MATH 111 - Integral Calculus
First Semester - 1446 H
Solution of the Final Exam

Dr Tariq A. Alfadhel

Question (1): [7 marks]

1. Find the value of c that satisfies the mean value theorem of the definite integral for the function $f(x) = 4x - x^2$ on the interval $[0, 3]$. [3]

   Solution : Using the formula $(b-a)\ f(c) = \displaystyle\int_a^b f(x)\ dx$.

   $$(3-0)\ (4c-c^2) = \int_0^3 (4x-x^2)\ dx = \left[2x^2 - \frac{x^3}{3}\right]_0^3$$

   $$3(4c-c^2) = \left(2(3)^2 - \frac{(3)^3}{3}\right) - \left(2(0)^2 - \frac{(0)^3}{3}\right) = 18 - 9 = 9$$

   $$3(4c-c^2) = 9 \implies 4c - c^2 = 3 \implies c^2 - 4c + 3 = 0$$

   $$\implies (c-3)(c-1) = 0 \implies c = 3\ ,\ c = 1.$$

   Note that $c = 1 \in (0,3)$ while $c = 3 \notin (0,3)$.

   The desired value is $c = 1$.

2. Find $F'(x)$, if $F(x) = \displaystyle\int_{\sec x}^{3^{2x}} \sqrt{2t^2-1}\ dt$. [2]

   Solution :

   $$F'(x) = \frac{d}{dx}\int_{\sec x}^{3^{2x}} \sqrt{2t^2-1}\ dt$$

   $$= \sqrt{2\left(3^{2x}\right)^2 - 1}\ \left(3^{2x}(2)\ \ln 3\right) - \sqrt{2\left(\sec x\right)^2 - 1}\ \left(\sec x\ \tan x\right)$$

   $$= 2\ln 3\left(3^{2x}\right)\sqrt{2\ \left(3^{4x}\right) - 1} - \sec x \tan x\sqrt{2\sec^2 x - 1}\ .$$

3. Find y' if $y = \cosh^{-1}\left(3^{x^2-1}\right) + \log|\tanh(2x)|$. [2]

   Solution :

   $$y' = \frac{1}{\sqrt{\left(3^{x^2-1}\right)^2 - 1}}\ \left(3^{x^2-1}\ (2x)\ \ln 3\right) + \frac{sech^2(2x)\ (2)}{\tanh(2x)}\ \frac{1}{\ln 10}$$

   $$= \frac{2x\ln 3\ \left(3^{x^2-1}\right)}{\sqrt{3^{2x^2-2} - 1}} + \frac{2\ sech^2(2x)}{\tanh(2x)\ \ln 10}\ .$$

Question (2): [14 marks]
Evaluate the following integrals :

1. $\displaystyle\int \frac{2}{\sqrt{-x^2+6x-8}}\, dx$. [3]

   Solution : By completing the square.

   $-x^2+6x-8 = -(x^2-6x)-8 = -(x^2-6x+9)-8+9$

   $= -(x-3)^2+1 = 1-(x-3)^2$.

   $$\int \frac{2}{\sqrt{-x^2+6x-8}}\, dx = 2\int \frac{1}{\sqrt{1-(x-3)^2}}\, dx = 2\sin^{-1}(x-3)+c \ .$$

2. $\displaystyle\int x^{-3}\ \ln|x|\ dx$. [3]

   Solution : Using integration by parts .

   $u = \ln x \qquad dv = x^{-3}\ dx$

   $du = \frac{1}{x}\ dx \qquad v = \frac{x^{-2}}{-2} = \frac{1}{-2x^2}$

   $$\int x^{-3}\ \ln x\ dx = \frac{1}{-2x^2}\ln x - \int \frac{1}{-2x^2}\ \frac{1}{x}\ dx$$

   $$= \frac{\ln x}{-2x^2} + \frac{1}{2}\int x^{-3}\ dx = \frac{\ln x}{-2x^2} + \frac{1}{2}\left(\frac{x^{-2}}{-2}\right) + c = \frac{\ln x}{-2x^2} - \frac{1}{4x^2} + c \ .$$

3. $\displaystyle\int \frac{\sqrt{x^2+9}}{x^4}\, dx$. [3]

   Solution : Using trigonometric substitutions.

