First Semester	**Final Exam**	**King Saud University**
(without calculators)	**Time allowed: 3 hours**	**College of Science**
Monday 2-7-1447	**240 Math**	**Math. Department**

<u>Q1</u>: Solve the following system:

$$x_1 + x_2 - x_3 = 1$$
$$x_2 - 3x_3 = 1$$
$$x_3 = 1$$

(i) by Gauss-Jordan elimination. (3 marks)

(ii) Let A be the coefficient matrix of the system. Find det(A) and A^{-1}. (5 marks)

<u>Q2</u>: Using the following matrix, find (6 marks)

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 5 & 6 & 1 \\ 1 & 3 & 4 & 2 \end{bmatrix}$$

(i) the basis of the column space of A.

(ii) the basis of the null space of A.

(iii) nullity(A^T).

<u>Q3</u>: Let W={(2x,0,3y,0)|x,y∈$\mathbb{R}$}.

(i) Show that W is a **<u>subspace</u>** of $\mathbb{R}^4$. (3 marks)

(ii) Find a basis of W. (2 marks)

(iii) **<u>Find</u>** dim(W). (1 mark)

<u>Q4</u>: If $A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & -3 \\ 0 & 0 & -1 \end{bmatrix}$, then

(i) find its eigenvalues. (1 mark)

(ii) Find the matrix P that diagonalizes A. (4 marks)

<u>Q5</u>: Let P_2 be the vector space of all polynomials of degree less than or equal to 2 with the standard inner product.

(i) Compute $<1,x^2>$. (1 mark)

(ii) Let $\{p_1=1+x+x^2, p_2=2-x+2x^2\}$ be a basis of a subspace of P_2. Apply the Gram-Schmidt process to transform that basis into an **<u>orthonormal basis</u>**. (4 marks)

<u>Q6</u>: Take V={$A = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$ |$a \in \mathbb{R}$} which is a subspace of M_{22} and let T:V→$\mathbb{R}$ be the map defined by T(A)=a for all matrices A in V. Show that:

(i) T is a linear transformation. (2 marks)

(ii) Find a basis of ker(T). (1 mark)

(iii) Find $[T]_{S,B}$ where S={1} and B={$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$}. (2 marks)

(iv) Find rank(T). (1 mark)

Q7: (i) If B={(1,2),(2,3)} is a basis of $\mathbb{R}^2$ and $(u)_B$=(1,2), then find u. (1 mark)

(ii) If A=$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, then why A couldn't be a transition matrix. (1 mark)

(iii) If u and v are orthogonal non-zero vectors in an inner product space V, then show that 2u and 3v are linearly independent. (2 marks)

Answers

Q1: Solve the following system:

$$x_1 + x_2 - x_3 = 1$$
$$x_2 - 3x_3 = 1$$
$$x_3 = 1$$

(i) by Gauss-Jordan elimination. (3 marks)

(ii) Let A be the coefficient matrix of the system. Find det(A) and A^{-1}. (5 marks)

Answer: (i) Using the augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow[3R_{32}]{1R_{31}} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow{(-1)R_{21}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\Rightarrow (x_1, x_2, x_3) = (-2, 4, 1)$$

And the system has a unique solution.

(ii) Using the above details of (i), det(A)=1, **or** Since A is an upper triangular, then:

$$|A| = \begin{vmatrix} 1 & 1 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

To find the inverse of A, we use the inversion algorithm

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[3R_{32}]{1R_{31}} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{(-1)R_{21}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & -2 \\ 0 & 1 & 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] = [I \mid A^{-1}]$$

Q2: Using the following matrix, find (6 marks)

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 5 & 6 & 1 \\ 1 & 3 & 4 & 2 \end{bmatrix}$$

(i) the basis of the column space of A.

(ii) the basis of the null space of A.

(iii) nullity(A^T).

