where

$$P_{ij} = \begin{cases} pr = (\xi_n = 2 - j), \ i \le 0 & \text{replenishment} \\ pr = (\xi_n = i - j), \ 0<i \le 2 & \text{without replenishment} \end{cases}$$

Q2: [3+6]

a)

$P_{00}^0 = \Pr\{X_0 = 0 \mid X_0 = 0\} = 1$

$P_{00}^1 = \Pr\{X_1 = 0 \mid X_0 = 0\} = 0$

$$P_{00}^2 = \Pr\{X_2 = 0 \mid X_0 = 0\} = [0 \quad 0.5 \quad 0.5]\begin{bmatrix} 0 \\ 0.5 \\ 0.5 \end{bmatrix} = 0.5$$

b)

$$\mathbf{P}= \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left\| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0.1 & 0.4 & 0.1 & 0.4 \\ 0.2 & 0.1 & 0.6 & 0.1 \\ 0 & 0 & 0 & 1 \end{array} \right\| \end{array}$$

$u_i = pr\{X_T = 0 | X_0 = i\}$ for i=1,2,

$and \;\; v_i$ =E[$T | X_0 = i$] for i=1,2.

(a)

$$u_1 = p_{10} + p_{11}u_1 + p_{12}u_2$$
$$u_2 = p_{20} + p_{21}u_1 + p_{22}u_2$$

$\Rightarrow$

$$u_1 = 0.1 + 0.4u_1 + 0.1u_2$$
$$u_2 = 0.2 + 0.1u_1 + 0.6u_2$$

$\Rightarrow$

$$6u_1 - u_2 = 1 \qquad (1)$$
$$u_1 - 4u_2 = -2 \qquad (2)$$

Solving (1) and (2), we get

$u_1 = \frac{6}{23}$ and $u_2 = \frac{13}{23}$

Starting in state 2, the probability that the Markov chain ends in state 0 is

$$u_2 = u_{20} = \frac{13}{23}$$

(b) Also, the mean time to absorption can be found as follows

$$v_1 = 1 + p_{11}v_1 + p_{12}v_2$$
$$v_2 = 1 + p_{21}v_1 + p_{22}v_2$$

$\Rightarrow$

$v_1 = 1 + 0.4v_1 + 0.1v_2$
$v_2 = 1 + 0.1v_1 + 0.6v_2$

$\Rightarrow$

$6v_1 - v_2 = 10 \qquad (1)$
$v_1 - 4v_2 = -10 \qquad (2)$

Solving (1) and (2), we get $v_2 = v_{20} = \frac{70}{23}$

Q3: [7]

We have $f_X(z) = \sum_{n=1}^{\infty} f^n(z) P_N(n)$

$\because$ The $n-fold$ convolution of $f(z)$ is the Gamma density function, $n \geq 1$

$$\therefore \quad f^n(z) = \begin{cases} \frac{\lambda^n}{\Gamma(n)} z^{n-1} e^{-\lambda z} & z \geq 0 \\ 0 & z < 0 \end{cases}$$

$\Rightarrow$

$$f^n(z) = \begin{cases} \frac{\lambda^n}{(n-1)!} z^{n-1} e^{-\lambda z} & z \geq 0 \\ 0 & z < 0 \end{cases}$$

$$\therefore \ f_X(z) = \lambda\beta e^{-\lambda z} \sum_{n=1}^{\infty} \frac{[\lambda(1-\beta)z]^{n-1}}{(n-1)!}$$
$$= \lambda\beta e^{-\lambda z}.e^{\lambda(1-\beta)z}$$
$$= \lambda\beta e^{-\lambda\beta z}, \quad z \geq 0$$

$\therefore$ X has an exponential distribution with parameter $\lambda\beta$.