Second Semester	**Final Exam**	**King Saud University**
(without calculators)	**Time allowed: 3 hours**	**College of Science**
Thursday 29-11-1445	**240 Math**	**Math. Department**

<u>Q1</u>: Find the inverse of $A = \begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix}$. (2 marks)

<u>Q2</u>: If A,B∈M_{22}, det(B)=2 and det(A)=3, then find det($2A^TB^{-1}$). (2 marks)

<u>Q3</u>: Let V be the subspace of $\mathbb{R}^4$ **<u>spanned</u>** by the set S={v_1=(1,1,1,0), v_2=(−1,0,0,1), v_3=(2,2,2,0), v_4=(−5,−4,−4,1)}.
(i) Find a **<u>subset</u>** of S that forms a basis of V. (3 marks)
(ii) show that u=(−6,0,0,7)∉V. (2 marks)

<u>Q4</u>: Let W={(a,a+1)∈ $\mathbb{R}^2$|a∈ $\mathbb{R}$}. Show that W is **<u>not a subspace</u>** of $\mathbb{R}^2$. (2 marks)

<u>Q5</u>: Let B={(1,2),(1,1)} and B'={u,v} be two bases of $\mathbb{R}^2$. If the transition matrix from B' to B is $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$, then find u. (2 marks).

<u>Q6</u>: Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. Show that A is diagonalizable and find the matrix P that diagonalizes A. (6 marks)

<u>Q7</u>: Let $\mathbb{R}^3$ be the Euclidean inner product space. Apply the Gram-Schmidt process to transform the following basis {u_1=(0,1,−1),u_2=(0,5,3),u_3=(1,0,0)} into an **<u>orthonormal basis</u>**. (6 marks)

<u>Q8</u>: Let M_{22} be the vector space of square matrices of order 2, and let T:M_{22}→$\mathbb{R}^2$ be the function defined by T(A)=0∈$\mathbb{R}^2$ for all A∈ M_{22}. Show that:
(i) T is a linear transformation. (2 marks)
(ii) Find a basis for ker(T). (2 marks)
(iii) Find $[T]_{S,B}$ where B and S are the standard bases of M_{22} and $\mathbb{R}^2$, respectively. (3 marks)
(iv) Find rank(T). (1 mark)

<u>Q9</u>: If A is a matrix such that $A\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$ and $A\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} -c \\ -d \end{bmatrix}$, where a, b, c and d are real numbers such that $ad \neq bc$, then:
(i) Show that A is diagonalizable. (2 marks)
(ii) Show that A is invertible. (1 mark)
(iii) Show that A^2=I. (2 marks)

Q10: Let V be an inner product space. If u and v are orthonormal vectors in V, then find d(u,v). (2 marks)

Q1: Find the inverse of $A=\begin{bmatrix}2&2\\2&3\end{bmatrix}$. (2 marks)

Answer:

$$\begin{bmatrix}2&2\\2&3\end{bmatrix}^{-1}=\frac{1}{6-4}\begin{bmatrix}3&-2\\-2&2\end{bmatrix}=\frac{1}{2}\begin{bmatrix}3&-2\\-2&2\end{bmatrix}=\begin{bmatrix}\frac{3}{2}&-1\\-1&1\end{bmatrix}$$

Q2: If A,B∈M_{22}, det(B)=2 and det(A)=3, then find det($2A^TB^{-1}$). (2 marks)

Answer: det($2A^TB^{-1}$)= 2^2det(A^T)det(B^{-1})=4det(A)(det(B))$^{-1}$=4(3)(1/2)=6.

Q3: Let V be the subspace of $\mathbb{R}^4$ **spanned** by the set S={v_1=(1,1,1,0), v_2=(−1,0,0,1), v_3=(2,2,2,0), v_4=(−5,−4,−4,1)}.

