Second Semester	**First Exam**	**King Saud University**
(without calculators)	**Time: 90 minutes**	**College of Science**
Monday 2-8-1445	**240 Math**	**Math. Department**

Q1: If $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 5 & 3 \\ 2 & 3 & 0 \end{bmatrix}$, then find the following:

(a) det(A) (2 marks)

(b) BB^T+I_2 (2 marks)

(c) tr(A^2) (3 marks)

(d) adj(A) **<u>in details</u>** (4 marks)

Q2: Find the inverse of the following matrix by using elementary row operations: (4 marks)

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 3 & 4 \\ 1 & 2 & 4 & 4 \\ 1 & 2 & 3 & 5 \end{bmatrix}$$

Q3: Solve the following linear system By Gaussian elimination: (4 marks)

$$\begin{aligned} 2x_1 + 4x_2 - 2x_3 &= 4 \\ x_1 + 3x_2 + 3x_3 &= 2 \\ x_1 + 3x_2 + 5x_3 &= 4 \end{aligned}$$

Q4: (a) Prove that if a square matrix A has a row of zeros, then |A|=0.

(1 mark)

(b) Prove that if A is an invertible symmetric matrix, then A^{-1} is symmetric.

(1 mark)

(c) If A is an 2×2 matrix such that det(A)=2, then find $|2AA^TA^{-1}|$ (2 marks)

(d) Solve the following system by Cramer's rule:

$$\begin{aligned} x_1 + 2bx_2 - 3cx_3 &= 1 \\ x_1 + ax_2 + 3dx_3 &= 1 \\ x_1 + mx_2 + nx_3 &= 1 \end{aligned}$$

Where the coefficient matrix of the system is invertible. (2 marks)

Solutions of the first mid-term exam

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Q1(a):

$$\det(A)=\begin{vmatrix}1&1&2\\1&0&1\\1&1&0\end{vmatrix}\underset{(-1)R13}{\overset{(-1)R_{12}}{=}}\begin{vmatrix}1&1&1\\0&-1&-1\\0&0&-2\end{vmatrix}=(1)(-1)(-2)=2$$

Q1(b):

$$BB^T+I_2=\begin{bmatrix}2&5&3\\2&3&0\end{bmatrix}\begin{bmatrix}2&2\\5&3\\3&0\end{bmatrix}+\begin{bmatrix}1&0\\0&1\end{bmatrix}$$

$$=\begin{bmatrix}38&19\\19&13\end{bmatrix}+\begin{bmatrix}1&0\\0&1\end{bmatrix}=\begin{bmatrix}39&19\\19&14\end{bmatrix}$$

Q1(c):

$$tr(A^2)=tr(AA)=tr\left(\begin{bmatrix}1&1&2\\1&0&1\\1&1&0\end{bmatrix}\begin{bmatrix}1&1&2\\1&0&1\\1&1&0\end{bmatrix}\right)$$

$$=tr\left(\begin{bmatrix}4&3&3\\2&2&2\\2&1&3\end{bmatrix}\right)=4+2+3=9$$

Q1(d):

Adjoint A is equal to the transpose of the matrix of cofactors C from A, where

$$C=\begin{bmatrix}C_{11}&C_{12}&C_{13}\\C_{21}&C_{22}&C_{23}\\C_{31}&C_{32}&C_{33}\end{bmatrix}$$

Such that

$$C_{11} = (-1)^{1+1} \det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = (-1)^2 (-1) = -1$$

$$C_{12} = (-1)^{1+2} \det \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = (-1)^3 (-1) = 1$$

$$C_{13} = (-1)^{1+3} \det \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = (-1)^4 (1) = 1$$

$$C_{21} = (-1)^{2+1} \det \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} = (-1)^3 (-2) = 2$$

$$C_{22} = (-1)^{2+2} \det \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} = (-1)^4 (-2) = -2$$

$$C_{23} = (-1)^{2+3} \det \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = (-1)^5 (0) = 0$$

$$C_{31} = (-1)^{3+1} \det \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = (-1)^4 (1) = 1$$

$$C_{32} = (-1)^{3+2} \det \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = (-1)^5 (-1) = 1$$

$$C_{33} = (-1)^{3+3} \det \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = (-1)^6 (-1) = -1$$

So

$$C = \begin{bmatrix} -1 & 1 & 1 \\ 2 & -2 & 0 \\ 1 & 1 & -1 \end{bmatrix}$$

and then

$$adj(A) = C^T = \begin{bmatrix} -1 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

