

MATH 111 - Integral Calculus
First Semester - 1446 H
Solution of the Second Exam
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Question (1): [4 marks]

Find $\frac{dy}{dx}$ of the following :

1. $y = \operatorname{csch}\left(\frac{2}{x}\right) + \tanh^{-1}(e^{5x})$. [2]

Solution :

$$\begin{aligned}\frac{dy}{dx} &= -\operatorname{csch}\left(\frac{2}{x}\right) \coth\left(\frac{2}{x}\right) \left(\frac{-2}{x^2}\right) + \frac{1}{1-(e^{5x})^2} (e^{5x} (5)) \\ &= \frac{2}{x^2} \operatorname{csch}\left(\frac{2}{x}\right) \coth\left(\frac{2}{x}\right) + \frac{5e^{5x}}{1-e^{10x}} .\end{aligned}$$

2. $y = \sinh^{-1}(\sqrt{x}) + \coth(4x^2)$. [2]

Solution :

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{\sqrt{1+(\sqrt{x})^2}} \left(\frac{1}{2\sqrt{x}}\right) + (-\operatorname{csch}^2(4x^2) (8x)) \\ &= \frac{1}{2\sqrt{x}\sqrt{1+x}} - 8x \operatorname{csch}^2(4x^2) .\end{aligned}$$

Question (2): [21 marks]

Evaluate the following integrals :

1. $\int \frac{\operatorname{sech}^2(x^{-2})}{x^3} dx$. [2]

Solution :

$$\begin{aligned}\int \frac{\operatorname{sech}^2(x^{-2})}{x^3} dx &= \int \operatorname{sech}^2(x^{-2}) x^{-3} dx \\ &= \frac{1}{-2} \int \operatorname{sech}^2(x^{-2}) (-2x^{-3}) dx = -\frac{1}{2} \tanh(x^{-2}) + c\end{aligned}$$

2. $\int \frac{3}{\sqrt{x^2 + 6x - 16}} dx$. [3]

Solution : By completing the square.

$$x^2 + 6x - 16 = (x^2 + 6x + 9) - 16 - 9 = (x + 3)^2 - (5)^2 .$$

$$\begin{aligned}\int \frac{3}{\sqrt{x^2 + 6x - 16}} dx &= 3 \int \frac{1}{\sqrt{(x+3)^2 - (5)^2}} dx \\ &= 3 \cosh^{-1} \left(\frac{x+3}{5} \right) + c .\end{aligned}$$

3. $\int e^x \cosh 2x dx$. [2]

Solution :

$$\begin{aligned}\int e^x \cosh 2x dx &= \int e^x \left(\frac{e^{2x} + e^{-2x}}{2} \right) dx = \int \left(\frac{e^{3x} + e^{-x}}{2} \right) dx \\ \int \left(\frac{e^{3x}}{2} + \frac{e^{-x}}{2} \right) dx &= \frac{1}{2} \frac{1}{3} \int e^{3x} (3) dx + \frac{1}{2} \frac{1}{-1} \int e^{-x} (-1) dx \\ &= \frac{1}{6} e^{3x} - \frac{1}{2} e^{-x} + c = \frac{e^{3x}}{6} - \frac{e^{-x}}{2} + c .\end{aligned}$$

4. $\int (3x - 2) \sinh x dx$. [2]

Solution : Using integration by parts.

$$\begin{aligned}u &= 3x - 2 & dv &= \sinh x dx \\ du &= 3 dx & v &= \cosh x\end{aligned}$$

$$\begin{aligned}\int (3x - 2) \sinh x dx &= (3x - 2) \cosh x - \int 3 \cosh x dx \\ &= (3x - 2) \cosh x - 3 \int \cosh x dx = (3x - 2) \cosh x - 3 \sinh x + c .\end{aligned}$$

5. $\int x^{-2} \ln x dx$. [2]

Solution : Using integration by parts.

$$\begin{aligned}u &= \ln x & dv &= x^{-2} dx \\ du &= \frac{1}{x} dx & v &= \frac{x^{-1}}{-1} = -\frac{1}{x}\end{aligned}$$

$$\begin{aligned}\int x^{-2} \ln x dx &= \frac{-1}{x} \ln x - \int \frac{-1}{x} \frac{1}{x} dx \\ &= \frac{-\ln x}{x} - \int \frac{-1}{x^2} dx = \frac{-\ln x}{x} - \frac{1}{x} + c .\end{aligned}$$

6. $\int \sin^4 x \cos^5 x \, dx$. [2]

Solution :

Using the substitution $u = \sin x$.

Hence $du = \cos x \, dx$.

$$\begin{aligned} \int \sin^4 x \cos^5 x \, dx &= \int \sin^4 x \cos^4 x \cos x \, dx = \int \sin^4 x (\cos^2 x)^2 \cos x \, dx \\ &= \int \sin^4 x (1 - \sin^2 x)^2 \cos x \, dx = \int u^4 (1 - u^2)^2 \, du = \int u^4 (1 - 2u^2 + u^4) \, du \\ &= \int (u^4 - 2u^6 + u^8) \, du = \frac{u^5}{5} - 2 \frac{u^7}{7} + \frac{u^9}{9} + c \\ &= \frac{\sin^5 x}{5} - 2 \frac{\sin^7 x}{7} + \frac{\sin^9 x}{9} + c \end{aligned}$$

