First Semester	**Second Exam**	**King Saud University**
(without calculators)	**Time allowed: 90 m**	**College of Science**
Monday 12-5-1447	**240 Math**	**Math. Department**

Q1: **(a)** Show that $\text{span}\{(3,3,5)\} < \text{span}\{(1,1,2), (2,2,3)\}$. (4 marks)
(b) Let $V=F(-\infty,\infty)$ and W is the set of all functions in V such that $f(1)=0$ for every f in W. Prove that W is a subspace of V. (3 marks)

Q2: **(a)** Use the Wronskian to show that $1, e^x, \sin(x)$ are linearly independent in the vector space $C^2(-\infty,\infty)$. (3 marks)

(b) show that the set $S=\{1+2x+2x^2, 2+x+x^2, 1+x\}$ forms a basis for P_2 and then find the vector w whereas $(w)_S=(1,2,3)$. (4 marks)

Q3: **(a)** Let $B=\{(1,3),(2,7)\}$ and $B'=\{u_1,u_2\}$ be two bases of $\mathbb{R}^2$. If the transition matrix from B' to B is $\begin{bmatrix} 5 & 14 \\ -2 & -6 \end{bmatrix}$, then find u_1. (2 marks).
(b) Find a subset of the row vectors of A that will be a basis for the row space of A:

$$A = \begin{bmatrix} 0 & 2 & 6 & -1 \\ 0 & 4 & 12 & -2 \\ 1 & 6 & 10 & 5 \end{bmatrix}$$

<u>Deduce</u> $\text{nullity}(A^T)$ without solving any linear system. (4 marks)
(c) Show that $\text{rank}(A)=\text{rank}(A^T)$ for any matrix A. (1 mark)
(d) If u and v are linearly independent, then show that ku and kv are linearly independent for some non-zero scalar k. (2 marks)

(e) Let $S=\{u_1,u_2,u_3,u_4,u_5\}$ be a subset of P_5. Is S a basis of P_5? Why? (1 mark)

(f) If $W=\text{span}\{u_1,u_2,u_3,u_4,u_5\}$, what can we say about $\dim(W)$? (1 mark)

Solutions:

A1(a): (3,3,5)=(1,1,2)+(2,2,3). So, span{(3,3,5)}≤span{(1,1,2), (2,2,3)}. But (1,1,2) is not a scalar multiple of (3,3,5), so they are linearly independent and hence span{(1,1,2), (2,2,3)} is not a sub space of span{(3,3,5)}.

Therefore, span{(3,3,5)}<span{(1,1,2), (2,2,3)}

A1(b): (1) Since 0(1)=0, so 0 is in W and W is not empty.

For all f and g in W and for all scalars k:

(2) (f+g)(1)=f(1)+g(1)=0+0=0. So, f+g belongs to W and W is closed under addition.

(3) (kf)(1)=kf(1)=k(0)=0. So, kf belongs to W and W is closed under scalar multiplication.

(1), (2) and (3) imply that W is a subspace of V.

A2(a):

$$W(x)=\begin{vmatrix}1 & e^x & \sin(x)\\ 0 & e^x & \cos(x)\\ 0 & e^x & -\sin(x)\end{vmatrix}=\begin{vmatrix}e^x & \cos(x)\\ e^x & -\sin(x)\end{vmatrix}=-e^x\sin(x)-e^x\cos(x)$$

$$W(0)=-e^0\sin(0)-e^0\cos(0)=0-1=-1\neq 0$$

So 1, e^x, sin(x) are linearly independent.

A2(b):

$$\begin{vmatrix}1 & 2 & 1\\ 2 & 1 & 1\\ 2 & 1 & 0\end{vmatrix}\underset{(-2)R13}{\overset{(-2)R_{12}}{=}}\begin{vmatrix}1 & 2 & 1\\ 0 & -3 & -1\\ 0 & -3 & -2\end{vmatrix}=\begin{vmatrix}-3 & -1\\ -3 & -2\end{vmatrix}=6-3=3\neq 0$$

So the set S={$1+2x+2x^2$,$2+x+x^2$,1+x} forms a basis for P_2. Now,

$$\begin{aligned}w&=1(1+2x+2x^2)+2(2+x+x^2)+3(1+x)\\&=1+2x+2x^2+4+2x+2x^2+3+3x=8+7x+4x^2\end{aligned}$$

A3(a): $[u_1]_B=\begin{bmatrix}5\\-2\end{bmatrix}$. So, u_1=5(1,3)−2(2,7)=(5,15)−(4,14)=(1,1).

A3(b):

$$A^T=\begin{bmatrix}0&0&1\\2&4&6\\6&12&10\\-1&-2&5\end{bmatrix}\xrightarrow{R_{12}}\begin{bmatrix}2&4&6\\0&0&1\\6&12&10\\-1&-2&5\end{bmatrix}\xrightarrow{\frac{1}{2}R_1}\begin{bmatrix}1&2&3\\0&0&1\\6&12&10\\-1&-2&5\end{bmatrix}$$

$$\xrightarrow[1R_{14}]{(-6)R_{13}}\begin{bmatrix}1&2&3\\0&0&1\\0&0&-8\\0&0&8\end{bmatrix}\xrightarrow{1R_{34}}\begin{bmatrix}1&2&3\\0&0&1\\0&0&-8\\0&0&0\end{bmatrix}\xrightarrow{(-1/8)R_3}\begin{bmatrix}1&2&3\\0&0&1\\0&0&1\\0&0&0\end{bmatrix}$$

$$\xrightarrow{(-1)R_{23}}\begin{bmatrix}1&2&3\\0&0&1\\0&0&0\\0&0&0\end{bmatrix}$$

Using the leading ones, {[0 2 6 -1]^T, [1 6 10 5]^T} is a basis of col(A^T). Hence, {[0 2 6 -1], [1 6 10 5] } is a basis of row(A)

Now, rank(A^T)+nullity(A^T)=m

So nullity(A^T)=m- rank(A^T)=3-2=1

A3(c): rank(A)=dim(row(A))=dim(col(A^T))=rank(A^T).

A3(d): Observe that:

$$\begin{aligned}&a(ku)+b(kv)=0\\&\Rightarrow k(au+bv)=0\\&k\neq 0\Rightarrow au+bv=0\\&L.I.\Rightarrow a=0\,\&\,b=0\end{aligned}$$

So, ku and kv are linearly independent.

A3(e) No, since |S|=5<6=dim(P_5).

A3(f) dim(W)≤5.