First Mid Term, S2-1442H
M 380 – Stochastic Processes
Time: 90 minutes

Answer the following questions:

Q1: [5+3]

a) Let $X=\begin{cases} 0 & \text{if } N=0 \\ \xi_1+\xi_2+\ \dots\ +\xi_N & \text{if } N>0 \end{cases}$ be a random sum and assume that $E(\xi_k)=\mu$, $E(N)=\upsilon$ and $Var(\xi_k)=\sigma^2$, $Var(N)=\tau^2$

Prove that $E(X)=\mu\upsilon$ and $Var(X)=\upsilon\sigma^2+\mu^2\tau^2$

b) Let us model the daily stock price change as $Z=\xi_0+\xi_1+\dots+\xi_N$, where $\xi_0,\xi_1,\dots,\xi_N$ are independent normally distributed random variables with common mean zero and variance 0.5, and N is the number of transactions during the day which has a Poisson distribution with mean 1.

(i) Determine the mean and variance of Z.

(ii) What is the distribution of Z?

Q2: [5+4]

a) Given the following joint distribution.

Y \ X	0	1
0	0.1	0.4
1	0.3	0.2

Compute $\rho(X,Y)$.

b) Let X have a binomial distribution with parameters p and N, where N has a binomial distribution with parameters q and M. What is the marginal distribution of X?

Q3: [1.5+1.5+5]

a) If $T \sim \exp(\lambda)$ prove that: $pr(T > t + s | T > s) = pr(T > t) \quad \forall\ t,\ s \geq 0$

b) The lifetime T of a certain component has an exponential distribution with parameter λ=0.02. Find pr$(T \leq 130 | T > 100)$

c) Let X have a probability density function defined as

$$f(\mathrm{x}) = \begin{cases} \mathrm{cx}^2 & \text{for } 0 \leq \mathrm{x} \leq 3 \\ 0 & \text{elsewhere} \end{cases}$$

(i) Find the value of c

(ii) Determine the distribution function, mean, variance, and median.

The Model Answer

Q1: [5+3]

a)

(i) To prove that $E(X)=\mu\upsilon$

$\because\ E(X)=\sum_{n=0}^{\infty} E[X|N=n]P_N(n)$ Def. of Total Expectation

$\therefore\ E(X)=\sum_{n=1}^{\infty} E[\xi_1+\xi_2+\ ...\ +\xi_N|N=n]P_N(n)$ Def. of Random Sum

$\therefore\ E(X)=\sum_{n=1}^{\infty} E[\xi_1+\xi_2+\ ...\ +\xi_n|N=n]P_N(n)$ Prop. of Conditional Expectation

$\therefore\ E(X)=\sum_{n=1}^{\infty} E[\xi_1+\xi_2+\ ...\ +\xi_n]P_N(n)$ where N is independent of $\xi_1,\ \xi_2,\ ...$

$\because\ E(\xi_k)=\mu,\quad k=1,2,\ ...\ ,n$

$\therefore\ E(X)=\sum_{n=1}^{\infty} n\mu P_N(n)$

$\therefore\ E(X)=\mu\sum_{n=1}^{\infty} nP_N(n)$

$\therefore\ E(X)=\mu E(N)=\mu\upsilon$

(ii) To prove that $Var(X)=\upsilon\sigma^2+\mu^2\tau^2$

$$Var(X)=E[(X-\mu\upsilon)^2]$$
$$=E[X-N\mu+N\mu-\upsilon\mu]^2$$
$$Var(X)=E[(X-N\mu)^2]+E[\mu^2(N-v)^2]+2E[\mu(X-N\mu)(N-v)] \qquad (1)$$

$$\because E[(X-N\mu)^2]=\sum_{n=0}^{\infty} E[(X-N\mu)^2|N=n]P_N(n)$$
$$=\sum_{n=1}^{\infty} E[(\xi_1+\xi_2+\ ...\ +\xi_n-n\mu)^2|N=n]P_N(n)$$
$$\therefore E[(X-N\mu)^2]=\sum_{n=1}^{\infty} E(\xi_1+\xi_2+\ ...\ +\xi_n-n\mu)^2]P_N(n)$$

