Second Semester	**Final Exam**	**King Saud University**
(without calculators)	**Time allowed: 3 hours**	**College of Science**
Sunday 6-8-1444	**240 Math**	**Math. Department**

<u>Q1</u>: Solve the following system:

$$x_1 + x_2 - 2x_3 = 1$$
$$x_2 - 3x_3 = 1$$
$$2x_3 = 0$$

(i) by Gauss-Jordan elimination. (3 marks)

(ii) by Cramer's rule. (5 marks)

<u>Q2</u>: Using the following matrix, find (5 marks)

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 5 & 6 & 1 \\ 1 & 3 & 4 & 2 \end{bmatrix}$$

(i) the basis of the row space of A.

(ii) the basis of the column space of A.

<u>Q3</u>: Let W={(x,0,x,0)|x $\in \mathbb{R}$}.

(i) Show that W is a **<u>subspace</u>** of $\mathbb{R}^4$. (3 marks)

(ii) Find a basis of W. (2 marks)

(iii) **<u>Find</u>** dim(W). (1 mark)

<u>Q4</u>: If $A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & -1 \end{bmatrix}$, then

(i) find its eigenvalues. (1 mark)

(ii) Show that A is **<u>not</u>** diagonalizable. (3 marks)

<u>Q5</u>: (i) Let P_2 be the vector space of all polynomials of degree less than or equal to 2 with the standard inner product. Compute $\langle x, x^2 \rangle$. (1 mark)

(ii) Let S={(1,1,1),(2,−1,2)} be a basis of a subspace of the Euclidean inner product space $\mathbb{R}^3$. Apply the Gram-Schmidt process to transform S into an **<u>orthonormal basis</u>**. (4 marks)

<u>Q6</u>: Take V={$A = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ |$a, b \in \mathbb{R}$} which is a subspace of M_{22} and let T:V→$\mathbb{R}$ be the map defined by T(A)=a+b for all matrices A in V. Show that:

(i) T is a linear transformation. (2 marks)

(ii) Find a basis of ker(T). (2 marks)

(iii) Find $[T]_{S,B}$ where S={1} and B={$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$}. (2 marks)

(iv) Find rank(T). (2 marks)

Q7: (i) If B={u,v,w} is a basis of a vector space V, then find the coordinate vector $(u+2w)_B$. (1 mark)

(ii) If A is a diagonalizable matrix of order 3 and has only one eigenvalue, then show that A is diagonal. (1 mark)

(iii) If B is a 5×7 matrix and its row echelon form (REF) has 4 leading ones, then find the dimension of the solution space of the system Bx=0. (1 mark)

(iv) Show that $3\sin(x) + 4\cos(x) \leq 5$ for every real number x in $\mathbb{R}$. (1 mark)

Q1: Solve the following system:

$$x_1 + x_2 - 2x_3 = 1$$
$$x_2 - 3x_3 = 1$$
$$2x_3 = 0$$

Answer: (i) By Gauss-Jordan elimination: (3 marks)

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 2 & 0 \end{array}\right] \xrightarrow[(-1)R_{21}]{\frac{1}{2}R_3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow[(-2)R_{31}]{3R_{32}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\Rightarrow (x_1, x_2, x_3) = (0,1,0)$$

(ii) By Cramer's rule and Ax=b: (5 marks)

$$|A| = \begin{vmatrix} 1 & 1 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & 2 \end{vmatrix} = 2$$

$$|A_1| = \begin{vmatrix} 1 & 1 & -2 \\ 1 & 1 & -3 \\ 0 & 0 & 2 \end{vmatrix} = 0$$

$$|A_2| = \begin{vmatrix} 1 & 1 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & 2 \end{vmatrix} = 2$$

$$|A_3| = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

$$\Rightarrow (x_1, x_2, x_3) = (\tfrac{0}{2}, \tfrac{2}{2}, \tfrac{0}{2}) = (0,1,0)$$

Q2: Using the following matrix, find (5 marks)

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 5 & 6 & 1 \\ 1 & 3 & 4 & 2 \end{bmatrix}$$

(i) the basis of the row space of A.

(ii) the basis of the column space of A.

