King Saud University
College of Sciences
Department of Mathematics

Final Exam, S1 1442
M 380 - Stochastic Processes
Time: 3 hours

Answer the following questions:

Q1: [5+4]

(a) Consider the Markov chain whose transition probability matrix is given by

$$P=\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left\|\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0.5 & 0.3 & 0.1 & 0.1 \\ 0.1 & 0.3 & 0.4 & 0.2 \\ 0 & 0 & 0 & 1 \end{array}\right\| \end{array}$$

(i) Starting in state 1, determine the probability that the Markov chain ends in state 0.

(ii) Determine the mean time to absorption.

(b) Demands on a first aid facility in a certain location occur according to a nonhomogeneous Poisson process having the rate function

$$\lambda(t)=\begin{cases} t & \text{for } 0 \le t<2 \\ 1 & \text{for } 2 \le t<3 \\ 5-t & \text{for } 3 \le t \le 6 \end{cases}$$

where t is measured in hours from the opening time of the facility. What is the probability that one demand occurs in the first 3h of operation and two in the second 3h?

Q2: [5+4]

(a) For the Markov process $\{X_t\}$, t=0,1,2,...,n with states $i_0, i_1, i_2, \ldots, i_{n-1}, i_n$

Prove that: $\Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \ldots, X_n = i_n\} = p_{i_0} P_{i_0 i_1} P_{i_1 i_2} \ldots P_{i_{n-1} i_n}$ where $p_{i_0} = \text{pr}\{X_0 = i_0\}$

(b) Consider the problem of sending a binary message, 0 or 1, through a signal channel consisting of several stages, where transmission through each stage is subject to a fixed probability of error α. Suppose that $X_0 = 0$ is the signal that is sent and let X_n be the signal that is received at the nth stage. Assume that $\{X_n\}$ is a Markov chain with transition probabilities $P_{00} = P_{11} = 1-\alpha$ and $P_{01} = P_{10} = \alpha$, where $0 < \alpha < 1$.

i) Determine $\Pr\{X_0 = 0, X_1 = 0, X_2 = 0\}$, the probability that no error occurs up to stage $n = 2$.
ii) Determine the probability that a correct signal is received at stage 2.

Q3: [5+5]

(a) A pure death process starting from $X(0) = 3$ has death parameters $\mu_0 = 0,\ \mu_1 = 2,\ \mu_2 = 3$ and $\mu_3 = 4$. Determine $P_n(t)$ for $n = 0, 1, 2, 3$.

(b) Let $X(t)$ be a Yule process that is observed at a random time U, where U is uniformly distributed over [0,1]. Show that $pr\{X(U) = k\} = p^k / (\beta k)$ for $k = 1, 2, \ldots$, with $p = 1 - e^{-\beta}$.

Q4: [6]

An airline reservation system has two computers, only one of which is in operation at any given time. A computer may break down on any given day with probability p. There is a duplicate repair facility that takes 2 days to restore a computer to normal. The facilities are such that both two computers can be repaired simultaneously. Form a Markov chain by taking as states the pairs (x,y), where x is the number of machines in operating condition at the end of a day and y is 1 if a day's labor has been expended on a machine not yet repaired and 0 otherwise. Also, find the system availability.

Q5: [6]

Suppose that the summands $\xi_1, \xi_2, \ldots$ are continuous random variables having a probability density function $f(z) = \begin{cases} \lambda e^{-\lambda z} & \text{for } z \geq 0 \\ 0 & \text{for } z < 0 \end{cases}$

and $P_N(n) = \beta(1-\beta)^{n-1}$ for $n = 1, 2, \ldots$

Find the probability density function for $X = \xi_1 + \xi_2 + \ldots + \xi_N$

Model Answer

Q1: [5+4]

(a)

$$\mathbf{P}= \begin{array}{c|cccc|} & 0 & 1 & 2 & 3 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0.5 & 0.3 & 0.1 & 0.1 \\ 2 & 0.1 & 0.3 & 0.4 & 0.2 \\ 3 & 0 & 0 & 0 & 1 \end{array}$$

$u_i = pr\{X_T = 0 | X_0 = i\}$ for i=1,2,

and v_i =E[$T | X_0 = i$] for i=1,2.

