First Semester	**Final Exam**	**King Saud University**
(without calculators)	**Time allowed: 3 hours**	**College of Science**
Monday 15-6-1446	**240 Math**	**Math. Department**

<u>Q1</u>: If $A=\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{bmatrix}$ and $C=\begin{bmatrix}2 & 4 & 6\\ 4 & 5 & 6\end{bmatrix}$ such that $AB=C$, then:

- (i) find the size of B. (1 mark)
- (ii) show that A and C are row equivalent. (1 mark)

<u>Q2</u>: If $A\in M_{22}$ and det(A)=3, then find:

- (i) $\det(A^T+A^T)$. (2 marks)
- (ii) RREF of A. (1 mark)
- (iii) the solution set of the system Ax=0. (1 mark)

<u>Q3</u>: Let V=span(S), where $S=\{v_1=(1,1,1,0),\ v_2=(-2,0,0,2)\}$.

- (i) **<u>Find</u>** dim(V). (2 marks)
- (ii) show that $u=(-6,0,0,7)\notin V$. (2 marks)

<u>Q4</u>: Let $W=\{(b,b)\,|\,b\in\mathbb{R}\}$. Show that W is a **<u>subspace</u>** of $\mathbb{R}^2$. (3 marks)

<u>Q5</u>: Let B={(1,0),(1,1)} and B'={u,v} be two bases of $\mathbb{R}^2$. If the transition matrix from B' to B is $\begin{bmatrix}2 & 1\\ 3 & 2\end{bmatrix}$, then find u. (2 marks).

<u>Q6</u>: Let $A=\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$. Show that A is diagonalizable and find the matrix P that diagonalizes A. (4 marks).

<u>Q7:</u> Let A be a square non-zero matrix of order 3 such that Ax=2x, Ay=3y and Az=z where x, y and z are column matrices. Then:

- (i) show that A is diagonalizable. (2 marks)
- (ii) if P is the matrix that diagonalizes A, then find the product $P^{-1}AP$. (1 mark)
- (iii) show that A is invertible. (1 mark)

<u>Q8</u>: Let $\mathbb{R}^4$ be the Euclidean inner product space. Applying Gram-Schmidt process to the basis $\{u_1,u_2,u_3,u_4=(1,1,1,1)\}$ to transform it into the following **<u>orthogonal basis</u>** $\{v_1,v_2,v_3,v_4\}$ where $v_1=(2,2,1,0)$, $v_2=(1,-2,2,0)$ and $v_3=(2,-1,-2,0)$. Find v_4. (4 marks)

<u>Q9</u>: Let M_{22} be the vector space of square matrices of order 2, and let $T:M_{22}\to\mathbb{R}^2$ be the function defined by $T\begin{pmatrix}a & b\\ c & d\end{pmatrix}=(a,a)$ for all $a,b,c,d\in\mathbb{R}$:

- (i) Show that T is a linear transformation. (2 marks)
- (ii) Find a basis for ker(T). (3 marks)
- (iii) Find $[T]_{B',B}$ where B and B' are the standard bases of M_{22} and $\mathbb{R}^2$, respectively. (2 marks)
- (iv) Find rank(T). (1 mark)

<u>Q10</u>: Solve the following:

- (i) Consider the Weighted Euclidian inner product on $\mathbb{R}^2$ defined by:

  $$\langle(u_1,u_2),(v_1,v_2)\rangle=2u_1v_1+3u_2v_2$$

  Find d((1,1),(1,2)). (1 mark)
- (ii) Let *u* and *v* be orthonormal vectors in an inner product space. Find $\|2u+3v\|^2$. (1 mark)
- (iii) If B is a 5×7 matrix with nullity(B)=4, then find $\text{rank}(B^T)$. (1 mark)
- (iv) Let $T:\mathbb{R}^3\to\mathbb{R}^2$ be a matrix transformation. Find the size of its standard matrix. (1 mark)
- (v) Let {u,v,w} be a linearly independent subset of a vector space V. Show that span{u}≠span{v,w}. (1 mark)

Q1: If $A=\begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}$ and $C=\begin{bmatrix}2&4&6\\4&5&6\end{bmatrix}$ such that $AB=C$, then:

(i) the size of B. (1 mark)

(ii) Show that A and C are row equivalent. (1 mark)

Answer: (i) Suppose B is of size m×n. Since $\begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}$ is of size 2×3 and the product $\begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}B$ is defined, so m=3. But the product $\begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}B$ is of size 2×n and is equal to $\begin{bmatrix}2&4&6\\4&5&6\end{bmatrix}$ which is of size 2×3. So, n=3. Hence, B is of size 3×3.

(ii) Since $\begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}\xrightarrow{2R_1}\begin{bmatrix}2&4&6\\4&5&6\end{bmatrix}$, so A and C are row equivalent.