Answer:

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 5 & 6 & 1 \\ 1 & 3 & 4 & 2 \end{bmatrix} \xrightarrow[(-1)R_{13}]{(-1)R_{12}} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 3 & 3 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \xrightarrow[(-2)R_{31}]{(\frac{1}{3})R_2} \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{(-1)R_{23}} \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{1R_{31}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(i) The basis of the row space is {[1 0 1 0],[0 1 1 0],[0 0 0 1]}

(ii) The basis of the column space is $\{[1\ 1\ 1]^T, [2\ 5\ 3]^T, [1\ 1\ 2]^T\}$

Q3: Let W={(x,0,x,0)|x ∈ ℝ}.

(i) Show that W is a **subspace** of $\mathbb{R}^4$. (3 marks)

(ii) Find a basis of W. (2 marks)

(iii) **Find** dim(W). (1 mark)

Answer: (i) 1- If x=0, then (0,0,0,0)∈W. So W≠∅.

2- Suppose u=(x,0,x,0),v=(y,0,y,0)∈W. Now,

u+v=(x,0,x,0)+(y,0,y,0)=(x+y,0,x+y,0). x+y ∈ ℝ implies that u+v∈W.

3- Suppose u=(x,0,x,0)∈W & k∈ℝ. Now, ku=(kx,0,kx,0). kx ∈ ℝ implies that ku∈W.

1,2 and 3 imply that W is a subspace of $\mathbb{R}^4$.

(ii) (x,0,x,0)=x(1,0,1,0). So, (1,0,1,0) spans W and also it is linearly independent since it is not a zero vector. Thus, {(1,0,1,0)} is a basis of W.

(iii) dim(W)=1.

Q4: If $A=\begin{bmatrix}1&2&2\\0&1&-3\\0&0&-1\end{bmatrix}$, then

(i) find its eigenvalues. (1 mark)

(ii) Show that A is **not** diagonalizable. (3 marks)

Answer: (i) since it is upper triangular, it has two eigenvalues which they are 1 and -1.

(ii)

$$\lambda I-A=\begin{bmatrix}\lambda-1&-2&-2\\0&\lambda-1&3\\0&0&\lambda+1\end{bmatrix}$$

$$\lambda=1\Rightarrow(1)I-A=\begin{bmatrix}0&-2&-2\\0&0&3\\0&0&2\end{bmatrix}\xrightarrow[(\frac{1}{3})R_2]{(\frac{-1}{2})R_1}\begin{bmatrix}0&1&1\\0&0&1\\0&0&2\end{bmatrix}\xrightarrow[(-2)R_{23}]{(-1)R_{21}}\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}$$

$$\Rightarrow y=z=0,x=t\ \&t=1\Rightarrow C_1=\begin{bmatrix}1\\0\\0\end{bmatrix}$$

So, the algebraic multiplicity of 1 is 2 which is not equal to its geometric multiplicity=1. Therefore, A is not diagonalizable.

Or

Suppose A is diagonalizable. So, it is similar to $D=\begin{bmatrix}1&0&0\\0&1&0\\0&0&-1\end{bmatrix}$ and then $A=P^{-1}DP$. Thus, $A^2=PD^2P^{-1}=PIP^{-1}=I$. But $A^2=\begin{bmatrix}1&2&2\\0&1&-3\\0&0&-1\end{bmatrix}\begin{bmatrix}1&2&2\\0&1&-3\\0&0&-1\end{bmatrix}=\begin{bmatrix}1&4&-6\\0&1&0\\0&0&1\end{bmatrix}\neq I$, a contradiction.

Thus, A is not diagonalizable.

Q5: (i) Let P_2 be the vector space of all polynomials of degree less than or equal to 2 with the standard inner product. Compute $<x,x^2>$. (1 mark)

(ii) Let S={(1,1,1),(2,−1,2)} be a basis of a subspace of the Euclidean inner product space $\mathbb{R}^3$. Apply the Gram-Schmidt process to transform S into an **orthonormal basis**. (4 marks)

Answer: (i) $<x,x^2>=0$.