(i)

$$u_1 = p_{10} + p_{11}u_1 + p_{12}u_2$$
$$u_2 = p_{20} + p_{21}u_1 + p_{22}u_2$$

$\Rightarrow$

$$u_1 = 0.5 + 0.3u_1 + 0.1u_2$$
$$u_2 = 0.1 + 0.3u_1 + 0.4u_2$$

$\Rightarrow$

$$7u_1 - u_2 = 5 \qquad (1)$$
$$3u_1 - 6u_2 = -1 \qquad (2)$$

Solving (1) and (2), we get

$$u_1 = \tfrac{31}{39} \text{ and } u_2 = \tfrac{22}{39}$$

Starting in state 1, the probability that the Markov chain ends in state 0 is

$$u_1 = u_{10} = \tfrac{31}{39} = 0.7949$$

(ii) Also, the mean time to absorption can be found as follows

$$v_1 = 1 + p_{11}v_1 + p_{12}v_2$$
$$v_2 = 1 + p_{21}v_1 + p_{22}v_2$$

$\Rightarrow$

$$v_1 = 1 + 0.3v_1 + 0.1v_2$$
$$v_2 = 1 + 0.3v_1 + 0.4v_2$$

$\Rightarrow$

$$7v_1 - v_2 = 10 \qquad (1)$$
$$3v_1 - 6v_2 = -10 \qquad (2)$$

Solving (1) and (2), we get

$v_1 = \frac{70}{39}$ and $v_2 = \frac{100}{39}$

$v_1 = v_{10} = \frac{70}{39} = 1.7949$

(b)

i)

$$\mu_1 = \int_0^3 \lambda(u)du$$
$$= \int_0^2 t dt + \int_2^3 1 dt$$
$$= \left[\frac{t^2}{2}\right]_0^2 + [t]_2^3$$
$$= 2 + 1 = 3$$

The prob. that one demand occurs in the first 3h of operation is

$$\Pr\{X(3) = 1\} = \Pr\{X(3) - X(0) = 1\}$$
$$= \frac{e^{-\mu_1}\mu_1^k}{k!}$$
$$= \frac{e^{-3} \times 3^1}{1!} = 0.1494$$
$$\approx 0.15$$

ii)

$$\mu_2 = \int_3^6 \lambda(u)du$$

$$= \int_3^6 (5-t)dt$$

$$= \left[5t - \frac{t^2}{2}\right]_3^6$$

$$=12\text{-}10.5=1.5$$

The prob. that two demands occur in the second 3h of operation is

$$\Pr\{X(6)-X(3)=2\}$$

$$= \frac{e^{-\mu_2}\mu_2^k}{k!}$$

$$= \frac{e^{-1.5}\times 1.5^2}{2!}$$

$$\approx 0.25$$

Q2: [5+4]

(a)

$$\because \Pr\{X_0=i_0, X_1=i_1, X_2=i_2, \ldots, X_n=i_n\}$$

$$= \Pr\{X_0=i_0, X_1=i_1, X_2=i_2, \ldots, X_{n-1}=i_{n-1}\}.\Pr\{X_n=i_n | X_0=i_0, X_1=i_1, X_2=i_2, \ldots, X_{n-1}=i_{n-1}\}$$

$$= \Pr\{X_0=i_0, X_1=i_1, X_2=i_2, \ldots, X_{n-1}=i_{n-1}\}.P_{i_{n-1}i_n}$$ Definition of Markov

By repeating this argument $n-1$ times

$$\therefore \Pr\{X_0=i_0, X_1=i_1, X_2=i_2, \ldots, X_n=i_n\}$$

$= p_{i_0} P_{i_0 i_1} P_{i_1 i_2} \ldots P_{i_{n-2} i_{n-1}} P_{i_{n-1} i_n}$ where $p_{i_0} = \Pr\{X_0 = i_0\}$ is obtained from the initial distribution of the process.

