First Semester	**Second Exam**	**King Saud University**
(without calculators)	**Time allowed: 1 h and 30 m**	**College of Science**
Monday 25-4-1446	**240 Math**	**Math. Department**

Q1: **(a)** Show that the vector (1,3,4)$\notin$span{(1,2,2), (2,4,8)}. (3 marks)
(b) Let V=F($-\infty,\infty$) and W is the set of all functions in V such that f(1)=1 for every f in W. Prove that W is **not** a subspace of V. (2 marks)

Q2: **(a)** Use the Wronskian to show that 1, x, sin(x) are linearly independent in the vector space $C^2(-\infty,\infty)$. (3 marks)

(b) show that the set S={$1+x+2x^2, 2+x+x^2, 1+x$} forms a basis for P_2 and then find the vector w whereas $(w)_S=(1,2,3)$. (4 marks)

Q3: **(a)** Let B={(1,3),(2,7)} and B'={(1,1),(2,0)} be two bases of $\mathbb{R}^2$. Find the transition matrix from B' to B. (2 marks).
(b) Find a basis for the column space of the matrix:

$$A=\begin{bmatrix} 1 & 2 & 6 & -1 \\ 2 & 4 & 4 & 6 \\ 3 & 6 & 10 & 5 \end{bmatrix}$$

and **deduce** nullity(A^T) without solving any linear system. (4 marks)

Q4: **(a)** Let S={$\mathbf{v_1},\mathbf{v_2},\mathbf{v_3},\mathbf{v_4}$} be a basis for a vector space **V**. Suppose **u** is a vector in **V** such that

$$\mathbf{u}=|2A|\mathbf{v_1}+2|2A|\mathbf{v}_2+3|2A|\mathbf{v}_3+4|2A|\mathbf{v}_4$$

where A is the identity matrix of order 2. Find:

(i) $(\mathbf{u})_S$ (1 mark)

(ii) dim(V). (1 mark)

(b) If S={$\mathbf{v_1},\mathbf{v_2},\ldots,\mathbf{v_n}$} is a basis for a vector space **V**, then prove that every vector **v** in **V** can be expressed in the form $\mathbf{v}=c_1\mathbf{v_1}+c_2\mathbf{v_2}+\ldots+c_n\mathbf{v_n}$ in exactly one way, where $c_1, c_2, \ldots, c_n$ are real numbers. (2 marks)

(c) Show that rank(A)=rank(A^T) for any matrix A. (1 mark)

(d) If u and v are linearly independent, then show that u+v and u-v are linearly independent. (2 marks)

Solutions:

A1(a):

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 2 & 8 & 4 \end{bmatrix} \xrightarrow[(-2)R_{13}]{(-2)R_{12}} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 4 & 1 \end{bmatrix}$$

So, it has no solution and hence $(1,3,4) \notin \text{span}\{(1,2,2), (2,4,8)\}$.

A1(b): The zero vector 0 in V does not belong to W since $0(1)=0\neq 1$.

A2(a):

$$W(x) = \begin{vmatrix} 1 & x & \sin(x) \\ 0 & 1 & \cos(x) \\ 0 & 0 & -\sin(x) \end{vmatrix} = -\sin(x)$$

$$W(\tfrac{\pi}{2}) = -1 \neq 0$$

So 1, x, sin(x) are linearly independent.

A2(b):

$$\begin{vmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{vmatrix} \underset{(-2)R13}{\overset{(-1)R_{12}}{=}} \begin{vmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & -3 & -2 \end{vmatrix} = 1(-1)(-2) = 2 \neq 0$$

So the set $S=\{1+x+2x^2, 2+x+x^2, 1+x\}$ forms a basis for P_2. Now,

$$w = 1 + x + 2x^2 + 2(2 + x + x^2) + 3(1 + x)$$
$$= 1 + x + 2x^2 + 4 + 2x + 2x^2 + 3 + 3x = 8 + 6x + 4x^2$$

A3(a):

$$[B \mid B'] = \left[\begin{array}{cc|cc} 1 & 2 & 1 & 2 \\ 3 & 7 & 1 & 0 \end{array}\right] \xrightarrow{(-3)R_{12}} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 2 \\ 0 & 1 & -2 & -6 \end{array}\right]$$

$$\xrightarrow{(-2)R_{21}} \left[\begin{array}{cc|cc} 1 & 0 & 5 & 14 \\ 0 & 1 & -2 & -6 \end{array}\right] = [I \mid P_{B'\to B}]$$

$$P_{B'\to B} = \begin{bmatrix} 5 & 14 \\ -2 & -6 \end{bmatrix}$$

A3(b):

$$A=\begin{bmatrix}1 & 2 & 6 & -1\\ 2 & 4 & 4 & 6\\ 3 & 6 & 10 & 5\end{bmatrix}\xrightarrow[(-3)R_{13}]{(-2)R_{12}}\begin{bmatrix}1 & 2 & 1 & 2\\ 0 & 0 & -8 & 8\\ 0 & 0 & -8 & 8\end{bmatrix}$$

$$\xrightarrow{(-1)R_{23}}\begin{bmatrix}1 & 2 & 1 & 2\\ 0 & 0 & -8 & 8\\ 0 & 0 & 0 & 0\end{bmatrix}\xrightarrow{(-1/8)R_{2}}\begin{bmatrix}1 & 2 & 1 & 2\\ 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 0\end{bmatrix}$$

Using the leading ones, $\{[1\ \ 2\ \ 3]^T, [6\ \ 4\ \ 10]^T\}$ is a basis of col(A).

Now, rank(A)+nullity(A^T)=m

So nullity(A^T)=m- rank(A)=3-2=1

A4(a): (i) $(u)_S$=(4,8,12,16)

(ii) dim(V)=4.

A4(b): Suppose v∈V has two expressions:

$v = c_1v_1 + c_2v_2 + \cdots + c_nv_n$ and $v = k_1v_1 + k_2v_2 + \cdots + k_nv_n$, so

$0= (c_1-k_1)v_1 + (c_2-k_2)v_2 + \cdots + (c_n-k_n)v_n$

But $S = \{v_1, v_2, \ldots, v_n\}$ is a basis, so it is linearly independent. Thus,

$c_1-k_1= c_2-k_2=\ldots= c_n-k_n=0$ and hence $c_i=k_i$ for all $i\in\{1,2,\ldots,n\}$ and hence v has exactly one expression.

A4(c): rank(A)=dim(row(A))=dim(col(A^T))=rank(A^T).

A4(d): Observe that:

$$\begin{aligned}&a(u+v)+b(u-v)=0\\ &\Rightarrow (a+b)u+(a-b)v=0\\ &L.I.\Rightarrow a+b=0 \,\&\, a-b=0\\ &\Rightarrow 2a=0\Rightarrow a=0\Rightarrow b=0\end{aligned}$$

So, u+v and u-v are linearly independent.