   Put $x = 3\tan\theta \implies \tan\theta = \frac{x}{3}$.

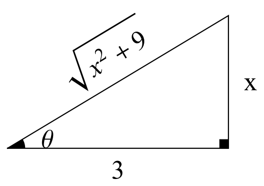
   $dx = 3\sec^2\theta\ d\theta$.

   $\sqrt{x^2+9} = \sqrt{9\tan^2\theta+9} = \sqrt{9\left(\tan^2\theta+1\right)} = \sqrt{9\sec^2\theta} = 3\sec\theta$.

   $$\int \frac{\sqrt{x^2+9}}{x^4}\, dx = \int \frac{3\sec\theta\ 3\sec^2\theta}{(3\tan\theta)^4}\, d\theta = \frac{3^2}{3^4}\int \frac{\sec^3\theta}{\tan^4\theta}\, d\theta$$

   $$= \frac{1}{9}\int \frac{\cos^4\theta}{\sin^4\theta\ \cos^3\theta}\, d\theta = \frac{1}{9}\int (\sin\theta)^{-4}\ \cos\theta\ d\theta = \frac{1}{9}\ \frac{(\sin\theta)^{-3}}{-3} + c$$

   $\tan\theta = \frac{x}{3}$.

   From the triangle :
   $\sin\theta = \frac{x}{\sqrt{x^2+9}}$

$$\int \frac{\sqrt{x^2+9}}{x^4}\, dx = \frac{1}{-27} \left(\frac{x}{\sqrt{x^2+9}} \right)^{-3} + c \ .$$

4. $\displaystyle\int \frac{3x+1}{x^3+x}\, dx$. [3]

Solution : Using the method of partial fractions.

$$\frac{3x+1}{x^3+x} = \frac{3x+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$\frac{3x+1}{x(x^2+1)} = \frac{A(x^2+1)}{x(x^2+1)} + \frac{x\,(Bx+C)}{x(x^2+1)}$$

$$3x+1 = A(x^2+1) + x(Bx+C)$$

$$3x+1 = Ax^2 + A + Bx^2 + Cx = (A+B)x^2 + Cx + A$$

By comparing the coefficients of the two polynomials in each side :

$$\begin{array}{lll} A+B=0 & \longrightarrow & (1) \\ C=3 & \longrightarrow & (2) \\ A=1 & \longrightarrow & (3) \end{array}$$

From equation (1) : $1+B=0 \implies B=-1$.

$$\int \frac{3x+1}{x^3+x}\, dx = \int \left(\frac{1}{x} + \frac{-x+3}{x^2+1} \right) dx$$

$$= \int \frac{1}{x}\, dx + \int \frac{-x}{x^2+1}\, dx + \int \frac{3}{x^2+1}\, dx$$

$$= \int \frac{1}{x}\, dx - \frac{1}{2} \int \frac{2x}{x^2+1}\, dx + 3 \int \frac{1}{x^2+(1)^2}\, dx$$

$$= \ln|x| - \frac{1}{2} \ln(x^2+1) + 3\tan^{-1}(x) + c \ .$$

5. $\displaystyle\int \frac{1}{\sqrt{x}\,(1+x)}\, dx$. [2]

Solution : Using the substitution $u = \sqrt{x} \implies u^2 = x$.

$2u\ du = dx$.

$$\int \frac{1}{\sqrt{x}\,(1+x)}\, dx = \int \frac{2u}{u(1+u^2)}\, du = 2 \int \frac{1}{1+u^2}\, du$$

$$= 2\tan^{-1}(u) + c = 2\tan^{-1}\left(\sqrt{x}\right) + c \ .$$

Question (3): [19 marks]

1. Evaluate the limit $\lim_{x\to\infty} e^{-x^3}\ (x^4+1)$. [2]

   Solution :

   $\lim_{x\to\infty} e^{-x^3}\ (x^4+1) \qquad (0.\infty)$

   $\lim_{x\to\infty} e^{-x^3}\ (x^4+1) = \lim_{x\to\infty} \frac{x^4+1}{e^{x^3}} \qquad \left(\frac{\infty}{\infty}\right)$

   Using L'Hôpital's rule.

   $\lim_{x\to\infty} \frac{4\ x^3}{e^{x^3}\ (3x^2)} = \lim_{x\to\infty} \frac{4x}{3e^{x^3}} \qquad \left(\frac{\infty}{\infty}\right)$

   Using L'Hôpital's rule.