Q2: If A∈M_{22} and det(A)=3, then find:

(i) det(A^T+A^T). (2 marks)

(ii) RREF of A. (1 mark)

(iii) the solution set of the system Ax=0. (1 mark)

Answer:

(i) det(A^T+A^T)=det($2A^T$)= 2^2det(A^T) =4det(A)=4(3)=12.

(ii) RREF of A is $\begin{bmatrix}1&0\\0&1\end{bmatrix}$ since det(A)$\neq$ 0 (by a theorem).

(iii) x=0 since det(A)$\neq$ 0 (a theorem) OR Since det(A)$\neq$ 0, A is invertible and x=A^{-1}0=0.

Q3: Let V=span(S), where S={v_1=(1,1,1,0), v_2=(−2,0,0,2)}.

(i) **Find** dim(V). (2 marks)

(ii) show that u=(−6,0,0,7)∉V. (2 marks)

Answer: (i) Since v_1 and v_2 span V and none of them is a scalar multiple of the other, So S is a basis of V and hince dim(V)=2.

(ii) Suppose u=av_1+bv_2, where a and b are scalars. So,

$$\begin{bmatrix}1&-2&-6\\1&0&0\\1&0&0\\0&2&7\end{bmatrix}\xrightarrow[(-1)R_{13}]{(-1)R_{12}}\begin{bmatrix}1&-2&-6\\0&2&6\\0&2&6\\0&2&7\end{bmatrix}\xrightarrow[(-1)R_{24}]{1R_{21},\ (-1)R_{23}}\begin{bmatrix}1&0&0\\0&2&6\\0&0&0\\0&0&1\end{bmatrix}$$

So, u∉V.

Q4: Let W={(b,b)|b∈ℝ}. Show that W is a **subspace** of $\mathbb{R}^2$. (3 marks)

Answer: 1- If b=0, then (0,0)∈W. So W≠∅.

2- Take u=(a,a),v=(b,b)∈W. Now, u+v=(a+b,a+b). So u+v∈W.

3- Take u=(b,b)∈W & k∈ℝ. Now, ku=(kb,kb). So ku∈W.

1,2 and 3 imply that W is a subspace of $\mathbb{R}^2$.

Q5: Let B={(1,0),(1,1)} and B'={u,v} be two bases of $\mathbb{R}^2$. If the transition matrix from B' to B is $\begin{bmatrix}2&1\\3&2\end{bmatrix}$, then find u. (2 marks).

Answer:

$$P_{B'\to B}=\begin{bmatrix}2&1\\3&2\end{bmatrix}=\left[[u]_B \quad | \quad [v]_B\right]$$

$$\Rightarrow [u]_B=\begin{bmatrix}2\\3\end{bmatrix}$$

$$\Rightarrow u=2(1,0)+3(1,1)=(2,0)+(3,3)=(5,3)$$

Q6: Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. Show that A is diagonalizable and find the matrix P that diagonalizes A. (4 marks).

Answer: The characteristic equation:

$$0 = \det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right) = \begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{vmatrix}$$

$$= (\lambda - 2)^2 - 1 = \lambda^2 - 4\lambda + 4 - 3 = \lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3)$$

and hence the Eigenvalues are λ=1,3. Since the Eigenvalues are distinct, A is diagonalizable. To find P, take the equation (λI−A)x=0 and substitute λ=1,3, respectively as follows:

$$\lambda I - A = \begin{bmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{bmatrix}$$

$$\lambda = 1 \Rightarrow (1)I - A = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \xrightarrow{(-1)R_{12}} \begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix} \xrightarrow{(-1)R_1} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow x = -y = -t \ \& t = -1 \Rightarrow C_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\lambda = 3 \Rightarrow (3)I - A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \xrightarrow{(1)R_{12}} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow x = y = t \ \& t = 1 \Rightarrow C_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

Q7: Let A be a square non-zero matrix order 3 such that Ax=2x, Ay=3y and Az=z where x, y and z are column matrices. Then:

(i) show that A is diagonalizable. (2 marks)

(ii) if P is the matrix that diagonalizes A, then find the product $P^{-1}AP$. (1 mark)

(iii) show that A is invertible. (1 mark)

Answer: (i) As Ax=2x, Ay=3y and Az=z, we have $\lambda = 2, \lambda = 3$ and $\lambda = 1$ are the eigenvalues of the matrix A. Since they are distinct, A is diagonalizable.

(ii) $P^{-1}AP = D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (the arrangement of the entries in the main diagonal is not important).

