First Semester	**First Exam**	**King Saud University**
(without calculators)	**Time: 8 to 9:30 am**	**College of Science**
Monday 27-3-1446	**240 Math**	**Math. Department**

Q1: If $A=\begin{bmatrix}2 & 2\\ 1 & -1\end{bmatrix}$ and $F=\begin{bmatrix}3 & 2 & -1\\ 2 & 1 & -1\\ 1 & 1 & 0\end{bmatrix}$, then find the following:

(a) P(A), where P(x)= x^2+2x + 3 (3 marks)

(b) det(F) (3 marks)

(c) adj(F) in details (4 marks)

Q2: Find the inverse of the following matrix by using elementary row operations: (4 marks)

$$A=\begin{bmatrix}1 & 2 & 3 & 4\\ 1 & 3 & 3 & 4\\ 1 & 2 & 4 & 4\\ 1 & 2 & 3 & 5\end{bmatrix}$$

Q3: Solve the following linear system By Gaussian Elimination: (4 marks)

$$\begin{aligned}2x_1+4x_2-2x_3&=4\\ x_1+3x_2+3x_3&=2\\ x_1+3x_2+5x_3&=4\end{aligned}$$

Q4: (a) Prove: if a square matrix A has a row of zeros, then |A|=0. (1 mark)

(b) Prove: if A is an invertible matrix, then AA^T is invertible. (1 mark)

Q5: Decide if the following statements are true or false. Justify your answer in all cases. (5 marks)

(i) The matrix $\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}$ is an elementary matrix.

(ii) If A and B are row equivalent matrices, then they have the same size.

(iii) If A and B are row equivalent square matrices, then |A|=|B|.

(iv) If A is a triangular matrix, then det(A)=tr(A).

(v) If A is a square matrix, then |3A|=3|A|.

Solutions

Q1: If $A = \begin{bmatrix} 2 & 2 \\ 1 & -1 \end{bmatrix}$ and $F = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 1 & -1 \\ 1 & 1 & 0 \end{bmatrix}$, then find the following:

(a) P(A), where P(x)= x^2+2x + 3 (3 marks)

$$P(A) = A^2 + 2A + 3I = \begin{bmatrix} 2 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 1 & -1 \end{bmatrix} + 2\begin{bmatrix} 2 & 2 \\ 1 & -1 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 6 & 2 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} 4 & 4 \\ 2 & -2 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 13 & 6 \\ 3 & 4 \end{bmatrix}$$

(b)) det(F) (3 marks)

$$\det(F) = \begin{vmatrix} 3 & 2 & -1 \\ 2 & 1 & -1 \\ 1 & 1 & 0 \end{vmatrix} \underset{(-3)R_{31}}{\overset{(-2)R_{32}}{=}} \begin{vmatrix} 0 & -1 & -1 \\ 0 & -1 & -1 \\ 1 & 1 & 0 \end{vmatrix} = (1)\begin{vmatrix} -1 & -1 \\ -1 & -1 \end{vmatrix} = 0$$

(c) adj(F) in details (4 marks)

$$adj\,(F) = adj\left(\begin{bmatrix} 3 & 2 & -1 \\ 2 & 1 & -1 \\ 1 & 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$$

$$= \begin{bmatrix} \begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 2 & -1 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} \\ -\begin{vmatrix} 2 & -1 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 3 & -1 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} \\ \begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix} & -\begin{vmatrix} 3 & -1 \\ 2 & -1 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ -1 & 1 & -1 \end{bmatrix}^T = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix}$$

Q2: Find the inverse of the following matrix by using elementary row operations: (4 marks)

$$[A\,|\,I]=\left[\begin{array}{cccc|cccc}1&2&3&4&1&0&0&0\\1&3&3&4&0&1&0&0\\1&2&4&4&0&0&1&0\\1&2&3&5&0&0&0&1\end{array}\right]\xrightarrow[(-1)R_{14}]{\substack{(-1)R_{12}\\(-1)R_{13}}}\left[\begin{array}{cccc|cccc}1&2&3&4&1&0&0&0\\0&1&0&0&-1&1&0&0\\0&0&1&0&-1&0&1&0\\0&0&0&1&-1&0&0&1\end{array}\right]$$

