First Semester 1447	**First Exam**	**King Saud University**
(without calculators)	**Time: 8 - 9:30 am**	**College of Science**
Monday 7-4-1447	**240 Math**	**Math. Department**

Q1: If $A=\begin{bmatrix}1&1\\1&1\end{bmatrix}$, $B^T=\begin{bmatrix}1&1\\0&1\\1&2\end{bmatrix}$, $C=\begin{bmatrix}1&1&1\\2&3&2\\1&1&2\end{bmatrix}$ and $P(x)=2x^2-2x-2$, then find the following:

(a) P(A) (3 marks)

(b) adj(C) **<u>in details</u>** (3 marks)

(c) the inverse of C (3 marks)

(d) the solution set of Bx=0 by Gauss-Jordan Elimination. (3 marks)

Q2: **<u>Find</u>** the determinant of A:

(3 marks)

$$A=\begin{bmatrix}1&2&2&2\\2&5&4&4\\3&6&6&7\\4&8&11&8\end{bmatrix}$$

Q3: (a) Prove that if A is an invertible matrix, then $\det(A^{-1})=(\det(A))^{-1}$. (2 marks)

(b) Prove that if A is an invertible symmetric matrix, then A^{-1} is symmetric.

(2 marks)

(c) If A is an invertible matrix of order 4, then find RREF(A). (1 mark)

(d) If A is an invertible matrix of order 2, then find $\det(2A^TA^{-1})$. (2 marks)

(e) If the solution set of the system Ax=b is $\{(2r+1,s-1,r,s):r,s\in\mathbb{R}\}$, where A is a **<u>square</u>** matrix, then **<u>find</u>** the solution set of the system Ax=0. Also, **<u>find</u>** RREF of the augmented matrix of the system Ax=b. (3 marks)

Solutions

Q1: If $A=\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$, $B^T=\begin{bmatrix}1 & 1\\ 0 & 1\\ 1 & 2\end{bmatrix}$, $C=\begin{bmatrix}1 & 1 & 1\\ 2 & 3 & 2\\ 1 & 1 & 2\end{bmatrix}$ and $P(x)=2x^2-2x-2$, then find the following:

(a) $P(A)=2A^2-2A-2I$

$$2\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}-2\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}-2\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$=2\begin{bmatrix}2 & 2\\ 2 & 2\end{bmatrix}-\begin{bmatrix}2 & 2\\ 2 & 2\end{bmatrix}-\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}$$

$$=\begin{bmatrix}4 & 4\\ 4 & 4\end{bmatrix}-\begin{bmatrix}2 & 2\\ 2 & 2\end{bmatrix}-\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}=\begin{bmatrix}0 & 2\\ 2 & 0\end{bmatrix}$$

(b) adj(C)=

$$C=\begin{bmatrix}1 & 1 & 1\\ 2 & 3 & 2\\ 1 & 1 & 2\end{bmatrix}$$

$$adj\,(A)=C^T=\begin{bmatrix}C_{11} & C_{12} & C_{13}\\ C_{21} & C_{22} & C_{23}\\ C_{31} & C_{32} & C_{33}\end{bmatrix}^T=\begin{bmatrix}4 & -2 & -1\\ -1 & 1 & 0\\ -1 & 0 & 1\end{bmatrix}^T=\begin{bmatrix}4 & -1 & -1\\ -2 & 1 & 0\\ -1 & 0 & 1\end{bmatrix}$$

(c) the inverse of C

$$[C\,|\,I]=\left[\begin{array}{ccc|ccc}1 & 1 & 1 & 1 & 0 & 0\\ 2 & 3 & 2 & 0 & 1 & 0\\ 1 & 1 & 2 & 0 & 0 & 1\end{array}\right]\xrightarrow[(-1)R_{13}]{(-2)R_{12}}\left[\begin{array}{ccc|ccc}1 & 1 & 1 & 1 & 0 & 0\\ 0 & 1 & 0 & -2 & 1 & 0\\ 0 & 0 & 1 & -1 & 0 & 1\end{array}\right]$$

$$\xrightarrow{(-1)R_{21}}\left[\begin{array}{ccc|ccc}1 & 0 & 1 & 3 & -1 & 0\\ 0 & 1 & 0 & -2 & 1 & 0\\ 0 & 0 & 1 & -1 & 0 & 1\end{array}\right]\xrightarrow{(-1)R_{31}}\left[\begin{array}{ccc|ccc}1 & 0 & 0 & 4 & -1 & -1\\ 0 & 1 & 0 & -2 & 1 & 0\\ 0 & 0 & 1 & -1 & 0 & 1\end{array}\right]=[I\,|\,C^{-1}]$$