(iii) Since all the eigenvalues are non-zero, A is invertible (by a theorem)

Q8: Let $\mathbb{R}^4$ be the Euclidean inner product space. Applying Gram-Schmidt process to the basis {u_1,u_2,u_3,u_4=(1,1,1,1)} to transform it into the following **orthogonal basis** {v_1,v_2,v_3,v_4} where v_1=(2,2,1,0), v_2=(1,−2,2,0) and v_3=(2,−1,−2,0). Find v_4. (4 marks)

Answer:

$$v_4 = u_4 - \frac{<u_4, v_3>}{\|v_3\|^2} v_3 - \frac{<u_4, v_2>}{\|v_2\|^2} v_2 - \frac{<u_4, v_1>}{\|v_1\|^2} v_1$$

$$= (1,1,1,1) - \frac{<(1,1,1,1),(2,-1,-2,0)>}{\|(2,-1,-2,0)\|^2}(2,-1,-2,0) - \frac{<(1,1,1,1),(1,-2,2,0)>}{\|(1,-2,2,0)\|^2}(1,-2,2,0)$$

$$- \frac{<(1,1,1,1),(2,2,1,0)>}{\|(2,2,1,0)\|^2}(2,2,1,0) = (1,1,1,1) - \tfrac{-1}{9}(2,-1,-2,0) - \tfrac{1}{9}(1,-2,2,0) - \tfrac{5}{9}(2,2,1,0)$$

$$= (1 - \tfrac{-2}{9} - \tfrac{1}{9} - \tfrac{10}{9}, 1 - \tfrac{1}{9} - \tfrac{-2}{9} - \tfrac{10}{9}, 1 - \tfrac{2}{9} - \tfrac{2}{9} - \tfrac{5}{9}, 1 - 0 - 0 - 0) = (0,0,0,1)$$

Q9: Let M_{22} be the vector space of square matrices of order 2, and let T: $M_{22} \to \mathbb{R}^2$ be the function defined by $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a, a)$ for all $a, b, c, d \in \mathbb{R}$:

(i) Show that T is a linear transformation. (2 marks)

(ii) Find a basis for ker(T). (3 marks)

(iii) Find $[T]_{B',B}$ where B and B' are the standard bases of M_{22} and $\mathbb{R}^2$, respectively. (2 marks)

(iv) Find rank(T). (1 mark)

Answer: For all $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, B = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \in M_{22}$, $k \in \mathbb{R}$:

(i) 1- T(A+B)= $T\begin{bmatrix} a+a' & b+b' \\ c+c' & d+d' \end{bmatrix} = (a+a', a+a') = (a,a) + (a' + a')$ =T(A)+T(B)

2- T(*k*A)= $T\begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix} = (ka, ka) = k(a,a)$ =*k*T(A)

So T is linear.

(ii) ker(T)={A∈ M_{22}|T(A)=0}=$\left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{22} | (a,a) = (0,0)\right\}$

=$\left\{\begin{bmatrix} 0 & b \\ c & d \end{bmatrix} \in M_{22} | \, b,c,d \in \mathbb{R}\right\} = \left\{b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} | b,c,d \in \mathbb{R}\right\}$

So, the set $S = \left\{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$ spans ker(T). Observe that

$$b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

implies that

$$\begin{bmatrix} 0 & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and hence b=c=d=0. So, S is linearly independent also. Thus, S is a basis of ker(T).

(iii) $T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = (1,1)$, $T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = (0,0)$, $T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = (0,0)$ and $T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = (0,0)$.

Now,

$$\left[T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)\right]_{B'} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \left[T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)\right]_{B'} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \left[T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right)\right]_{B'} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \left[T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)\right]_{B'} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Therefore, $[T]_{B',B} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$.

(iv) Since dim(ker(T))=3, so nullity(T)=3 and hence rank(T)=dim(M_{22})-nullity(T)=4-3=1.

Q10: Solve the following:

(i) Consider the Weighted Euclidian inner product on $\mathbb{R}^2$ defined by:

$$\langle(u_1,u_2),(v_1,v_2)\rangle = 2u_1v_1 + 3u_2v_2$$

Find d((1,1),(1,2)). (1 mark)

(ii) Let *u* and *v* be orthonormal vectors in an inner product space. Find $\|2u+3v\|^2$. (1 mark)

(iii) If *B* is a 5×7 matrix with nullity(B)=4, then find rank(B^T). (1 mark)

(iv) Let T:$\mathbb{R}^3 \to \mathbb{R}^2$ be a matrix transformation. Find the size of its standard matrix. (1 mark)

(v) Let {u,v,w} be a linearly independent subset of a vector space V. Show that span{u}≠span{v,w}. (1 mark)

Answer: (i) d((1,1),(1,2))=‖(1,1)−(1,2)‖= ‖(0,−1)‖=$(2(0)^2+3(1)^2)^{0.5}=\sqrt{3}$.

Answer: (ii) $\|2u+3v\|^2$=<2u+3v,2u+3v>=4<u,u>+6<u,v>+6<v,u>+9<v,v>=4+9=13

since <u,v>=<v,u>=0 and <u,u>=<v,v>=1.

Answer: (iii) rank(B^T)=rank(B)=7− nullity(B)= 7−4=3

Answer: (iv) The size is 2×3.

Answer: (v) Since {u,v,w} is a linearly independent subset, so dim(span{u})=1≠2=dim(span{v,w}) and hence span{u}≠span{v,w}.