

MATH 111 - Integral Calculus
First Semester - 1447 H
Solution of the First Exam
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Question (1): [9 marks]

1. Use Riemann Sum to evaluate the definite integral $\int_0^1 (3x^2 - x) dx$. [3]

Solution : $[a, b] = [0, 1]$, $f(x) = 3x^2 - x$.

$$\Delta x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$$

$$x_k = a + k \Delta x = 0 + k \left(\frac{1}{n} \right) = \frac{k}{n}$$

$$f(x_k) = 3 \left(\frac{k}{n} \right)^2 - \frac{k}{n} = \frac{3k^2}{n^2} - \frac{k}{n}$$

$$R_n = \sum_{k=1}^n f(x_k) \Delta x = \sum_{k=1}^n \left(\frac{3k^2}{n^2} - \frac{k}{n} \right) \left(\frac{1}{n} \right)$$

$$= \sum_{k=1}^n \left(\frac{3k^2}{n^3} - \frac{k}{n^2} \right) = \sum_{k=1}^n \frac{3k^2}{n^3} - \sum_{k=1}^n \frac{k}{n^2}$$

$$= \frac{3}{n^3} \sum_{k=1}^n k^2 - \frac{1}{n^2} \sum_{k=1}^n k = \frac{3}{n^3} \frac{n(n+1)(2n+1)}{6} - \frac{1}{n^2} \frac{n(n+1)}{2}$$

$$= \frac{(n+1)(2n+1)}{2n^2} - \frac{n+1}{2n}$$

$$\int_0^1 (3x^2 - x) dx = \lim_{n \rightarrow \infty} R_n = \lim_{n \rightarrow \infty} \left[\frac{(n+1)(2n+1)}{2n^2} - \frac{n+1}{2n} \right]$$

$$= \frac{2}{2} - \frac{1}{2} = 1 - \frac{1}{2} = \frac{1}{2} .$$

2. Find $F'(x)$, if $F(x) = \int_{\cos(\sqrt{x+1})}^{e^{x^2}} \frac{1}{1-t^2} dt$. [2]

Solution :

$$F'(x) = \frac{d}{dx} \int_{\cos(\sqrt{x+1})}^{e^{x^2}} \frac{1}{1-t^2} dt$$

$$= \frac{1}{1-(e^{x^2})^2} (e^{x^2} (2x)) - \frac{1}{1-(\cos(\sqrt{x+1}))^2} \left(-\sin(\sqrt{x+1}) \frac{1}{2\sqrt{x+1}} \right)$$

$$= \frac{2x e^{x^2}}{1-e^{2x^2}} + \frac{1}{2\sqrt{x+1} \sin(\sqrt{x+1})} .$$

Find $\frac{dy}{dx}$ of the following :

3. $y = \sin^{-1}(5^{3x+1}) + \log_2(\sqrt[5]{x^4 + 2})$. [2]

Solution :

$$y = \sin^{-1}(5^{3x+1}) + \log_2(\sqrt[5]{x^4 + 2}) = \sin^{-1}(5^{3x+1}) + \log_2(x^4 + 2)^{\frac{1}{5}}$$

$$= \sin^{-1}(5^{3x+1}) + \frac{1}{5} \log_2(x^4 + 2)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - (5^{3x+1})^2}} (5^{3x+1} (3 \ln 5)) + \frac{1}{5} \frac{4x^3}{x^4 + 2} \frac{1}{\ln 2}$$

$$= \frac{3 \ln 5 \cdot 5^{3x+1}}{\sqrt{1 - 5^{6x+2}}} + \frac{4x^3}{5 \ln 2 (x^4 + 2)} .$$

4. $y = (\sin x)^{\sqrt{x}}$. [2]

Solution :

$$y = (\sin x)^{\sqrt{x}} \implies \ln |y| = \ln |(\sin x)^{\sqrt{x}}| = \sqrt{x} \ln |\sin x|$$

Differentiate both sides.

$$\frac{y'}{y} = \frac{1}{2\sqrt{x}} \ln |\sin x| + \sqrt{x} \left(\frac{\cos x}{\sin x} \right)$$

$$y' = y \left[\frac{\ln |\sin x|}{2\sqrt{x}} + \sqrt{x} \cot x \right]$$

$$= (\sin x)^{\sqrt{x}} \left[\frac{\ln |\sin x|}{2\sqrt{x}} + \sqrt{x} \cot x \right] .$$

Question (2): [16 marks]

Evaluate the following integrals :

1. $\int \frac{\cos x}{(1 + \sin x)^4} dx$. [2]