Answer: Firstly, we find RREF of A.

$$A=\begin{bmatrix}1&2&3&1\\1&5&6&1\\1&3&4&2\end{bmatrix}\xrightarrow[(-1)R_{13}]{(-1)R_{12}}\begin{bmatrix}1&2&3&1\\0&3&3&0\\0&1&1&1\end{bmatrix}\xrightarrow{\frac{1}{3}R_2}\begin{bmatrix}1&2&3&1\\0&1&1&0\\0&1&1&1\end{bmatrix}$$

$$\xrightarrow[(-1)R_{23}]{(-2)R_{21}}\begin{bmatrix}1&0&1&1\\0&1&1&0\\0&0&0&1\end{bmatrix}\xrightarrow{(-1)R_{31}}\begin{bmatrix}1&0&1&0\\0&1&1&0\\0&0&0&1\end{bmatrix}$$

(i) $\{[1\ \ 1\ \ 1]^T, [2\ \ 5\ \ 3]^T, [1\ \ 1\ \ 2]^T\}$ is a basis of col(A).

(ii)

$$x_1=-x_3=-t$$
$$x_2=-x_3=-t$$
$$x_3=t\in\mathbb{R}$$
$$x_4=0$$

Now, null(A)={t(-1,-1,1,0)|t$\in\mathbb{R}$}. So, {(-1,-1,1,0)} is a basis of null(A).

(iii) From (i), clearly rank(A)=3. So, nullity(A^T)=m-rank(A)=3-3=0.

Q3: Let W={(2x,0,3y,0)|x,y$\in\mathbb{R}$}.

(i) Show that W is a **subspace** of $\mathbb{R}^4$. (3 marks)

(ii) Find a basis of W. (2 marks)

(iii) **Find** dim(W). (1 mark)

Answer: (i) 1- W is not empty. By taking x=y=0, (0,0,0,0)$\in$W.

2- For all u and v in W, u=($2x_1$,0,$3y_1$,0) and v=($2x_2$,0,$3y_2$,0) and then:

u+v=($2x_1+2x_2$,0, $3y_1+3y_2$,0)=($2(x_1+x_2)$,0,$3(y_1+y_2)$,0)$\in$W (since x_1+x_2,$y_1+y_2\in\mathbb{R}$).

3- For all u=(2x,0,3y,0) in W, and k$\in\mathbb{R}$, we have that:

ku=k(2x,0,3y,0)= (2kx,0,3ky,0)$\in$W (since kx,ky$\in\mathbb{R}$).

1, 2 and 3 implies that W is a subspace of $\mathbb{R}^4$.

(ii) W={(2x,0,3y,0)|x,y$\in\mathbb{R}$}= W={x(2,0,0,0)+y(0,0,3,0)|x,y$\in\mathbb{R}$}. So, S={(2,0,0,0),(0,0,3,0)} spans W. S is Linearly independent since neither of the vectors is a scalar multiple of the other. Thus, S is a basis of W.

(iii) From (ii), dim(W)=2.

Q4: If $A=\begin{bmatrix}1&2&2\\0&0&-3\\0&0&-1\end{bmatrix}$, then

(i) find its eigenvalues. (1 mark)

(ii) Find the matrix P that diagonalizes A. (4 marks)

Answer: (i) Since it is upper triangular, its eigenvalues are 1,0,-1.

(ii) Since the eigenvalues are all different, then A is diagonalizable. To find P that diagonalizes A, we need the following steps:

$$\lambda I - A = \begin{bmatrix} \lambda-1 & -2 & -2 \\ 0 & \lambda & 3 \\ 0 & 0 & \lambda+1 \end{bmatrix}$$

$$\lambda = 1 \Rightarrow (1)I - A = \begin{bmatrix} 0 & -2 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow[\frac{1}{2}R_3]{(-\frac{1}{2})R_1} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow y = -z = 0, x = t \;\&\; t = 1 \Rightarrow C_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda = 0 \Rightarrow (0)I - A = -A = \begin{bmatrix} -1 & -2 & -2 \\ 0 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[2R_{31}]{(-3)R_{32}} \begin{bmatrix} -1 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow x = -2y = -2t, z = 0 \;\&\; t = 1 \Rightarrow C_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

$$\lambda = -1 \Rightarrow (-1)I - A = \begin{bmatrix} -2 & -2 & -2 \\ 0 & -1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{(-2)R_{21}} \begin{bmatrix} -2 & 0 & -8 \\ 0 & -1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[(-1)R_2]{(-\frac{1}{2})R_1} \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow x = -4z = -4t, y = 3z = 3t \;\&\; t = 1 \Rightarrow C_3 = \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix}$$

$$\Rightarrow P = \begin{bmatrix} 1 & -2 & -4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

Q5: Let P_2 be the vector space of all polynomials of degree less than or equal to 2 with the standard inner product.