(i) Find a **subset** of S that forms a basis of V. (3 marks)

(ii) show that u=(−6,0,0,7)∉V. (2 marks)

Answer: (i) Putting the vectors as columns in the following matrix:

$$\begin{bmatrix}1&-1&2&-5\\1&0&2&-4\\1&0&2&-4\\0&1&0&1\end{bmatrix}\xrightarrow[(-1)R_{13}]{(-1)R_{12}}\begin{bmatrix}1&-1&2&-5\\0&1&0&1\\0&1&0&1\\0&1&0&1\end{bmatrix}\xrightarrow[(-1)R_{24}]{(-1)R_{23}}\begin{bmatrix}1&-1&2&-5\\0&1&0&1\\0&0&0&0\\0&0&0&0\end{bmatrix}$$

So, S={v_1, v_2} is a basis of V.

(ii) Suppose u=av_1+bv_2, where a, b and c are scalars. So,

$$\begin{bmatrix}1&-1&-6\\1&0&0\\1&0&0\\0&1&7\end{bmatrix}\xrightarrow[(-1)R_{13}]{(-1)R_{12}}\begin{bmatrix}1&-1&-6\\0&1&6\\0&1&6\\0&1&7\end{bmatrix}\xrightarrow[(-1)R_{24}]{\substack{1R_{21}\\(-1)R_{23}}}\begin{bmatrix}1&0&0\\0&1&6\\0&0&0\\0&0&1\end{bmatrix}$$

So, it has no solution and hence u∉V.

Q4: Let W={(a,a+1)∈ $\mathbb{R}^2$|a∈ $\mathbb{R}$}. Show that W is **not a subspace** of $\mathbb{R}^2$. (2 marks)

Answer: since (1,2) belongs to W, but 3(1,2)=(3,6) doesn't belong to W, so W is not a subspace of $\mathbb{R}^2$.

Q5: Let B={(1,2),(1,1)} and B'={u,v} be two bases of $\mathbb{R}^2$. If the transition matrix from B' to B is $\begin{bmatrix}1&2\\2&3\end{bmatrix}$, then find u. (2 marks).

Answer:

$$P_{B'\to B}=\begin{bmatrix}1&2\\2&3\end{bmatrix}=\left[[u]_B \quad | \quad [v]_B\right]$$

$$\Rightarrow [u]_B=\begin{bmatrix}1\\2\end{bmatrix}$$

$$\Rightarrow u=(1)(1,2)+(2)(1,1)=(1,2)+(2,2)=(3,4)$$

Q6: Let $A=\begin{bmatrix}1&2\\2&1\end{bmatrix}$. Show that A is diagonalizable and find the matrix P that diagonalizes A. (6 marks)

Answer: Observe that:

$$0=\det(\lambda I-A)=\begin{vmatrix}\lambda-1&-2\\-2&\lambda-1\end{vmatrix}$$
$$=(\lambda-1)^2-4=\lambda^2-2\lambda+1-4$$
$$=\lambda^2-2\lambda-3=(\lambda-3)(\lambda+1)$$

So, $\lambda=-1$ and $\lambda=3$ are the eigenvalues of A and since they are different, A is diagonalizable. Now, we will find the eigenvectors by the equation $(\lambda I-A)x=0$. When $\lambda=-1$, observe that

$$\begin{bmatrix} \lambda-1 & -2 \\ -2 & \lambda-1 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ -2 & -2 \end{bmatrix} \xrightarrow{(-1)R_{12}} \begin{bmatrix} -2 & -2 \\ 0 & 0 \end{bmatrix} \xrightarrow{\left(-\frac{1}{2}\right)R_1} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

So, $x = -y = -t$, where $t \in \mathbb{R}$ and $(x, y) = (-t, t) = t(-1,1)$. Hence, $(-1,1)$ is an eigenvector of A corresponding to $\lambda = -1$. When $\lambda = 3$, observe that

$$\begin{bmatrix} \lambda-1 & -2 \\ -2 & \lambda-1 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \xrightarrow{1R_{12}} \begin{bmatrix} 2 & -2 \\ 0 & 0 \end{bmatrix} \xrightarrow{\left(\frac{1}{2}\right)R_1} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

