Second Semester 1446	**First Exam**	**King Saud University**
(without calculators)	**Time: 8 - 9:30 am**	**College of Science**
Monday 4-8-1446	**240 Math**	**Math. Department**

Q1: If $A=\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$, $B^T=\begin{bmatrix}1 & 2\\ 0 & 2\\ -1 & 0\end{bmatrix}$, $C=\begin{bmatrix}1 & 1 & 1\\ 2 & 3 & 2\\ 1 & 1 & 2\end{bmatrix}$ and $P(x)=2x^2-2x-2$, then find the following:

(a) P(A) (3 marks)

(b) tr(adj(A)) (2 marks)

(c) the inverse of C (3 marks)

(d) the solution set of Bx=0 by Gauss-Jordan Elimination. (3 marks)

Q2: **<u>Find</u>** the determinant of A and then **<u>find</u>** the cofactor C_{12}:

(4 marks)

$$A=\begin{bmatrix}1 & 2 & 2 & 2\\ 2 & 5 & 4 & 4\\ 3 & 6 & 6 & 7\\ 4 & 8 & 10 & 8\end{bmatrix}$$

Q3: (a) Prove that if A is an invertible matrix, then $\det(A^{-1})=(\det(A))^{-1}$. (2 marks)

(b) Prove that if A is an invertible symmetric matrix, then A^{-1} is symmetric.

(2 marks)

(c) If A is an invertible matrix of order 4, then find RREF(A). (1 mark)

(d) If A is an invertible matrix of order 2, then find $\det(2(A^T))\det(A^{-1})$. (2 marks)

(e) If the solution set of the system Ax=b is $\{(2r+1,s-1,r,s):r,s\in\mathbb{R}\}$, where A is a square matrix, then **<u>find</u>** the solution set of the system Ax=0. Also, **<u>find</u>** RREF of the augmented matrix of the system Ax=b. (3 marks)

Solutions

Q1: If $A=\begin{bmatrix}1&1\\1&1\end{bmatrix}$, $B^T=\begin{bmatrix}1&2\\0&2\\-1&0\end{bmatrix}$, $C=\begin{bmatrix}1&1&1\\2&3&2\\1&1&2\end{bmatrix}$ and $P(x)=2x^2-2x-2$, then find the following:

(a) $P(A)=2A^2-2A-2I$

$$2\begin{bmatrix}1&1\\1&1\end{bmatrix}\begin{bmatrix}1&1\\1&1\end{bmatrix}-2\begin{bmatrix}1&1\\1&1\end{bmatrix}-2\begin{bmatrix}1&0\\0&1\end{bmatrix}$$
$$=2\begin{bmatrix}2&2\\2&2\end{bmatrix}-\begin{bmatrix}2&2\\2&2\end{bmatrix}-\begin{bmatrix}2&0\\0&2\end{bmatrix}$$
$$=\begin{bmatrix}4&4\\4&4\end{bmatrix}-\begin{bmatrix}2&2\\2&2\end{bmatrix}-\begin{bmatrix}2&0\\0&2\end{bmatrix}=\begin{bmatrix}0&2\\2&0\end{bmatrix}$$

(b) tr(adj(A))=

$$tr\left(adj\left(\begin{bmatrix}1&1\\1&1\end{bmatrix}\right)\right)=tr\left(\begin{bmatrix}1&-1\\-1&1\end{bmatrix}\right)=1+1=2$$

$$adj(A)=C^T=\begin{bmatrix}C_{11}&C_{12}\\C_{21}&C_{22}\end{bmatrix}^T$$

(c) the inverse of C

$$[C\mid I]=\left[\begin{array}{ccc|ccc}1&1&1&1&0&0\\2&3&2&0&1&0\\1&1&2&0&0&1\end{array}\right]\xrightarrow[(-1)R_{13}]{(-2)R_{12}}\left[\begin{array}{ccc|ccc}1&1&1&1&0&0\\0&1&0&-2&1&0\\0&0&1&-1&0&1\end{array}\right]$$
$$\xrightarrow{(-1)R_{21}}\left[\begin{array}{ccc|ccc}1&0&1&3&-1&0\\0&1&0&-2&1&0\\0&0&1&-1&0&1\end{array}\right]\xrightarrow{(-1)R_{31}}\left[\begin{array}{ccc|ccc}1&0&0&4&-1&-1\\0&1&0&-2&1&0\\0&0&1&-1&0&1\end{array}\right]=[I\mid C^{-1}]$$