$\because \mathrm{Var}(\xi_k) = \mathrm{E}(\xi_k - \mu)^2 = \sigma^2, \quad k = 1, 2, ..., n$

$$\therefore \mathrm{E}[(\mathrm{X} - N\mu)^2] = \sum_{n=1}^{\infty} n\sigma^2 P_N(n)$$

$$= \sigma^2 \sum_{n=1}^{\infty} nP_N(n)$$

$$\therefore \mathrm{E}[(\mathrm{X} - N\mu)^2] = \upsilon\sigma^2, \text{ where } \sum_{n=1}^{\infty} nP_N(n) = \upsilon \qquad (2)$$

$$\mathrm{E}[\mu^2 (N - v)^2] = \mu^2 \mathrm{E}[(N - v)^2]$$

$$\therefore \mathrm{E}[\mu^2 (N - v)^2] = \mu^2 Var(N) = \mu^2 \tau^2 \qquad (3)$$

Also,

$$\mathrm{E}[\mu(\mathrm{X} - N\mu)(N - v)] = \mu \sum_{n=1}^{\infty} \mathrm{E}[(\mathrm{X} - n\mu)(n - \upsilon) | N = n] P_N(n)$$

$$= \mu \sum_{n=1}^{\infty} (n - \upsilon) \mathrm{E}[(\mathrm{X} - n\mu) | N = n] P_N(n)$$

$$= 0 \qquad (4)$$

where $\mathrm{E}[(\mathrm{X} - n\mu) | N = n] = \mathrm{E}(\mathrm{X} - n\mu)$ independent prop.

$$= \mathrm{E}(\xi_1 + \xi_2 + \ ... + \xi_n - n\mu)$$

$$= n\mu - n\mu = 0$$

Substitute (2), (3) and (4) in (1), we get

$\mathrm{Var}(\mathrm{X}) = \upsilon\sigma^2 + \mu^2\tau^2$

b)

$E(\xi_k) = \mu = 0, \ \mathrm{Var}(\xi_k) = \sigma^2 = 0.5$

$E(N) = v = 1, \quad \mathrm{Var}(N) = \tau^2 = 1$

$\because Z = \xi_0 + \xi_1 + ... + \xi_N$

$\therefore E(Z) = \mu(v+1) = 0(2) = 0$ and

$\mathrm{Var}(Z) = (v+1)\sigma^2 + \mu^2\tau^2 = 2(0.5) = 1$

$\Rightarrow \ Z \sim N(0,1)$

$\therefore \ Z$ has the standard normal distribution.

Q2: [5+4]

a)

Y \ X	0	1	$P_Y(y)$
0	0.1	0.4	0.5
1	0.3	0.2	0.5
$P_X(x)$	0.4	0.6	Sum=1

$$E(X)=0.6,\ E(X^2)=0.6,\ \ Var(X)=0.24$$
$$E(Y)=0.5,\ E(Y^2)=0.5,\ \ Var(Y)=0.25$$
$$E(XY)=0.2,\ Cov(X,Y)=-0.10,\ \ \rho(X,Y)=-0.4$$

b)

$\because\ X \sim \ \text{Bin}(p,N)$ and $N \sim \ \text{Bin}(q,M)$,

We have the conditional prob. mass function

$$p_{X|N}(x|n)=\binom{n}{x}p^{x}(1-p)^{n-x},\ \ x=0,1,2,\ldots,n$$

and the marginal prob. mass function

$$p_N(n)=\binom{M}{n}q^{n}(1-q)^{M-n},\ \ n=0,1,2,\ldots,M$$

Apply the law of total prob. as follows

$$\begin{aligned}pr(X=x)&=\sum_{n=0}^{M}p_{X|N}(x|n)\,p_N(n)\\&=\sum_{n=x}^{M}\frac{n!}{x!(n-x)!}p^{x}(1-p)^{n-x}.\frac{M!}{n!(M-n)!}q^{n}(1-q)^{M-n}\end{aligned}$$

