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| **Second Semester** | **Second Exam** | **King Saud University** |
| **(without calculators)** | **Time allowed: 1 h and 30 m** | **College of Science** |
| **Monday 20-10-1445** | **240 Math** | **Math. Department** |

**Q1**: **(a)** Find $(w)_S$, where w=(2,6,2) and S={(1,2,2), (2,4,8), (1,0,0)} a basis of $\mathbb{R}^3$. (3 marks)
**(b)** Let $V=M_{22}$ and W is the set of all diagonal matrices matrices of degree 2. Prove that W is a subspace of V. (3 marks)

**Q2**: **(a)** Use the Wronskian to show that 1, $e^x$, $x^3$ are linearly independent in the vector space $C^2(-\infty,\infty)$. (2 marks)

**(b)** show that the set S={(1,1,1), (2,1,2), (1,0,0)} forms a basis for $\mathbb{R}^3$. (2 marks)

**Q3**: **(a)** If $P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$. Find two bases B and B' of $\mathbb{R}^2$ Such that P is the transition matrix from B to B'. (2 marks).
**(b)** Find a basis for the column space of the matrix:

$$A = \begin{bmatrix} 1 & 2 & 6 & -1 \\ 1 & 3 & 6 & 1 \\ 1 & 2 & 6 & 0 \end{bmatrix}$$

and <u>**deduce**</u> rank($A^T$) without using the matrix $A^T$. (4 marks)

**Q4**: Let $\mathbb{R}^2$ be the Euclidean vector space. Show that the function $f:\mathbb{R}^2\times\mathbb{R}^2\to\mathbb{R}$ defined by f(u,v)=0 for all $u,v\in\mathbb{R}^2$, is <u>**not**</u> an inner product function. (2 marks)

**Q5(a)** If $a,b\in\mathbb{R}$, then show that $(a\cos(\theta)+b\sin(\theta))^2\leq a^2+b^2$. (2 marks)

**(b)** Let v∈V, where V is a vector space with a basis S={$\mathbf{v_1}$,$\mathbf{v_2}$}. Show that we can write v as a linear combination of the basis vectors in a unique way.

(2 marks)

**(c)** If u and v are orthogonal in an inner product space, then show that $\|u+v\|^2=\|u\|^2+\|v\|^2$. (1 mark)

**(d)** If $\|u\|=\|v\|=\langle u,v\rangle=2$, then compute d(u,v). (2 marks)

# Solutions:

A1(a):

$$\left[\begin{array}{ccc|c}1&2&1&2\\2&4&0&6\\2&8&0&2\end{array}\right]\xrightarrow[(-2)R_{13}]{(-2)R_{12}}\left[\begin{array}{ccc|c}1&2&1&2\\0&0&-2&2\\0&4&-2&-2\end{array}\right]\xrightarrow{-\frac{1}{2}R_2}\left[\begin{array}{ccc|c}1&2&1&2\\0&0&1&-1\\0&4&-2&-2\end{array}\right]$$

$$\xrightarrow[(2)R_{23}]{(-1)R_{21}}\left[\begin{array}{ccc|c}1&2&0&3\\0&0&1&-1\\0&4&0&-4\end{array}\right]\xrightarrow{\frac{1}{4}R_3}\left[\begin{array}{ccc|c}1&2&0&3\\0&0&1&-1\\0&1&0&-1\end{array}\right]$$

$$\xrightarrow{(-2)R_{31}}\left[\begin{array}{ccc|c}1&0&0&5\\0&0&1&-1\\0&1&0&-1\end{array}\right]\xrightarrow{R_{23}}\left[\begin{array}{ccc|c}1&0&0&5\\0&1&0&-1\\0&0&1&-1\end{array}\right]$$

$$\Rightarrow (w)_S=(5,-1,-1)$$

A1(b): For all A=$\begin{bmatrix}a&0\\0&a'\end{bmatrix}$,B=$\begin{bmatrix}b&0\\0&b'\end{bmatrix}$ and k∈$\mathbb{R}$:

1- W is not empty since 0∈W
2- A+B =$\begin{bmatrix}a+b&0\\0&a'+b'\end{bmatrix}$. So A+B∈W.
3- kA=$\begin{bmatrix}ka&0\\0&ka'\end{bmatrix}$. So kA∈W

   1, 2 and 3 implies that W is a subspace of V=$M_{22}$.

A2(a):

$$W(x)=\begin{vmatrix}1&e^x&x^3\\0&e^x&3x^2\\0&e^x&6x\end{vmatrix}=6xe^x-3x^2e^x$$

$$W(1)=6e-3e=3e\neq 0$$

So 1, $e^x$, $x^3$ are linearly independent.

A2(b):

$$\begin{vmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 1 & 2 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 2-1=1\neq 0$$

So the vectors (1,1,2), (2,1,1), (1,1,0) form a basis for $\mathbb{R}^3$.

A3(a):

B={(1,1),(1,2)} and B'={(1,0),(0,1)}

A3(b):

$$A = \begin{bmatrix} 1 & 2 & 6 & -1 \\ 1 & 3 & 6 & 1 \\ 1 & 2 & 6 & 0 \end{bmatrix} \xrightarrow[(-1)R_{13}]{(-1)R_{12}} \begin{bmatrix} 1 & 2 & 6 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Using the leading ones, $\{[1\ \ 1\ \ 1]^T, [2\ \ 3\ \ 2]^T, [-1\ \ 1\ \ 0]^T\}$ is a basis of col(A).

Now, $\text{rank}(A^T)= \text{rank}(A) =3$.

A4: f((1,1),(1,1)=0, but (1,1)≠0=(0,0). So it is not an inner product function.

A5(a): Consider the Euclidean inner product on $\mathbb{R}^2$. Take the two vectors u=(a,b) and v=(cos(θ),sin(θ)). By Cauchy-Schwarz Inequality:

$$|\langle u,v\rangle| \leq \|u\|\|v\|$$

or equivalently:

$$\langle u,v\rangle^2 \leq \|u\|^2\|v\|^2$$

So,

$$(a\cos(\theta)+b\sin(\theta))^2=\langle u,v\rangle^2\leq\|u\|^2\|v\|^2=(a^2+b^2)(\cos^2(\theta)+\sin^2(\theta))= a^2+b^2$$

A5(b): Suppose $v\in V$ has two expressions:

$v = c_1v_1 + c_2v_2$ and $v = k_1v_1 + k_2v_2$, so

$0= (c_1-k_1)v_1 + (c_2-k_2)v_2$

But $S = \{v_1, v_2\}$ is a basis, so it is linearly independent. Thus, $c_1 - k_1 = c_2 - k_2 = 0$ and hence $c_1 = k_1$ and $c_2 = k_2$. Hence v can be written as a linear combination of the basis vectors in a unique way.

A5(c): Since u and v are orthogonal, then $\langle u,v\rangle=0$ and hence:

$$\|u+v\|^2=\langle u+v,u+v\rangle=\langle u,u\rangle+\langle u,v\rangle+\langle v,u\rangle+\langle v,v\rangle$$

$$=\|u\|^2+2\langle u,v\rangle+\|v\|^2=\|u\|^2+\|v\|^2$$

A5(d):

$$(d(u,v))^2= \|u-v\|^2=\langle u-v,u-v\rangle$$

$$=\langle u,u\rangle-2\langle u,v\rangle+\langle v,v\rangle=\|u\|^2-2\langle u,v\rangle+\|v\|^2=4-2(2)+4=4$$

So, $d(u,v)=(4)^{0.5}=2$.