So, $x = y = t$, where $t \in \mathbb{R}$ and $(x, y) = (t, t) = t(1,1)$. Hence, $(1,1)$ is an eigenvector of A corresponding to $\lambda = 3$. Therefore,

$$P = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

Q7: Let $\mathbb{R}^3$ be the Euclidean inner product space. Apply the Gram-Schmidt process to transform the following basis $\{u_1=(0,1,-1), u_2=(0,5,3), u_3=(1,0,0)\}$ into an **orthonormal basis**. (6 marks)

Answer:

$$u_1 = (0,1,-1), u_2 = (0,5,3), u_3 = (1,0,0),$$

$$v_1 = u_1 = (0,1,-1)$$

$$v_2 = u_2 - \frac{<u_2, v_1>}{\|v_1\|^2} v_1$$

$$= (0,5,3) - \frac{<(0,5,3),(0,1,-1)>}{\|(0,1,-1)\|^2}(0,1,-1) = (0,5,3) - \tfrac{2}{2}(0,1,-1)$$

$$= (0,5,3) - (0,1,-1) = (0,4,4)$$

$$v_3 = u_3 - \frac{<u_3, v_2>}{\|v_2\|^2} v_2 - \frac{<u_3, v_1>}{\|v_1\|^2} v_1$$

$$= (1,0,0) - \frac{<(1,0,0),(0,1,-1)>}{\|(0,1,-1)\|^2}(0,1,-1) - \frac{<(1,0,0),(0,4,4)>}{\|(0,4,4)\|^2}(0,4,4)$$

$$= (1,0,0) - \tfrac{0}{2}(0,1,-1) - \tfrac{0}{18}(0,4,4) = (1,0,0)$$

$$w_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{2}}(0,1,-1)$$

$$w_2 = \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{32}}(0,4,4) = \frac{1}{4\sqrt{2}}(0,4,4) = \frac{1}{\sqrt{2}}(0,1,1)$$

$$w_3 = \frac{v_3}{\|v_3\|} = (1,0,0)$$

Q8: Let M_{22} be the vector space of square matrices of order 2, and let T: $M_{22} \to \mathbb{R}^2$ be the function defined by $T(A)=0 \in \mathbb{R}^2$ for all $A \in M_{22}$. Show that:

(i) T is a linear transformation. (2 marks)

(ii) Find a basis for ker(T). (2 marks)

(iii) Find $[T]_{S,B}$ where B and S are the standard bases of M_{22} and $\mathbb{R}^2$, respectively. (3 marks)

(iv) Find rank(T). (1 mark)

Answer: For all $A, B \in M_{22}$, $k \in \mathbb{R}$:

(i) 1- $T(A+B)=0=0+0=T(A)+T(B)$

2- $T(kA)=0=k0=kT(A)$

So T is linear.

(ii) $\ker(T)=\{A \in M_{22} \mid T(A)=0\}=M_{22}$

So, the set $B = \left\{\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right\}$ which is the standard basis of M_{22} is a basis of ker(T).

(iii) $T\left(\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\right) = (0,0)$, $T\left(\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}\right) = (0,0)$, $T\left(\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}\right) = (0,0)$ and $T\left(\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right) = (0,0)$.

Now,

$$\left[T\left(\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\right)\right]_S = \begin{bmatrix}0\\0\end{bmatrix}, \left[T\left(\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}\right)\right]_S = \begin{bmatrix}0\\0\end{bmatrix}, \left[T\left(\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}\right)\right]_S = \begin{bmatrix}0\\0\end{bmatrix}, \left[T\left(\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right)\right]_S = \begin{bmatrix}0\\0\end{bmatrix}$$

Therefore, $[T]_{S,B} = \begin{bmatrix}0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{bmatrix}$.

(iv) Since dim(ker(T))=dim(M_{22})=4, so nullity(T)=4 and hence rank(T)=dim(M_{22})-nullity(T)=4-4=0.