(b)

i)

$$p_0 = pr(X_0 = 0) = 1$$

$$\Pr\{X_0=0, X_1=0, X_2=0\} = p_0 P_{00} P_{00}$$

$$=1\times(1\text{-}\alpha)\times(1\text{-}\alpha)$$

$$=(1\text{-}\alpha)^2$$

ii)

$$\Pr\left\{X_0=0, X_1=0, X_2=0\right\}+\Pr\left\{X_0=0, X_1=1, X_2=0\right\}$$
$$=p_0P_{00}P_{00}+p_0P_{01}P_{10}$$
$$=(1-\alpha)^2+\alpha^2$$
$$=1-2\alpha+2\alpha^2$$

Q3: [5+5]

(a) The transition probabilities are given by

$$p_N(t)=e^{-\mu_N t} \qquad (1)$$

and for $n<N$

$$p_n(t)=pr\left\{X(t)=n \middle| X(0)=N\right\}$$
$$=\mu_{n+1}\mu_{n+2}\ \dots\ \mu_N\left[A_{n,n}e^{-\mu_n t}+\ \dots +A_{k,n}e^{-\mu_k t}+\ \dots +A_{N,n}e^{-\mu_N t}\right] \quad (2)$$

where $A_{k,n}=\prod_{i=N}^{n}\frac{1}{(\mu_i-\mu_k)}\ ,\quad i\neq k,\ n\leq k\leq N,\ i=N,N\text{-}1,\ \dots,\ n \quad (3)$

For N=3 $\quad (1)\Rightarrow \text{p}_3\text{(t)=e}^{-\mu_3 t}$

$\therefore\ \text{p}_3\text{(t)=e}^{-4t} \qquad$ (I)

For n=2 $\quad (2)\Rightarrow \text{p}_2\text{(t)=}\mu_3\left[A_{2,2}e^{-\mu_2 t}+A_{3,2}e^{-\mu_3 t}\right]$

$$(3)\Rightarrow\ A_{2,2}=\prod_{i=3}^{2}\frac{1}{(\mu_i-\mu_2)}\ ,\ i\neq 2$$
$$=\frac{1}{\mu_3-\mu_2}=1,$$

$$A_{3,2}=\prod_{i=3}^{2}\frac{1}{(\mu_i\text{ - }\mu_3)},\ i\neq 3$$
$$=\frac{1}{\mu_2-\mu_3}=-1$$

$\therefore\ \text{p}_2\text{(t)=4}\left[e^{-3t}-e^{-4t}\right] \qquad$ (II)

For n=1 $\quad (2)\Rightarrow \text{p}_1\text{(t)=}\mu_2\mu_3\left[A_{1,1}e^{-\mu_1 t}+A_{2,1}e^{-\mu_2 t}+A_{3,1}e^{-\mu_3 t}\right]$

$$(3) \Rightarrow \; A_{1,1} = \prod_{i=3}^{1} \frac{1}{(\mu_i - \mu_1)} \; , \; i \neq 1$$

$$= \frac{1}{(\mu_3 - \mu_1)(\mu_2 - \mu_1)} = \frac{1}{2},$$

$$A_{2,1} = \prod_{i=3}^{1} \frac{1}{(\mu_i - \mu_2)} \; , \; i \neq 2$$

$$= \frac{1}{(\mu_3 - \mu_2)(\mu_1 - \mu_2)} = -1,$$

$$A_{3,1} = \prod_{i=3}^{1} \frac{1}{(\mu_i - \mu_3)} \; , \; i \neq 3$$

$$= \frac{1}{(\mu_2 - \mu_3)(\mu_1 - \mu_3)} = \frac{1}{2}$$

$$\mathrm{p}_1(\mathrm{t}) = 12\left[\frac{1}{2}e^{-2t} - e^{-3t} + \frac{1}{2}e^{-4t}\right]$$

$$\therefore \; \mathrm{p}_1(\mathrm{t}) \; = 6\left[e^{-2t} - 2e^{-3t} + e^{-4t}\right] \qquad \text{(III)}$$

Using (I), (II) and (III) we can get $\mathrm{p}_0(\mathrm{t})$ as follows

$$\therefore \; \mathrm{p}_0(\mathrm{t}) = 1 - \left[\mathrm{p}_1(\mathrm{t}) + \mathrm{p}_2(\mathrm{t}) + \mathrm{p}_3(\mathrm{t})\right]$$

$$= 1 - \left[6e^{-2t} - 12e^{-3t} + 6e^{-4t} + 4e^{-3t} - 4e^{-4t} + \mathrm{e}^{-4t}\right])$$

$$= 1 - 6e^{-2t} + 8e^{-3t} - 3e^{-4t} \qquad \text{(IV)}$$

(b) For Yule process,

$$p_n(t) = e^{-\beta t}(1-e^{-\beta t})^{n-1}, \quad n \geq 1$$

$\Rightarrow$

$$\because pr\{X(U)=k\} = \int_0^1 e^{-\beta u}(1-e^{-\beta u})^{k-1} du$$

$$= \frac{1}{\beta}\int_0^1 (1-e^{-\beta u})^{k-1}.\ \beta e^{-\beta u} du$$

$$= \frac{1}{\beta}\left[\frac{(1-e^{-\beta u})^k}{k}\right]_0^1$$

$$= \frac{1}{\beta k}\left[(1-e^{-\beta})^k\right]$$

$\therefore pr\{X(U)=k\} = \dfrac{p^k}{\beta k}$, $k = 1, 2, \ ...$ where $p = 1-e^{-\beta}$