Solution :

$$\int \frac{\cos x}{(1 + \sin x)^4} dx = \int (1 + \sin x)^{-4} \cos x dx$$

$$= \frac{(1 + \sin x)^{-3}}{-3} + c = \frac{1}{-3(1 + \sin x)^3} + c .$$

2. $\int \frac{\sec(x^{\frac{2}{5}}) \tan(x^{\frac{2}{5}})}{x^{\frac{3}{5}}} dx$. [2]

Solution : .

$$\begin{aligned} \int \frac{\sec(x^{\frac{2}{5}}) \tan(x^{\frac{2}{5}})}{x^{\frac{3}{5}}} dx &= \int \sec\left(x^{\frac{2}{5}}\right) \tan\left(x^{\frac{2}{5}}\right) x^{-\frac{3}{5}} dx \\ &= \frac{5}{2} \int \sec\left(x^{\frac{2}{5}}\right) \tan\left(x^{\frac{2}{5}}\right) \left(\frac{2}{5} x^{-\frac{3}{5}}\right) dx = \frac{5}{2} \sec\left(x^{\frac{2}{5}}\right) + c . \end{aligned}$$

3. $\int \frac{1}{\sqrt{x}(1+\sqrt{x})} dx$. [2]

Solution :

$$\begin{aligned} \int \frac{1}{\sqrt{x}(1+\sqrt{x})} dx &= 2 \int \frac{1}{2\sqrt{x}(1+\sqrt{x})} dx \\ &= 2 \int \frac{\left(\frac{1}{2\sqrt{x}}\right)}{1+\sqrt{x}} dx = 2 \ln|1+\sqrt{x}| + c . \end{aligned}$$

4. $\int \frac{7^{\cot(3x)}}{\sin^2(3x)} dx$. [2]

Solution :

$$\begin{aligned} \int \frac{7^{\cot(3x)}}{\sin^2(3x)} dx &= \int 7^{\cot(3x)} \frac{1}{\sin^2(3x)} dx \\ &= \frac{-1}{3} \int 7^{\cot(3x)} (-\csc^2(3x) (3)) dx = -\frac{1}{3} \frac{7^{\cot(3x)}}{\ln 7} + c . \end{aligned}$$

5. $\int_0^1 \frac{x}{x+1} dx$. [2]

Solution :

$$\begin{aligned} \int_0^1 \frac{x}{x+1} dx &= \int_0^1 \frac{(x+1)-1}{x+1} dx = \int_0^1 \left(\frac{x+1}{x+1} - \frac{1}{x+1} \right) dx \\ &= \int_0^1 \left(1 - \frac{1}{x+1} \right) dx = [x - \ln|x+1|]_0^1 \\ &= (1 - \ln 2) - (0 - \ln(1)) = 1 - \ln 2 . \end{aligned}$$

6. $\int \frac{x-5}{\sqrt{1-x^2}} dx$. [2]

Solution :

$$\begin{aligned} \int \frac{x-5}{\sqrt{1-x^2}} dx &= \int \left(\frac{x}{\sqrt{1-x^2}} - \frac{5}{\sqrt{1-x^2}} \right) dx \\ &= \frac{1}{-2} \int (1-x^2)^{-\frac{1}{2}} (-2x) dx - 5 \int \frac{1}{\sqrt{1-x^2}} dx \end{aligned}$$

$$= \frac{1}{-2} \frac{(1-x^2)^{\frac{1}{2}}}{\frac{1}{2}} - 5 \sin^{-1} x + c = -\sqrt{1-x^2} - 5 \sin^{-1} x + c .$$

7. $\int \frac{1}{x\sqrt{x^6-9}} dx$. [2]

Solution :

$$\begin{aligned} \int \frac{1}{x\sqrt{x^6-9}} dx &= \int \frac{1}{x\sqrt{(x^3)^2-(3)^2}} dx = \int \frac{x^2}{x^3\sqrt{(x^3)^2-(3)^2}} dx \\ &= \frac{1}{3} \int \frac{3x^2}{x^3\sqrt{(x^3)^2-(3)^2}} dx = \frac{1}{3} \frac{1}{3} \sec^{-1} \left(\frac{x^3}{3} \right) + c . \end{aligned}$$

8. $\int \frac{1}{e^{-x}+e^x} dx$. [2]

Solution :

$$\begin{aligned} \int \frac{1}{e^{-x}+e^x} dx &= \int \frac{e^x}{e^x(e^{-x}+e^x)} dx = \int \frac{e^x}{1+e^{2x}} dx \\ &= \int \frac{e^x}{(1)^2+(e^x)^2} dx = \tan^{-1}(e^x) + c . \end{aligned}$$