Q9: If A is a matrix such that $A\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}a\\b\end{bmatrix}$ and $A\begin{bmatrix}c\\d\end{bmatrix} = \begin{bmatrix}-c\\-d\end{bmatrix}$, where a, b, c and d are real numbers such that $ad \neq bc$, then:

(i) Show that A is diagonalizable. (2 marks)

(ii) Show that A is invertible. (1 mark)

(iii) Show that $A^2 = I$. (2 marks)

Answer: (i) Since $A\begin{bmatrix}a\\b\end{bmatrix} = 1\begin{bmatrix}a\\b\end{bmatrix}$ and $A\begin{bmatrix}c\\d\end{bmatrix} = (-1)\begin{bmatrix}c\\d\end{bmatrix}$, so 1 and -1 are two eigenvalues of A. But from the products of the matrices, A is of size 2×2. Hence, it should have at most two eigenvalues. Since the eigenvalues are different, A is diagonalizable.

(ii) Since the eigenvalues are non-zero, A is invertible (in fact, det(A)=1(−1)=−1). (in fact, we don't need the condition $ad \neq bc$, but to make the parts independent from each other)

It can be solved by another way: $A\begin{bmatrix}a\\b\end{bmatrix} = 1\begin{bmatrix}a\\b\end{bmatrix}$ and $A\begin{bmatrix}c\\d\end{bmatrix} = (-1)\begin{bmatrix}c\\d\end{bmatrix}$ implies that:

$$A\begin{bmatrix}a & c\\b & d\end{bmatrix} = \begin{bmatrix}a & -c\\b & -d\end{bmatrix}$$

As $ad \neq bc$ then $det\begin{bmatrix}a & -c\\b & -d\end{bmatrix} \neq 0$ and hence $\det(A)\, det\begin{bmatrix}a & c\\b & d\end{bmatrix} \neq 0$. Therefore, |A|≠0 and A is invertible.

(iii) A is diagonalizable, so there exists an invertible matrix P such that $P^{-1}AP = D = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$. Thus, $A = PDP^{-1}$ and hence $A^2 = PD^2P^{-1} = PIP^{-1} = I$. (in fact, we don't need the condition $ad \neq bc$, but to make the parts independent from each other)

It can be solved by another way: As $ad \neq bc$ then $det\begin{bmatrix}a & c\\b & d\end{bmatrix} \neq 0$ and hence

$$A\begin{bmatrix}a & c\\b & d\end{bmatrix} = \begin{bmatrix}a & -c\\b & -d\end{bmatrix} \Rightarrow A = \begin{bmatrix}a & -c\\b & -d\end{bmatrix}\begin{bmatrix}a & c\\b & d\end{bmatrix}^{-1}$$

$$= \frac{1}{ad-bc}\begin{bmatrix}a & -c\\b & -d\end{bmatrix}\begin{bmatrix}d & -c\\-b & a\end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix}ad+bc & -2ac\\2bd & -ad-bc\end{bmatrix}$$

$$\Rightarrow A^2 = \left(\frac{1}{ad-bc}\right)^2\begin{bmatrix}ad+bc & -2ac\\2bd & -ad-bc\end{bmatrix}^2$$

$$= \left(\frac{1}{ad-bc}\right)^2\begin{bmatrix}(ad+bc)^2 - 4abcd & -2ac(ad+bc) + 2a(ad+bc)\\2bd(ad+bc) - 2bd(ad+bc) & (ad+bc)^2 - 4abcd\end{bmatrix}$$

$$= \left(\frac{1}{ad-bc}\right)^2\begin{bmatrix}(ad-bc)^2 & 0\\0 & (ad-bc)^2\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

Q10: Let V be an inner product space. If u and v are orthonormal vectors in V, then find d(u,v). (2 marks)

Answer: $d(u,v)^2 = \|u-v\|^2 = \langle u-v, u-v\rangle = \langle u,u\rangle - 2\langle u,v\rangle + \langle v,v\rangle = \|u\|^2 + \|v\|^2 = 1+1 = 2$, Since $\langle u,v\rangle = 0$. So, $d(u,v) = (2)^{0.5} = \sqrt{2}$.