(d) Solution of Bx=0 by Gauss-Jordan Elimination

$$\begin{bmatrix} 1 & 0 & -1 \\ 2 & 2 & 0 \end{bmatrix} \xrightarrow{(-2)R_{12}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 2 \end{bmatrix} \xrightarrow{(\frac{1}{2})R_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$x - z = 0 \;\&\; y + z = 0$$

$$z = t \Rightarrow x = t, y = -t, t \in \mathbb{R}$$

Q2: Find the determinant of A, and then find the cofactor C_{12}:

(4 marks)

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 5 & 4 & 4 \\ 3 & 6 & 6 & 7 \\ 4 & 8 & 10 & 8 \end{bmatrix}$$

$$\det(A) = \begin{vmatrix} 1 & 2 & 2 & 2 \\ 2 & 5 & 4 & 4 \\ 3 & 6 & 6 & 7 \\ 4 & 8 & 10 & 8 \end{vmatrix} \underset{(-4)R_{14}}{\overset{(-2)R_{12}, (-3)R_{13}}{=}} \begin{vmatrix} 1 & 2 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \end{vmatrix} \overset{R_{34}}{=} - \begin{vmatrix} 1 & 2 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = -1(1)(2)(1) = -2$$

$$C_{12} = (-1)^{1+2} \begin{vmatrix} 2 & 4 & 4 \\ 3 & 6 & 7 \\ 4 & 10 & 8 \end{vmatrix} = -4 \begin{vmatrix} 1 & 2 & 2 \\ 3 & 6 & 7 \\ 2 & 5 & 4 \end{vmatrix}$$

$$\underset{(-2)R_{13}}{\overset{(-3)R_{12}}{=}} -4 \begin{vmatrix} 1 & 2 & 2 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} \overset{R_{23}}{=} 4 \begin{vmatrix} 1 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 4(1)(1)(1) = 4$$

Q3: (a) Prove that if A is an invertible matrix, then $\det(A^{-1}) = (\det(A))^{-1}$. (2 marks)

$$AA^{-1} = I \Rightarrow |AA^{-1}| = |I| = 1$$

$$\Rightarrow |A||A^{-1}| = 1 \Rightarrow |A^{-1}| = \frac{1}{|A|}$$

$$as\ |A| \neq 0$$

(b) Prove that if A is an invertible symmetric matrix, then A^{-1} is symmetric.

(2 marks)

$$(A^{-1})^T = (A^T)^{-1} = A^{-1}.$$

(c) If A is an invertible matrix of order 4, then find RREF(A).

RREF(A)=I_4.

(d) If A is an invertible matrix of order 2, then find $\det(2(A^T))\det(A^{-1})$. (2 marks)

$$\begin{aligned}&det\left(2\left(A^T\right)\right)\det(A^{-1}) = 2^2 det\left(A^T\right)(\det(A))^{-1}\\&= 4det\left(A\right)(\det(A))^{-1} = 4\end{aligned}$$

(e) If the solution set of the system Ax=b is $\{(2r+1,s-1,r,s):r,s\in\mathbb{R}\}$ where A is a square matrix, then find the solution set of the system Ax=0. Also, find RREF of the augmented matrix of the system Ax=b. (3 marks)

The system Ax=b is consistent and has two parameters. So, we can write the solution of the system as (2r+1,s-1,r,s)= (2r,s,r,s)+ (1,-1,0,0) where (1,-1,0,0) is a special solution of Ax=b and (2r,s,r,s) is the general solution of Ax=0. So, the solution set of Ax=0 is $\{(2r,s, r,s):r,s\in\mathbb{R}\}$. The solution set shows that A has 4 variables and hence A is of order 4. Thus, [A|b] is of size 4×5 and

$$\begin{aligned}&x_1 = 2r+1 = 2x_3+1 \Rightarrow x_1 - 2x_3 = 1\\&x_2 = s-1 = x_4 - 1 \Rightarrow x_2 - x_4 = -1\\&RREF([A\,|\,b]) = \begin{bmatrix}1 & 0 & -2 & 0 & 1\\0 & 1 & 0 & -1 & -1\\0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0\end{bmatrix}\end{aligned}$$