(ii)

$$u_1=(1,1,1),u_2=(2,-1,2)$$

$$v_1=u_1=(1,1,1)$$

$$v_2=u_2-\frac{<u_2,v_1>}{\|v_1\|^2}v_1=(2,-1,2)-\frac{<(2,-1,2),(1,1,1)>}{\|(1,1,1)\|^2}(1,1,1)$$

$$=(2,-1,2)-\frac{3}{3}(1,1,1)=(2,-1,2)-(1,1,1)=(1,-2,1)$$

$$w_1=\frac{v_1}{\|v_1\|}=\frac{1}{\sqrt{3}}(1,1,1)$$

$$w_2=\frac{v_2}{\|v_2\|}=\frac{1}{\sqrt{6}}(1,-2,1)$$

Q6: Take V={$A = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ | $a, b \in \mathbb{R}$} which is a subspace of M_{22} and let T:V→ℝ be the map defined by T(A)=a+b for all matrices A in V. Show that:

(i) T is a linear transformation. (2 marks)

(ii) Find a basis of ker(T). (2 marks)

(iii) Find $[T]_{S,B}$ where S={1} and B={$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$}. (2 marks)

(iv) Find rank(T). (2 marks)

Answer: (i) For all $A = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$,$B = \begin{bmatrix} a' & b' \\ 0 & 0 \end{bmatrix}$∈ V, k∈ ℝ:

1- T(A+B)=$T\left(\begin{bmatrix} a+a' & b+b' \\ 0 & 0 \end{bmatrix}\right)$=$a + a' + b + b'$ = $(a+b) + (a'+b')$=T(A)+T(B)

2- T(kA)= $T\left(\begin{bmatrix} ka & kb \\ 0 & 0 \end{bmatrix}\right)$=$ka + kb$= $k(a+b)$=kT(A)

So T is linear.

(ii) ker(T)={A∈ V | T(A)=0}=$\left\{\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \in \text{V} | a+b=0\right\} = \left\{\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \in \text{V} | b=-a\right\}$

$= \left\{\begin{bmatrix} a & -a \\ 0 & 0 \end{bmatrix} \in \text{V} | a \in \mathbb{R}\right\} = \left\{a\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \in \text{V} | a \in \mathbb{R}\right\}$.

So, $\left\{\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\right\}$ is a basis of ker(T).

(iii) $T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = 1$ and $T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = 1$.

Now,

$\left[T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)\right]_S = 1$ & $\left[T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)\right]_S = 1$. Therefore, $[T]_{S,B}$ =$[1 \quad 1]$.

(iv) From (ii), nullity(T)=1 and hence rank(T)=dim(V)-nullity(T)=2−1=1.

Q7: (i) If B={u,v,w} is a basis of a vector space V, then find the coordinate vector $(u+2w)_B$. (1 mark)

Answer: As u+2w=1u+0v+2w and writing a vector as a linear combination of the vectors in B is unique, so $(u+2w)_B$=(1,0,2)

(ii) If A is a diagonalizable matrix of order 3 and has only one eigenvalue, then show that A is diagonal. (1 mark)

Answer: Suppose λ is the eigenvalue of A. Since A is diagonalizable, A=PDP^{-1}. So,

$$A = P\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}P^{-1} = P\left(\lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)P^{-1} = \lambda P\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}P^{-1}$$

$$= \lambda PP^{-1} = \lambda I = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

(iii) If B is a 5×7 matrix and its row echelon form (REF) has 4 leading ones, then find the dimension of the solution space of the system Bx=0. (1 mark)

Answer: nullity(B)=n−rank(B) =7−4=3

(iv) Show that $3\sin(x) + 4\cos(x) \leq 5$ for every real number x in ℝ. (1 mark)

Answer: Let $\mathbb{R}^2$ be the Euclidean inner product space. Using Cauchy-Schwarz inequality:

$$\langle(3,4), (\sin(x), \cos(x))\rangle \leq |\langle(3,4), (\sin(x), \cos(x))\rangle|$$

$$\leq \|(3,4)\|\|(\sin(x), \cos(x))\| = \sqrt{25}\sqrt{\sin^2(x) + \cos^2(x)} = 5(1) = 5$$