MATH 111 - Integral Calculus
First Semester - 1447 H
Solution of the Second Exam
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Question (1): [$2+2$ marks]

Find $\dfrac{dy}{dx}$ of the following :

1. $y = \sinh\left(\sqrt{x^2+1}\right) + \cosh^{-1}\left(e^x\right)$.

   Solution :

   $$\frac{dy}{dx} = \cosh\left(\sqrt{x^2+1}\right) \left(\frac{2x}{2\sqrt{x^2+1}}\right) + \frac{1}{\sqrt{\left(e^x\right)^2-1}} \left(e^x\right)$$

   $$= \frac{x \cosh\left(\sqrt{x^2+1}\right)}{\sqrt{x^2+1}} + \frac{e^x}{\sqrt{e^{2x}-1}} \; .$$

2. $y = \operatorname{sech}\left(5x^2\right) + \tanh^{-1}\left(\sin x\right)$.

   Solution :

   $$\frac{dy}{dx} = -\operatorname{sech}\left(5x^2\right) \; \tanh\left(5x^2\right) \; (10x) + \frac{1}{1-\left(\sin x\right)^2} \left(\cos x\right)$$

   $$= -10x \operatorname{sech}\left(5x^2\right) \; \tanh\left(5x^2\right) + \frac{\cos x}{1-\sin^2 x} \; .$$

Question (2): [$2+2+2+3+2+3+2+3+2$ marks]

Evaluate the following integrals :

1. $\displaystyle\int x \; \tanh(4x^2-1) \; dx$.

   Solution :

   $$\int x \; \tanh(4x^2-1) \; dx = \frac{1}{8} \int \tanh(4x^2-1) \; (8x) \; dx$$

   $$= \frac{1}{8} \; \ln\left|\cosh(4x^2-1)\right| + c$$

2. $\displaystyle\int \frac{1}{\sqrt{5^{2x}+1}} \; dx$

   Solution :

   $$\int \frac{1}{\sqrt{5^{2x}+1}} \; dx = \int \frac{1}{\sqrt{\left(5^x\right)^2+(1)^2}} \; dx$$

$$= \frac{1}{\ln 5} \int \frac{5^x \ \ln 5}{5^x \ \sqrt{\left(5^x\right)^2 + (1)^2}} \ dx = -\frac{1}{\ln 5} \ \text{csch}^{-1}\left(5^x\right) + c$$

3. $\displaystyle\int \sin^2 x \ \cos^5 x \ dx$

Solution :

Using the substitution $u = \sin x$.

Hence $du = \cos x \ dx$.

$$\int \sin^2 x \ \cos^5 x \ dx = \int \sin^2 x \ \cos^4 x \ \cos x \ dx = \int \sin^2 x \ \left(\cos^2 x\right)^2 \ \cos x \ dx$$

$$= \int \sin^2 x \ \left(1 - \sin^2 x\right)^2 \ \cos x \ dx = \int u^2 \left(1 - u^2\right)^2 \ du$$

$$= \int u^2 \left(1 - 2u^2 + u^4\right) \ du = \int \left(u^2 - 2u^4 + u^6\right) \ du$$

$$= \frac{u^3}{3} - 2 \ \frac{u^5}{5} + \frac{u^7}{7} + c = \frac{\sin^3 x}{3} - 2 \ \frac{\sin^5 x}{5} + \frac{\sin^7 x}{7} + c$$

4. $\displaystyle\int \frac{1}{(x-3)\sqrt{7 - x^2 + 6x}} \ dx$.

Solution : By completing the square.

$$7 - x^2 + 6x = 7 - \left(x^2 - 6x\right) = 7 - \left(x^2 - 6x + 9\right) + 9$$

$$= 16 - \left(x^2 - 6x + 9\right) = (4)^2 - (x-3)^2 \ .$$

$$\int \frac{1}{(x-3)\sqrt{7 - x^2 + 6x}} \ dx = \int \frac{1}{(x-3)\sqrt{(4)^2 - (x-3)^2}} \ dx$$

$$= -\frac{1}{4} \ \text{sech}^{-1}\left(\frac{x-3}{4}\right) + c \ .$$

5. $\displaystyle\int \sin^{-1} x \ dx$.