$$\xrightarrow{(-2)R_{21}}\left[\begin{array}{cccc|cccc}1&0&3&4&3&-2&0&0\\0&1&0&0&-1&1&0&0\\0&0&1&0&-1&0&1&0\\0&0&0&1&-1&0&0&1\end{array}\right]\xrightarrow{(-3)R_{31}}\left[\begin{array}{cccc|cccc}1&0&0&4&6&-2&-3&0\\0&1&0&0&-1&1&0&0\\0&0&1&0&-1&0&1&0\\0&0&0&1&-1&0&0&1\end{array}\right]$$

$$\xrightarrow{(-4)R_{41}}\left[\begin{array}{cccc|cccc}1&0&3&0&10&-2&-3&-4\\0&1&0&0&-1&1&0&0\\0&0&1&0&-1&0&1&0\\0&0&0&1&-1&0&0&1\end{array}\right]=[I\,|\,A^{-1}]$$

$$A^{-1}=\begin{bmatrix}10&-2&-3&-4\\-1&1&0&0\\-1&0&1&0\\-1&0&0&1\end{bmatrix}$$

Q3: Solve the following linear system By Gaussian Elimination: (4 marks)

$$\begin{aligned}2x_1+4x_2-2x_3&=4\\x_1+3x_2+3x_3&=2\\x_1+3x_2+5x_3&=4\end{aligned}$$

We will solve the system by reducing the augmented matrix of the system in the row echelon form (R.E.F.) and then solving the corresponding system of equations:

$$[A\,|\,b]=\left[\begin{array}{ccc|c}2&4&-2&4\\1&3&3&2\\1&3&5&4\end{array}\right]\xrightarrow{\frac{1}{2}R_1}\left[\begin{array}{ccc|c}1&2&-1&2\\1&3&3&2\\1&3&5&4\end{array}\right]$$

$$\xrightarrow[(-1)R_{13}]{(-1)R_{12}}\left[\begin{array}{ccc|c}1&2&-1&2\\0&1&4&0\\0&1&6&2\end{array}\right]\xrightarrow{(-1)R_{23}}\left[\begin{array}{ccc|c}1&2&-1&2\\0&1&4&0\\0&0&2&2\end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_3}\left[\begin{array}{ccc|c}1&2&-1&2\\0&1&4&0\\0&0&1&1\end{array}\right]$$

So

$$\begin{aligned}
&x_3 = 1, \\
&x_2 + 4x_3 = 0 \\
&\Rightarrow x_2 = -4x_3 = -4(1) = -4, \\
&x_1 + 2x_2 - x_3 = 2 \\
&\Rightarrow x_1 = -2x_2 + x_3 + 2 \\
&-2(-4) + 1 + 3 = 11
\end{aligned}$$

Q4: (a) Prove: if a square matrix A has a row of zeros, then |A|=0. (1 mark)

Suppose A is of order n and the row of zeros is the row number i. Computing the determinant using the cofactor expansion, we get that:

$$\det(A) = \sum_{j=1}^{n} a_{ij} C_{ij} = \sum_{j=1}^{n} 0 C_{ij} = 0$$

(b) Prove: if A is an invertible matrix, then AA^T is invertible. (1 mark)

Since A is invertible, then A^T is invertible (theorem) and hence the product AA^T is invertible since it is a product of two invertible matrices (theorem).

Q5:Decide if the following statements are true or false. Justify your answer in all cases. (5 marks)

(i) The matrix $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ is an elementary matrix. (false)

To be an elementary matrix, it should be obtained from the identity by only one row operation. But here, we need two row operations to get this matrix.

(ii) If A and B are row equivalent matrices, then they have the same size. (true)

Row operations do not change the size of the matrix.

(iii) If A and B are row equivalent square matrices, then |A|=|B|. (false)

Take A=$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ and B=$\begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$, then A and B are row equivalent, but |A|=4≠8=|B|

(iv) If A is a triangular matrix, then det(A)=tr(A). (false)

Take A=$\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$, then det(A)=6≠5=tr(A).

(v) If A is a square matrix, then |3A|=3|A|. (false)

Take A=$\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$, then |3A|=$\begin{vmatrix} 9 & 3 \\ 0 & 6 \end{vmatrix}$=54≠3|A|=3(6)=18.