Q2:

$$[A\,|\,I]=\left[\begin{array}{cccc|cccc}1&2&3&4&1&0&0&0\\1&3&3&4&0&1&0&0\\1&2&4&4&0&0&1&0\\1&2&3&5&0&0&0&1\end{array}\right]\xrightarrow[(-1)R_{14}]{(-1)R_{12}\;(-1)R_{13}}\left[\begin{array}{cccc|cccc}1&2&3&4&1&0&0&0\\0&1&0&0&-1&1&0&0\\0&0&1&0&-1&0&1&0\\0&0&0&1&-1&0&0&1\end{array}\right]$$

$$\xrightarrow{(-2)R_{21}}\left[\begin{array}{cccc|cccc}1&0&3&4&3&-2&0&0\\0&1&0&0&-1&1&0&0\\0&0&1&0&-1&0&1&0\\0&0&0&1&-1&0&0&1\end{array}\right]\xrightarrow{(-3)R_{31}}\left[\begin{array}{cccc|cccc}1&0&0&4&6&-2&-3&0\\0&1&0&0&-1&1&0&0\\0&0&1&0&-1&0&1&0\\0&0&0&1&-1&0&0&1\end{array}\right]$$

$$\xrightarrow{(-4)R_{41}}\left[\begin{array}{cccc|cccc}1&0&3&0&10&-2&-3&-4\\0&1&0&0&-1&1&0&0\\0&0&1&0&-1&0&1&0\\0&0&0&1&-1&0&0&1\end{array}\right]=[I\,|\,A^{-1}]$$

$$A^{-1}=\begin{bmatrix}10&-2&-3&-4\\-1&1&0&0\\-1&0&1&0\\-1&0&0&1\end{bmatrix}$$

Q3:

We will solve the system by reducing the augmented matrix of the system in the row echelon form (R.E.F.) and then solving the corresponding system of equations:

$$[A\,|\,b]=\left[\begin{array}{ccc|c}2&4&-2&4\\1&3&3&2\\1&3&5&4\end{array}\right]\xrightarrow{\frac{1}{2}R_1}\left[\begin{array}{ccc|c}1&2&-1&2\\1&3&3&2\\1&3&5&4\end{array}\right]$$

$$\xrightarrow[(-1)R_{13}]{(-1)R_{12}}\left[\begin{array}{ccc|c}1&2&-1&2\\0&1&4&0\\0&1&6&2\end{array}\right]\xrightarrow{(-1)R_{23}}\left[\begin{array}{ccc|c}1&2&-1&2\\0&1&4&0\\0&0&2&2\end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_3}\left[\begin{array}{ccc|c}1&2&-1&2\\0&1&4&0\\0&0&1&1\end{array}\right]$$

$$\Rightarrow x_1=2-2x_2+x_3, x_2=-4x_3, x_3=1$$

$$\Rightarrow x_2=-4(1)=-4, x_1=2-2(-4)+1=2+8+1=11$$

Q4(a):

Suppose A is of order n and the row of zeros is the row number i. Computing the determinant using the cofactor expansion, we get that:

$$\det(A) = \sum_{j=1}^{n} a_{ij} C_{ij} = \sum_{j=1}^{n} 0 C_{ij} = 0$$

Q4(b):

From a theorem, we have that

$$\left(A^{-1}\right)^T = \left(A^T\right)^{-1}$$

But A is symmetric, so

$$\left(A^T\right)^{-1} = (A)^{-1}$$

Hence

$$\left(A^{-1}\right)^T = (A)^{-1}$$

Q4(c): $|2AA^TA^{-1}|=2^2|A||A^T||A^{-1}|=4|A||A||A|^{-1}=4|A|=4(2)=8$

Q4(d): Observe that:

$$A = \begin{bmatrix} 1 & 2b & -3c \\ 1 & a & 3d \\ 1 & m & n \end{bmatrix}, A_1 = \begin{bmatrix} 1 & 2b & -3c \\ 1 & a & 3d \\ 1 & m & n \end{bmatrix} = A, A_2 = \begin{bmatrix} 1 & 1 & -3c \\ 1 & 1 & 3d \\ 1 & 1 & n \end{bmatrix}, A_3 = \begin{bmatrix} 1 & 2b & 1 \\ 1 & a & 1 \\ 1 & m & 1 \end{bmatrix}$$

$$|A| \neq 0 \Rightarrow x_1 = \frac{|A_1|}{|A|} = 1, x_2 = \frac{|A_2|}{|A|} = \frac{0}{|A|} = 0, x_3 = \frac{|A_3|}{|A|} = \frac{0}{|A|} = 0$$

$$\Rightarrow (x_1, x_2, x_3) = (1, 0, 0)$$