Q4: [6]

(a)

$$\begin{array}{c} \\ (2,0) \\ (1,0) \\ (1,1) \\ (0,1) \end{array} \begin{array}{c} \begin{array}{cccc} (2,0) & (1,0) & (1,1) & (0,1) \end{array} \\ \left\| \begin{array}{cccc} q & p & 0 & 0 \\ 0 & 0 & q & p \\ q & p & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right\| \end{array}$$

In the long run, the limiting distribution is $\pi = (\pi_0, \pi_1, \pi_2, \pi_3)$

$$q\pi_0 + q\pi_2 = \pi_0 \Rightarrow \pi_2 = \tfrac{p}{q}\pi_0, \ \ p = 1-q \qquad (1)$$

$$p\pi_0 + p\pi_2 = \pi_1 \Rightarrow \pi_1 = p\pi_0 + \tfrac{p^2}{q}\pi_0$$

$$= \tfrac{pq+p^2}{q}\pi_0$$

$$= \tfrac{p(q+p)}{q}\pi_0$$

$$= \tfrac{p}{q}\pi_0, \ \ p+q=1 \qquad (2)$$

Also, $p\pi_1 = \pi_3$

$$\therefore \ \pi_3 = \tfrac{p^2}{q}\pi_0 \qquad (3)$$

And $\because \ \pi_0 + \pi_1 + \pi_2 + \pi_3 = 1 \qquad (4)$

$\therefore$ By substituting (1), (2), (3) in (4)

$$\Rightarrow \pi_0 + \tfrac{p}{q}\pi_0 + \tfrac{p}{q}\pi_0 + \tfrac{p^2}{q}\pi_0 = 1$$

$$\Rightarrow \pi_0(\tfrac{1+p+p^2}{q}) = 1, \ p+q=1$$

$$\therefore \quad \pi_0 = \frac{q}{1+p+p^2} \qquad (5)$$

$$(1),\ (2) \Rightarrow \pi_1 = \pi_2 = \tfrac{p}{q}\pi_0$$

$$= \frac{p}{1+p+p^2} \qquad (6)$$

$$(3) \Rightarrow \pi_3 = \frac{p^2}{1+p+p^2} \qquad (7)$$

$\therefore$ The limiting distribution is determined by equations (5), (6) and (7).

The availability for two repair facilities is

$$\text{Ava} = 1 - \pi_3$$

$$= 1 - \frac{p^2}{1+p+p^2}$$

$$= \frac{1+p}{1+p+p^2} \qquad (8)$$

Q5: [6]

We have $f_X(z) = \sum_{n=1}^{\infty} f^n(z) P_N(n)$

$\because$ The $n-fold$ convolution of $f(z)$ is the Gamma density function, $n \geq 1$

$$\therefore \quad f^n(z) = \begin{cases} \dfrac{\lambda^n}{\Gamma(n)} z^{n-1} e^{-\lambda z} & z \geq 0 \\ 0 & z < 0 \end{cases}$$

$\Rightarrow$

$$f^n(z) = \begin{cases} \dfrac{\lambda^n}{(n-1)!} z^{n-1} e^{-\lambda z} & z \geq 0 \\ 0 & z < 0 \end{cases}$$

and $\because$ $P_N(n) = \beta(1-\beta)^{n-1}$ for $n = 1, 2, \ \ldots$

$$\therefore\ f_X(z) = \lambda\beta e^{-\lambda z}\sum_{n=1}^{\infty}\frac{[\lambda(1-\beta)z]^{n-1}}{(n-1)!}$$

$$=\lambda\beta e^{-\lambda z}.e^{\lambda(1-\beta)z}$$

$$=\lambda\beta e^{-\lambda\beta z}, \quad z \geq 0$$

$\therefore\ X$ has an exponential distribution with parameter $\lambda\beta$.