$$=\frac{M!\,(pq)^{x}(1-q)^{M-x}}{x!(M-x)!}\sum_{n=x}^{M}\binom{M-x}{n-x}(1-p)^{n-x}\left(\frac{q}{1-q}\right)^{n-x}$$

$$=\frac{M!\,(pq)^{x}(1-q)^{M-x}}{x!(M-x)!}\sum_{n=x}^{M}\binom{M-x}{n-x}\left(\frac{q(1-p)}{1-q}\right)^{n-x}$$

$$=\frac{M!\,(pq)^{x}(1-q)^{M-x}}{x!(M-x)!}\sum_{r=0}^{M-x}\binom{M-x}{r}1^{M-x-r}\left(\frac{q(1-p)}{1-q}\right)^{r},\quad r=n-x$$

$$\therefore pr(X=x)=\frac{M!\,(pq)^{x}(1-q)^{M-x}}{x!(M-x)!}\left(1+\frac{q(1-p)}{1-q}\right)^{M-x}$$

$$=\frac{M!\,(pq)^{x}(1-pq)^{M-x}}{x!(M-x)!}$$

$\therefore pr(X=x)=\binom{M}{x}(pq)^{x}(1-pq)^{M-x},\ x=0,1,2,\ldots,M$ is the marginal prob. mass function of X

$\therefore\ X\sim \mathrm{Bin}(M,pq)$

Q3: [1.5+1.5+5]

a) If $T\sim\exp(\lambda)$ prove that: $pr(T>t+s|T>s)=pr(T>t)\quad \forall\ t,\ s\geq 0$

Proof:

$$pr(T>t+s|T>s)=\frac{pr(T>t+s,T>s)}{pr(T>s)}$$

$$=\frac{pr(T>t+s)}{pr(T>s)}$$

$\because\ T\sim\ \exp(\lambda)$

$$\therefore\ pr(T>t+s|T>s)=\frac{e^{-\lambda(t+s)}}{e^{-\lambda s}}$$

$$=e^{-\lambda t}=R(t)$$

$$=pr(T>t)$$

b)

$$
\begin{aligned}
pr(T \le 130 | T > 100) &= 1 - pr(T > 130 | T > 100) \\
&= 1 - pr(T > 30 + 100 | T > 100) \\
&= 1 - pr(T > 30) \\
&= pr(T \le 30) \\
&= 1 - e^{-\lambda t} \\
&= 1 - e^{-0.02(30)}
\end{aligned}
$$

$$\therefore pr(T \le 130 | T > 100) = 1 - e^{-0.6} \simeq 0.45$$

c)

(i)

$$\because \int_0^3 cx^2 dx = 1$$

$$\therefore \left[\frac{cx^3}{3} \right]_0^3 = 1$$

$$\therefore c = \frac{1}{9}$$

$$\Rightarrow f(x) = \frac{1}{9} x^2, \ 0 \le x \le 3$$

(ii) Mean is

$$\mu = \int_0^3 \frac{1}{9} x^3 dx$$

$$= \frac{1}{9} \left[\frac{x^4}{4} \right]_0^3 = 2.25$$

To get variance

$$E(X^2) = \int_0^3 \frac{1}{9} x^4 dx$$

$$= \frac{1}{9} \left[\frac{x^5}{5} \right]_0^3 = 5.4$$

$\Rightarrow$

$$\sigma^2 = E(X^2) - \mu^2$$
$$= 5.4 - 2.25^2 = 0.3375$$

The distribution function (CDF) is given by

$$F(x) = \int_{-\infty}^{x} f(t)\, dt$$
$$= \int_{0}^{x} \frac{1}{9} t^2 dt$$
$$= \frac{1}{27} x^3, \ 0 \le x \le 3$$

To get the median m, solve the equation

$$F(m) = 0.5$$
$$\Rightarrow \ m^3 = \frac{27}{2}$$
$$\therefore \quad m = \frac{3}{\sqrt[3]{2}} \approx 2.3811$$