7. $\int \frac{\sqrt{x^2 - 9}}{x} \, dx$. [3]

Solution : Using trigonometric substitutions.

$$\text{Put } x = 3 \sec \theta \implies \sec \theta = \frac{x}{3} \implies \cos \theta = \frac{3}{x} .$$

$$dx = 3 \sec \theta \tan \theta \, d\theta .$$

$$\sqrt{x^2 - 9} = \sqrt{9 \sec^2 \theta - 9} = \sqrt{9(\sec^2 \theta - 1)} = \sqrt{9 \tan^2 \theta} = 3 \tan \theta$$

$$= \int \frac{\sqrt{x^2 - 9}}{x} \, dx = \int \frac{(3 \tan \theta)(3 \sec \theta \tan \theta)}{3 \sec \theta} \, d\theta = 3 \int \tan^2 \theta \, d\theta$$

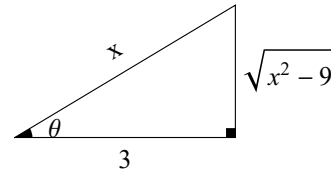
$$= 3 \int (\sec^2 \theta - 1) \, d\theta = 3(\tan \theta - \theta) + c = 3 \tan \theta - 3\theta + c$$

$$\cos \theta = \frac{3}{x} .$$

From the triangle :

$$\tan \theta = \frac{\sqrt{x^2 - 9}}{\frac{3}{x}}$$

$$\theta = \sec^{-1} \left(\frac{x}{3} \right)$$



$$\int \frac{\sqrt{x^2 - 9}}{x} \, dx = 3 \left(\frac{\sqrt{x^2 - 9}}{3} \right) - 3 \sec^{-1} \left(\frac{x}{3} \right) + c$$

$$= \sqrt{x^2 - 9} - 3 \sec^{-1} \left(\frac{x}{3} \right) + c .$$

8. $\int \frac{5x^2 + x + 8}{x^3 + 4x} dx$. [3]

Solution : Using the method of partial fractions.

$$\frac{5x^2 + x + 8}{x^3 + 4x} = \frac{5x^2 + x + 8}{x(x^2 + 4)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4}$$

$$\frac{5x^2 + x + 8}{x(x^2 + 4)} = \frac{A(x^2 + 4)}{x(x^2 + 4)} + \frac{x(Bx + C)}{x(x^2 + 4)}$$

$$5x^2 + x + 8 = A(x^2 + 4) + x(Bx + C)$$

$$5x^2 + x + 8 = Ax^2 + 4A + Bx^2 + Cx = (A + B)x^2 + Cx + 4A$$

By comparing the coefficients of the two polynomials in each side :

$$A + B = 5 \quad \longrightarrow \quad (1)$$

$$C = 1 \quad \longrightarrow \quad (2)$$

$$4A = 8 \quad \longrightarrow \quad (3)$$

From equation (3) : $4A = 8 \implies A = \frac{8}{4} = 2$.

From equation (1) : $2 + B = 5 \implies B = 3$.

$$\begin{aligned} \int \frac{5x^2 + x + 8}{x^3 + 4x} dx &= \int \left(\frac{2}{x} + \frac{3x + 1}{x^2 + 4} \right) dx \\ &= \int \frac{2}{x} dx + \int \frac{3x}{x^2 + 4} dx + \int \frac{1}{x^2 + 4} dx \\ &= 2 \int \frac{1}{x} dx + \frac{3}{2} \int \frac{2x}{x^2 + 4} dx + \int \frac{1}{x^2 + (2)^2} dx \\ &= 2 \ln |x| + \frac{3}{2} \ln(x^2 + 4) + \frac{1}{2} \tan^{-1} \left(\frac{x}{2} \right) + c . \end{aligned}$$

9. $\int \frac{dx}{\sqrt[4]{x} + \sqrt{x}}$. [2]

Solution : $\int \frac{dx}{\sqrt[4]{x} + \sqrt{x}} = \int \frac{1}{x^{\frac{1}{4}} + x^{\frac{1}{2}}} dx$

Using the substitution $x = u^4$, then $u = x^{\frac{1}{4}}$.

$$dx = 4u^3 du .$$

$$\begin{aligned} \int \frac{dx}{x^{\frac{1}{4}} + x^{\frac{1}{2}}} &= \int \frac{4u^3}{(u^4)^{\frac{1}{4}} + (u^4)^{\frac{1}{2}}} du = \int \frac{4u^3}{u + u^2} du \\ &= \int \frac{4u^3}{u(1 + u)} du = 4 \int \frac{u^2}{u + 1} du \end{aligned}$$

Using long division of polynomials :

$$\begin{aligned}4 \int \frac{u^2}{u+1} du &= 4 \int \left(u - 1 + \frac{1}{u+1} \right) du \\&= 4 \left(\frac{u^2}{2} - u + \ln |u+1| \right) + c = 2u^2 - 4u + 4 \ln |u-1| + c \\ \int \frac{dx}{x^{\frac{1}{4}} + x^{\frac{1}{2}}} &= 2 \left(x^{\frac{1}{4}} \right)^2 - 4x^{\frac{1}{4}} + 4 \ln \left| x^{\frac{1}{4}} + 1 \right| + c \\&= 2x^{\frac{1}{2}} - 4x^{\frac{1}{4}} + 4 \ln \left| x^{\frac{1}{4}} + 1 \right| + c .\end{aligned}$$