   $\lim_{x\to\infty} \frac{4}{3e^{x^3}(3x^2)} = \lim_{x\to\infty} \frac{4}{9x^2\ e^{x^3}} = 0$.

   Therefore, $\lim_{x\to\infty} e^{-x^3}\ (x^4+1) = 0$.

   Note that $\lim_{x\to\infty} e^{x^3} = \infty$ and $\lim_{x\to\infty} 9x^2 = \infty$.

2. Discus whether the improper integral $\int_2^\infty \frac{1}{x\,(\ln x)^2}\, dx$ converges or diverges. [3]

   Solution :

   $\int_2^\infty \frac{1}{x\,(\ln x)^2}\, dx = \lim_{t\to\infty} \int_2^t \frac{1}{x\,(\ln x)^2}\, dx = \lim_{t\to\infty} \int_2^t (\ln x)^{-2}\, \frac{1}{x}\, dx$

   $= \lim_{t\to\infty} \left[\frac{(\ln x)^{-1}}{-1}\right]_2^t = \lim_{t\to\infty} \left[\frac{-1}{\ln x}\right]_2^t = \lim_{t\to\infty} \left[\frac{-1}{\ln t} - \frac{-1}{\ln 2}\right]$

   $= 0 - \frac{-1}{\ln 2} = \frac{1}{\ln 2}$.

   Hence, the improper integral converges.

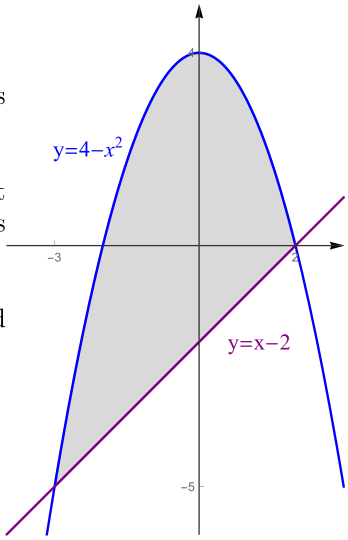
3. Sketch the region region bounded by the curves $y = 4 - x^2$ and $y = x - 2$, and find its area. [3]

   Solution :

$y = 4 - x^2$ represents a parabola opens downwards, and its vertex is $(0, 4)$.

$y = x - 2$ represents a straight line passing through $(0, -2)$, and its slope equals 1.

Points of intersection of $y = 4 - x^2$ and $y = x - 2$:
$x - 2 = 4 - x^2 \implies x^2 + x - 6 = 0$
$\implies (x + 3)(x - 2) = 0$
$\implies x = -3$, $x = 2$.



$$\mathbf{A} = \int_{-3}^{2} \left[(4 - x^2) - (x - 2) \right] \, dx = \int_{-3}^{2} (4 - x^2 - x + 2) \, dx$$

$$= \int_{-3}^{2} (-x^2 - x + 6) \, dx = \left[-\frac{x^3}{3} - \frac{x^2}{2} + 6x \right]_{-3}^{2}$$

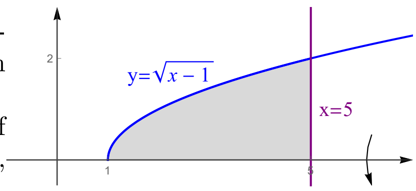
$$= \left(-\frac{(2)^3}{3} - \frac{(2)^2}{2} + 6(2) \right) - \left(-\frac{(-3)^3}{3} - \frac{(-3)^2}{2} + 6(-3) \right)$$

$$= \left(\frac{-8}{3} - 2 + 12 \right) - \left(9 - \frac{9}{2} - 18 \right) = 19 - \frac{8}{3} + \frac{9}{2} = \frac{125}{6} \; .$$

4. Sketch the region bounded bu the curves $y = \sqrt{x - 1}$, $y = 0$ and $x = 5$, and find the volume of the solid generated by revolving this region about the $x-$axis. [3]

Solution :

$y = 0$ represents the x-axis.
$x = 5$ represents a straight line parallel to the y-axis, and passing through $(5, 0)$.
$y = \sqrt{x - 1}$ represents the upper half of the parabola $x = y^2$ opens to the right, and its vertex is $(0, 1)$.



