Second Semester	**Second Exam**	**King Saud University**
(without calculators)	**Time allowed: 1 h and 30 m**	**College of Science**
Monday 9-10-1446	**240 Math**	**Math. Department**

Q1: **(a)** Show that the vector (1,1,1) is not generated by S={(1,2,2), (2,4,8)}. (3 marks)

(b) Let V=F($-\infty,\infty$) and W is the set of all functions in V such that f(1)=1 for every f in W. Prove that W is **<u>not</u>** a subspace of V. (2 marks)

Q2: **(a)** Use the Wronskian to show that 1, x, e^x are linearly independent in the vector space $C^2(-\infty,\infty)$. (3 marks)

(b) The set S={$1+x+2x^2, 2+x+x^2, 1+x$} forms a basis for P_2. Find the vector w whereas $(w)_S=(1,2,3)$. (2 marks)

Q3: **(a)** Let B={(1,3),(2,7)} and B'={(1,1),(2,0)} be two bases of $\mathbb{R}^2$. Find the transition matrix from B' to B. (3 marks).

(b) Find a basis for the column space of the matrix:

$$A = \begin{bmatrix} 1 & 2 & 6 & -1 \\ 2 & 4 & 13 & -1 \\ 3 & 6 & 26 & 5 \end{bmatrix}$$

and **<u>deduce</u>** nullity(A^T) without solving any linear system. (4 marks)

Q4: **(a)** Show that rank(A)=rank(A^T) for any matrix A. (1 mark)

(b) If S={$\mathbf{v_1},\mathbf{v_2},\ldots,\mathbf{v_n}$} is a basis for a vector space **V**, then prove that every vector **v** in **V** can be expressed in the form $\mathbf{v}=c_1\mathbf{v_1}+c_2\mathbf{v_2}+\ldots+c_n\mathbf{v_n}$ in exactly one way, where $c_1, c_2, \ldots, c_n$ are real numbers. (2 marks)

Q5: Choose the correct answer: (5 marks)

(i) Suppose that S={w_1,w_2,w_3} is a basis of a subspace W of $\mathbb{R}^4$. Then dim(W) is

(a) 4 (b) 3 (c) 2 (d) 1

(ii) Which of the following can be considered a transition matrix?

(a) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

(iii) Let P_2 be the vector space of all polynomials of degree ≤2 and S={v,u} a subset of P_2. If E is the set of all linear combinations of the vectors in S, Then the set E is:

(a) linearly independent (b) a basis of P_2 (c) spans P_2 (d) a vector space

(iv) Suppose {v_1, v_2, v_3} is a basis of $\mathbb{R}^3$. Then $\boldsymbol{\alpha}_1v_1+\boldsymbol{\alpha}_2v_2+\boldsymbol{\alpha}_3v_3=0$ implies that:

(a) $\boldsymbol{\alpha}_1=\boldsymbol{\alpha}_2=\boldsymbol{\alpha}_3=0$ (b) $v_1=v_2=v_3=0$ (c) $\boldsymbol{\alpha}_1=v_1, \boldsymbol{\alpha}_2=v_2, \boldsymbol{\alpha}_3=v_3$ (d) {v_1, v_2, v_3}= $\mathbb{R}^3$

(v) If W_1={(2,2,2)}, W_2={(1,1,1)}, W_3={(0,0,0)} and W_4={(1,0,0),(0,1,0),(0,0,1)} are four subsets of $\mathbb{R}^3$, which one of them is a subspace of $\mathbb{R}^3$?

(a) W_1 (b) W_2 (c) W_3 (d) W_4

Solutions:

A1(a):

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 1 \\ 2 & 8 & 1 \end{bmatrix} \xrightarrow[(-2)R_{13}]{(-2)R_{12}} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & -1 \\ 0 & 4 & -1 \end{bmatrix}$$

So, it has no solution and hence (1,1,1) is not generated by the set S={(1,2,2), (2,4,8)}.

A1(b): The zero vector 0 in V does not belong to W since 0(1)=0≠1.

A2(a):

$$W(x) = \begin{vmatrix} 1 & x & e^x \\ 0 & 1 & e^x \\ 0 & 0 & e^x \end{vmatrix} = e^x$$

$$W(0) = e^0 = 1 \neq 0$$

So 1, x, e^x are linearly independent.

A2(b):

Since the set S={$1+x+2x^2$,$2+x+x^2$,$1+x$} forms a basis for P_2, then

$$w = 1(1+x+2x^2) + 2(2+x+x^2) + 3(1+x)$$
$$= 1+x+2x^2+4+2x+2x^2+3+3x = 8+6x+4x^2$$

A3(a):

$$[B \mid B'] = \left[\begin{array}{cc|cc} 1 & 2 & 1 & 2 \\ 3 & 7 & 1 & 0 \end{array}\right] \xrightarrow{(-3)R_{12}} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 2 \\ 0 & 1 & -2 & -6 \end{array}\right]$$

$$\xrightarrow{(-2)R_{21}} \left[\begin{array}{cc|cc} 1 & 0 & 5 & 14 \\ 0 & 1 & -2 & -6 \end{array}\right] = [I \mid P_{B' \to B}]$$

$$P_{B' \to B} = \begin{bmatrix} 5 & 14 \\ -2 & -6 \end{bmatrix}$$

A3(b):

$$A=\begin{bmatrix}1 & 2 & 6 & -1\\ 2 & 4 & 13 & -1\\ 3 & 6 & 26 & 5\end{bmatrix}\xrightarrow[(-3)R_{13}]{(-2)R_{12}}\begin{bmatrix}1 & 2 & 1 & 2\\ 0 & 0 & 1 & 1\\ 0 & 0 & 8 & 8\end{bmatrix}\xrightarrow{(-8)R_{23}}\begin{bmatrix}1 & 2 & 1 & 2\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}$$

Using the leading ones, $\{[1\ \ 2\ \ 3]^T, [6\ \ 13\ \ 26]^T\}$ is a basis of col(A).

Now, rank(A)+nullity(A^T)=m

So nullity(A^T)=m- rank(A)=3-2=1

A4(a): rank(A)=dim(row(A))=dim(col(A^T))=rank(A^T).

A4(b): Suppose $v \in V$ has two expressions:

$v = c_1v_1 + c_2v_2 + \cdots + c_nv_n$ and $v = k_1v_1 + k_2v_2 + \cdots + k_nv_n$, so

$0 = (c_1-k_1)v_1 + (c_2-k_2)v_2 + \cdots + (c_n-k_n)v_n$

But $S = \{v_1, v_2, \ldots, v_n\}$ is a basis, so it is linearly independent. Thus,

$c_1-k_1 = c_2-k_2 = \ldots = c_n-k_n = 0$ and hence $c_i = k_i$ for all $i \in \{1,2,\ldots,n\}$ and hence v has exactly one expression.

A5

(i) b
(ii) d
(iii) d
(iv) a
(v) c