MATH 111 - Integral Calculus
First Semester - 1446 H
Solution of the First Exam
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Question (1): [9 marks]

1. Use Riemann Sum to evaluate the definite integral $\int_0^2 (x^2+3)\ dx$. [3]

   Solution : $[a,b]=[0,2]$, $f(x)=x^2+3$.

   $$\Delta_x = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n}$$

   $$x_k = a + k\ \Delta_x = 0 + k\left(\frac{2}{n}\right) = \frac{2k}{n}$$

   $$f(x_k) = \left(\frac{2k}{n}\right)^2 + 3 = \frac{4k^2}{n^2} + 3$$

   $$R_n = \sum_{k=1}^{n} f(x_k)\ \Delta_x = \sum_{k=1}^{n}\left(\frac{4k^2}{n^2}+3\right)\ \left(\frac{2}{n}\right)$$

   $$= \sum_{k=1}^{n}\left(\frac{8k^2}{n^3}+\frac{6}{n}\right) = \sum_{k=1}^{n}\frac{8k^2}{n^3} + \sum_{k=1}^{n}\frac{6}{n}$$

   $$= \frac{8}{n^3}\sum_{k=1}^{n}k^2 + \frac{6}{n}\sum_{k=1}^{n}1 = \frac{8}{n^3}\ \frac{n(n+1)(2n+1)}{6} + \frac{6}{n}\ (n)$$

   $$= \frac{4}{3}\left(\frac{(n+1)(2n+1)}{n^2}\right) + 6$$

   $$\int_0^2 (x^2+3)\ dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\left[\frac{4}{3}\left(\frac{(n+1)(2n+1)}{n^2}\right)+6\right]$$

   $$= \frac{4}{3}\ (2) + 6 = \frac{8}{3} + 6 = \frac{26}{3}\ .$$

2. Find $F'(x)$, if $F(x) = \int_{\tan\left(\frac{x}{2}\right)}^{3^{5x}} \sqrt{t^2+1}\ dt$. [2]

   Solution :

   $$F'(x) = \frac{d}{dx}\int_{\tan\left(\frac{x}{2}\right)}^{3^{5x}} \sqrt{t^2+1}\ dt$$

   $$= \sqrt{\left(3^{5x}\right)^2+1}\ \left(3^{5x}(5)\ \ln 3\right) - \sqrt{\left(\tan\left(\frac{x}{2}\right)\right)^2+1}\ \left(\sec^2\left(\frac{x}{2}\right)\ \left(\frac{1}{2}\right)\right)$$

   $$= 5\ 3^{5x}\ \ln 3\ \sqrt{3^{10x}+1} - \frac{1}{2}\ \sec^2\left(\frac{x}{2}\right)\ \sqrt{\tan^2\left(\frac{x}{2}\right)+1}\ .$$

Find $\dfrac{dy}{dx}$ of the following :

3. $y = \left[\cos^{-1}(3x)\right] \ \log|\csc x - \cot x|$. [2]

Solution :

$$\frac{dy}{dx} = \left(\frac{-1}{\sqrt{1-(3x)^2}}\ (3)\right)\ \log|\csc x - \cot x|$$

$$+ \left[\cos^{-1}(3x)\right]\ \left(\frac{-\csc x \cot x - (-\csc^2)x)}{\csc x - \cot x}\ \frac{1}{\ln 10}\right)$$

$$= \frac{-3\ \log|\csc x - \cot x|}{\sqrt{1-9x^2}} + \frac{\csc x\ \cos^{-1}(3x)}{\ln 10}\ .$$

4. $y = (\cot x)^{\csc x} + 3^{x^2}$. [2]

Solution :

Let $y = f(x) + g(x)$, where $f(x) = (\cot x)^{\csc x}$ and $g(x) = 3^{x^2}$.

Then $\dfrac{dy}{dx} = y' = f'(x) + g'(x)$

First - $g'(x) = 3^{x^2}\ (2x)\ln 3 = 2x\ 3^{x^2}\ \ln 3$

Second - Finding $f'(x)$

$f(x) = (\cot x)^{\csc x} \implies \ln|f(x)| = \ln|(\cot x)^{\csc x}| = \csc x\ \ln|\cot x|$

Differentiate both sides.

$$\frac{f'(x)}{f(x)} = (-\csc x\ \cot x)\ \ln|\cot x| + \csc x\left(\frac{-\csc^2 x}{\cot x}\right)$$

$$f'(x) = f(x)\left[-\csc x\ \cot x\ \ln|\cot x| - \frac{\csc^3 x}{\cot x}\right]$$

$$= (\cot x)^{\csc x}\ \left[-\csc x\ \cot x\ \ln|\cot x| - \frac{\csc^3 x}{\cot x}\right]$$

Therefore, $\dfrac{dy}{dx} = (\cot x)^{\csc x}\ \left[-\csc x\ \cot x\ \ln|\cot x| - \dfrac{\csc^3 x}{\cot x}\right] + 2x\ 3^{x^2}\ \ln 3$

Question (2): [16 marks]
Evaluate the following integrals :

1. $\displaystyle\int \left(x^{\frac{2}{5}}\ \sqrt[5]{\sin(x^3)}\right)^5\ dx$. [2]

Solution :

$$\int \left(x^{\frac{2}{5}}\ \sqrt[5]{\sin(x^3)}\right)^5\ dx = \int \left(x^{\frac{2}{5}}\right)^5\ \left(\sqrt[5]{\sin(x^3)}\right)^5\ dx = \int x^2\ \sin(x^3)\ dx$$

$$= \frac{1}{3} \int \sin(x^3)\ (3x^2)\ dx = \frac{1}{3}\ \left(-\cos(x^3)\right) + c = -\frac{1}{3}\ \cos(x^3) + c\ .$$