(i) Compute $<1,x^2>$. (1 mark)

(ii) Let $\{p_1=1+x+x^2, p_2=2-x+2x^2\}$ be a basis of a subspace of P_2. Apply the Gram-Schmidt process to transform that basis into an **orthonormal basis**. (4 marks)

Answer: (i) $<1,x^2>=<1+0+0,0+0+x^2>=0+0+0=0$.

(ii)

$$p_1 = 1 + x + x^2, p_2 = 2 - x + 2x^2$$

$$v_1 = p_1 = 1 + x + x^2$$

$$v_2 = p_2 - \frac{<p_2, v_1>}{\|v_1\|^2} v_1 = 2 - x + 2x^2 - \frac{<2 - x + 2x^2, 1 + x + x^2>}{\|1 + x + x^2\|^2}(1 + x + x^2)$$

$$= 2 - x + 2x^2 - \frac{3}{3}(1 + x + x^2) = 2 - x + 2x^2 - (1 + x + x^2) = 1 - 2x + x^2$$

$$w_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{3}}(1 + x + x^2)$$

$$w_2 = \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{6}}(1 - 2x + x^2)$$

Q6: Take V={$A = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} | a \in \mathbb{R}$} which is a subspace of M_{22} and let T:V→$\mathbb{R}$ be the map defined by T(A)=a for all matrices A in V. Show that:

(i) T is a linear transformation. (2 marks)

(ii) Find a basis of ker(T). (1 mark)

(iii) Find $[T]_{S,B}$ where S={1} and B={$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$}. (2 marks)

(iv) Find rank(T). (1 mark)

Answer: (i) For all $A = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$,$B = \begin{bmatrix} a' & 0 \\ 0 & 0 \end{bmatrix}$∈ V, k∈$\mathbb{R}$:

1- T(A+B)=$T\left(\begin{bmatrix} a + a' & 0 \\ 0 & 0 \end{bmatrix}\right)$=$a + a'$ =T(A)+T(B)

2- T(kA)= $T\left(\begin{bmatrix} ka & 0 \\ 0 & 0 \end{bmatrix}\right)$=$ka$=kT(A)

So T is linear.

(ii) ker(T)={A∈ V|T(A)=0}=$\left\{\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \in \text{V} | a = 0\right\} = \left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right\}$

So, the empty set ∅ is a basis of ker(T).

(iii) $T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = 1$ and $\left[T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)\right]_S = [1]_S = 1$

Therefore, $[T]_{S,B}$ =$[1]$.

(iv) From (ii), nullity(T)=0 and hence rank(T)=dim(V)-nullity(T)=1−0=1.

Q7: (i) If B={(1,2),(2,3)} is a basis of $\mathbb{R}^2$ and $(u)_B$=(1,2), then find u. (1 mark)

(ii) If A=$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, then why A couldn't be a transition matrix. (1 mark)

(iii) If u and v are orthogonal non-zero vectors in an inner product space V, then show that 2u and 3v are linearly independent. (2 marks)

Answer: (i) u=1(1,2)+2(2,3)=(1,2)+(4,6)=(5,8).

(ii) Since det(A)=4-4=0.

(iii) Since u and v are orthogonal non-zero vectors, so they are linearly independent.

Now, for some scalars k and m, suppose that k(2u)+m(3v)=0. So, (2k)u+(3m)v=0. But u and v are linearly independent, hence 2k=3m=0 and then k=m=0. Therefore, 2u and 3v are linearly independent.