Using the disk method.

$$\mathbf{V} = \pi \int_{1}^{5} \left(\sqrt{x - 1} \right)^2 \, dx = \pi \int_{1}^{5} (x - 1) \, dx = \left[\frac{x^2}{2} - x \right]_{1}^{5}$$

$$= \left(\frac{25}{2} - 5 \right) - \left(\frac{1}{2} - 1 \right) = \frac{25}{2} - 5 - \frac{1}{2} + 1 = 12 - 4 = 8 \; .$$

5. Find the arc length of the function $y = \ln|\sec x|$ from $x = 0$ to $x = \frac{\pi}{4}$. [3]

Solution :

$y' = \dfrac{\sec x \tan x}{\sec x} = \tan x$.

$\mathbf{L} = \displaystyle\int_0^{\frac{\pi}{4}} \sqrt{1 + (\tan x)^2}\; dx = \int_0^{\frac{\pi}{4}} \sqrt{1 + \tan^2 x}\; dx = \int_0^{\frac{\pi}{4}} \sqrt{\sec^2 x}\; dx$

$= \displaystyle\int_0^{\frac{\pi}{4}} |\sec x|\; dx = \int_0^{\frac{\pi}{4}} \sec x\; dx = [\ln|\sec x + \tan x|]_0^{\frac{\pi}{4}}$

$= \ln\left|\sec\left(\frac{\pi}{4}\right) + \tan\left(\frac{\pi}{4}\right)\right| - \ln|\sec(0) + \tan(0)|$

$= \ln\left|\sqrt{2} + 1\right| - \ln|1 + 0| = \ln\left|1 + \sqrt{2}\right|$.

6. Convert the polar equation $r = \dfrac{1}{\sin\theta - 2\cos\theta}$ into a Cartesian equation.[1]

Solution :

$r = \dfrac{1}{\sin\theta - 2\cos\theta} \implies r(\sin\theta - 2\cos\theta) = 1$

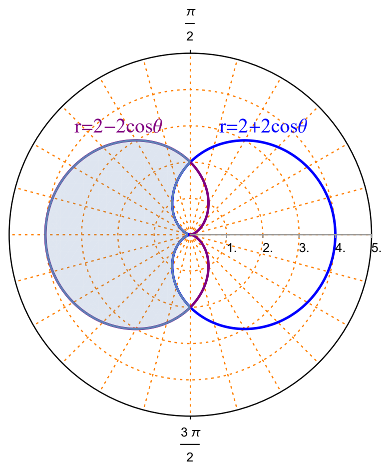
$\implies r\sin\theta - 2(r\cos\theta) = 1 \implies y - 2x = 1 \implies y = 2x + 1$.

7. Sketch the region inside the graph of the polar equation $r = 2 - 2\cos\theta$ and outside the graph of $r = 2 + 2\cos\theta$. Then compute its area. [4]

Solution :

Points of intersection of $r = 2 - 2\cos\theta$ and $r = 2 + 2\cos\theta$:
$2 + 2\cos\theta = 2 - 2\cos\theta$
$\implies 4\cos\theta = 0$
$\implies \cos\theta = 0$
$\implies \theta = \dfrac{\pi}{2}\ ,\ \theta = \dfrac{3\pi}{2}$.

Note that the shaded region is symmetric with respect to the polar axis.



$$\mathbf{A} = 2\left(\frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} \left[(2 - 2\cos\theta)^2 - (2 + 2\cos\theta)^2\right]\; d\theta\right)$$

$$= \int_{\frac{\pi}{2}}^{\pi} \left[4 - 8\cos\theta + 4\cos^2\theta - \left(4 + 8\cos\theta + 4\cos^2\theta\right)\right] \ d\theta$$

$$= \int_{\frac{\pi}{2}}^{\pi} \left(4 - 8\cos\theta + 4\cos^2\theta - 4 - 8\cos\theta - 4\cos^2\theta\right) \ d\theta$$

$$= \int_{\frac{\pi}{2}}^{\pi} -16\cos\theta \ d\theta = -16\left[\sin\theta\right]_{\frac{\pi}{2}}^{\pi}$$

$$= -16\left[\sin(\pi) - \sin\left(\frac{\pi}{2}\right)\right] = -16(0-1) = 16 \ .$$