2. $\displaystyle\int \frac{-1}{\sqrt{e^{6x} - 4}}\ dx$. [2]

Solution : .

$$\int \frac{-1}{\sqrt{e^{6x} - 4}}\ dx = \int \frac{-1}{\sqrt{\left(e^{3x}\right)^2 - (2)^2}}\ dx = -\int \frac{e^{3x}}{e^{3x}\ \sqrt{\left(e^{3x}\right)^2 - (2)^2}}\ dx$$

$$= -\frac{1}{3} \int \frac{e^{3x}\ (3)}{e^{3x}\ \sqrt{\left(e^{3x}\right)^2 - (2)^2}}\ dx = -\frac{1}{3} \left(\frac{1}{2}\ \sec^{-1}\left(\frac{e^{3x}}{2}\right)\right) + c\ .$$

$$= -\frac{1}{6}\ \sec^{-1}\left(\frac{e^{3x}}{2}\right) + c\ .$$

3. $\displaystyle\int_0^1 e^{2\ln x}\ 3^{3x^3}\ dx$. [2]

Solution :

$$\int_0^1 e^{2\ln x}\ 3^{3x^3}\ dx = \int_0^1 e^{\ln x^2}\ 3^{3x^3}\ dx = \int_0^1 x^2\ 3^{3x^3}\ dx$$

$$= \frac{1}{9} \int_0^1 3^{3x^3} (9x^2)\ dx = \frac{1}{9} \left[\frac{3^{3x^3}}{\ln 3}\right]_0^1 = \frac{1}{9} \left[\frac{3^{3(1)^3}}{\ln 3} - \frac{3^{3(0)^3}}{\ln 3}\right]$$

$$= \frac{1}{9} \left[\frac{3^3}{\ln 3} - \frac{3^0}{\ln 3}\right] = \frac{1}{9}\ \left(\frac{27 - 1}{\ln 3}\right) = \frac{26}{9 \ln 3}\ .$$

4. $\displaystyle\int \frac{x + 1}{\sqrt{1 - x^2}}\ dx$. [2]

Solution :

$$\int \frac{x + 1}{\sqrt{1 - x^2}}\ dx = \int \left[\frac{x}{\sqrt{1 - x^2}} + \frac{1}{\sqrt{1 - x^2}}\right]\ dx$$

$$= \int \frac{x}{\sqrt{1 - x^2}}\ dx + \int \frac{1}{\sqrt{1 - x^2}}\ dx$$

$$= -\frac{1}{2} \int \left(1 - x^2\right)^{-\frac{1}{2}} (-2x)\ dx + \int \frac{1}{\sqrt{1 - x^2}}\ dx$$

$$= -\frac{1}{2}\ \frac{\left(1 - x^2\right)^{\frac{1}{2}}}{\frac{1}{2}} + \sin^{-1} x + c = -\sqrt{1 - x^2} + \sin^{-1} x + c\ .$$

5. $\displaystyle\int \frac{\sec\left(\ln(x^2)\right)}{x}\ dx$. [2]

Solution :

$$\int \frac{\sec\left(\ln(x^2)\right)}{x}\ dx = \int \sec\left(2\ln|x|\right)\ \frac{1}{x}\ dx$$

$$= \frac{1}{2}\int \sec\left(2\ln|x|\right)\ \frac{2}{x}\ dx = \frac{1}{2}\ln\left|\sec\left(2\ln|x|\right) + \tan\left(2\ln|x|\right)\right| + c$$

6. $\displaystyle\int \frac{\tan\left(\sqrt[3]{x}\right)}{x^{\frac{2}{3}}}\ dx$. [2]

Solution :

$$\int \frac{\tan\left(\sqrt[3]{x}\right)}{x^{\frac{2}{3}}}\ dx = \int \tan\left(x^{\frac{1}{3}}\right)\ x^{-\frac{2}{3}}\ dx$$

$$= 3\int \tan\left(x^{\frac{1}{3}}\right)\left(\frac{1}{3}\ x^{-\frac{2}{3}}\right)\ dx = 3\ln\left|\sec\left(\sqrt[3]{x}\right)\right| + c$$

7. $\displaystyle\int \frac{\left(1+\sec^{-1}x\right)^5}{\sqrt{x^4-x^2}}\ dx$. [2]

Solution :

$$\int \frac{\left(1+\sec^{-1}x\right)^5}{\sqrt{x^4-x^2}}\ dx = \int \frac{\left(1+\sec^{-1}x\right)^5}{\sqrt{x^2\left(x^2-1\right)}}\ dx$$

$$= \int \frac{\left(1+\sec^{-1}x\right)^5}{|x|\sqrt{x^2-1}}\ dx = \int \left(1+\sec^{-1}x\right)^5\left(\frac{1}{x\sqrt{x^2-1}}\right)\ dx$$

$$= \frac{\left(1+\sec^{-1}x\right)^6}{6} + c\ ,\ \text{where } x > 0\ .$$

8. $\displaystyle\int \cos x\ \left(\sin^2 x\right)^{-1}\ dx$. [2]

First solution :

$$\int \cos x\ \left(\sin^2 x\right)^{-1}\ dx = \int (\sin x)^{-2}\cos x\ dx = \frac{(\sin x)^{-1}}{-1} + c = -\csc x + c$$

Second solution :

$$\int \cos x\ \left(\sin^2 x\right)^{-1}\ dx = \int \frac{\cos x}{\sin^2 x}\ dx = \int \frac{1}{\sin x}\,\frac{\cos x}{\sin x}\ dx$$

$$= \int \csc x\ \cot x\ dx = -\csc x + c\ .$$