or det(C)=1 and hence:

$$C^{-1}=\frac{1}{|C|}adj(C)=\frac{1}{1}\begin{bmatrix}4 & -1 & -1\\ -2 & 1 & 0\\ -1 & 0 & 1\end{bmatrix}=\begin{bmatrix}4 & -1 & -1\\ -2 & 1 & 0\\ -1 & 0 & 1\end{bmatrix}$$

(d) Solution of Bx=0 by Gauss-Jordan Elimination

$$\begin{bmatrix}1 & 0 & 1\\ 1 & 1 & 2\end{bmatrix}\xrightarrow{(-1)R_{12}}\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 1\end{bmatrix}$$

$$x+z=0\ \&\ y+z=0$$

$$z=t\Rightarrow x=y=-t, t\in\mathbb{R}$$

$$S.S.=\{(-t,-t,t)\,|\,t\in\mathbb{R}\}$$

Q2: Find the determinant of A:

(3 marks)

$$A=\begin{bmatrix}1 & 2 & 2 & 2\\ 2 & 5 & 4 & 4\\ 3 & 6 & 6 & 7\\ 4 & 8 & 11 & 8\end{bmatrix}$$

$$\det(A)=\begin{vmatrix}1 & 2 & 2 & 2\\ 2 & 5 & 4 & 4\\ 3 & 6 & 6 & 7\\ 4 & 8 & 11 & 8\end{vmatrix}\overset{\substack{(-2)R_{12}\\(-3)R_{13}\\(-4)R_{14}}}{=}\begin{vmatrix}1 & 2 & 2 & 2\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 3 & 0\end{vmatrix}\overset{R_{34}}{=}-\begin{vmatrix}1 & 2 & 2 & 2\\ 0 & 1 & 0 & 0\\ 0 & 0 & 3 & 0\\ 0 & 0 & 0 & 1\end{vmatrix}=-1(1)(3)(1)=-3$$

Q3: (a) Prove that if A is an invertible matrix, then $\det(A^{-1})=(\det(A))^{-1}$. (2 marks)

$$AA^{-1}=I\Rightarrow|AA^{-1}|=|I|=1$$

$$\Rightarrow|A||A^{-1}|=1\Rightarrow|A^{-1}|=\frac{1}{|A|}$$

$$as\ |A|\neq 0$$

(b) Prove that if A is an invertible symmetric matrix, then A^{-1} is symmetric.

(2 marks)

$$(A^{-1})^T=(A^T)^{-1}=A^{-1}.$$

(c) If A is an invertible matrix of order 4, then find RREF(A). (1 mark)

RREF(A)=I_4.

(d) If A is an invertible matrix of order 2, then find $\det(2A^TA^{-1})$. (2 marks)

$$det\left(2A^TA^{-1}\right) = det\left(2\left(A^T\right)\right)\det(A^{-1}) = 2^2 det\left(A^T\right)(\det(A))^{-1}$$
$$= 4det\left(A\right)(\det(A))^{-1} = 4$$

(e) If the solution set of the system Ax=b is {(2r+1,s-1,r,s):r,s∈ℝ} where A is a **square** matrix, then find the solution set of the system Ax=0. Also, find RREF of the augmented matrix of the system Ax=b. (3 marks)

The system Ax=b is consistent and has two parameters. So, we can write the solution of the system as (2r+1,s-1,r,s)= (2r,s,r,s)+ (1,-1,0,0) where (1,-1,0,0) is a special solution of Ax=b and (2r,s,r,s) is the general solution of Ax=0. So, the solution set of Ax=0 is {(2r,s, r,s):r,s∈ℝ}. The solution set shows that A has 4 variables and hence A is of order 4. Thus, [A|b] is of size 4×5 and

$$x_1 = 2r + 1 = 2x_3 + 1 \Rightarrow x_1 - 2x_3 = 1$$
$$x_2 = s - 1 = x_4 - 1 \Rightarrow x_2 - x_4 = -1$$
$$RREF([A|b]) = \begin{bmatrix} 1 & 0 & -2 & 0 & 1 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$