Solution : Using integration by parts.

$$u = \sin^{-1} x \qquad\qquad dv = dx$$
$$du = \frac{1}{\sqrt{1 - x^2}} \ dx \qquad v = x$$

$$\int \sin^{-1} x \ dx = x \ \sin^1 x - \int x \ \frac{1}{\sqrt{1 - x^2}} \ dx$$

$$= x \ \sin^1 x - \int x \ \left(1 - x^2\right)^{-\frac{1}{2}} \ dx$$

$$= x \ \sin^1 x - \frac{1}{-2} \int \left(1 - x^2\right)^{-\frac{1}{2}} \ (-2x) \ dx$$

$$= x \ \sin^1 x + \frac{1}{2} \ \frac{\left(1 - x^2\right)^{\frac{1}{2}}}{\frac{1}{2}} + c = x \sin^{-1} x + \sqrt{1 - x^2} + c.$$

6. $\displaystyle\int \frac{1}{(9 + x^2)^{\frac{3}{2}}} \ dx$.

Solution : Using trigonometric substitutions.

Put $x = 3 \tan\theta \implies \tan\theta = \dfrac{x}{3}$.

$dx = 3 \sec^2\theta \ d\theta.$

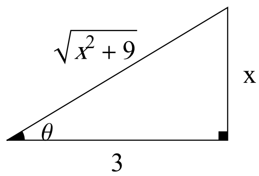
$\left(9 + x^2\right)^{\frac{3}{2}} = \left(9 + 9\tan^2\theta\right)^{\frac{3}{2}} = \left[9\left(1 + \tan^2\theta\right)\right]^{\frac{3}{2}} = \left[(3)^2 \ \sec^2\theta\right]^{\frac{3}{2}} = 3^3 \ \sec^3\theta.$

$$\int \frac{1}{(9 + x^2)^{\frac{3}{2}}} \ dx = \int \frac{3\sec^2\theta}{3^3 \ \sec^3\theta} \ d\theta = \frac{1}{3^2} \int \frac{1}{\sec\theta} \ d\theta$$

$$= \frac{1}{9} \int \cos\theta \ d\theta = \frac{1}{9} \sin\theta + c$$

$\tan\theta = \dfrac{x}{3}$.

From the triangle :
$\sin\theta = \dfrac{x}{\sqrt{x^2 + 9}}$



$$\int \frac{1}{(9 + x^2)^{\frac{3}{2}}} \ dx = \frac{1}{9} \ \frac{x}{\sqrt{x^2 + 9}} + c$$

7. $\displaystyle\int x^3 \ \ln|2x| \ dx$

Solution : Using integration by parts.

$u = \ln|2x| \qquad dv = x^3 \ dx$

$du = \dfrac{2}{2x} \ dx = \dfrac{1}{x} \ dx \qquad v = \dfrac{x^4}{4}$

$$\int x^3 \ \ln|2x| \ dx = \frac{x^4}{4} \ \ln|2x| - \int \frac{x^4}{4} \ \frac{1}{x} \ dx$$

$$= \frac{x^4}{4} \ \ln|2x| - \frac{1}{4} \int x^3 \ dx = \frac{x^4}{4} \ \ln|2x| - \frac{1}{4} \ \frac{x^4}{4} + c$$

8. $\displaystyle\int \frac{2x + 1}{x^2(1 + x^2)} \ dx$

Solution : Using the method of partial fractions.

$$\frac{2x+1}{x^2(1+x^2)} = \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{Bx+C}{1+x^2}$$

$$\frac{2x+1}{x^2(1+x^2)} = \frac{A_1\ x(1+x^2)}{x^2(1+x^2)} + \frac{A_2(1+x^2)}{x^2(1+x^2)} + \frac{(Bx+C)x^2}{x^2(1+x^2)}$$

$$2x+1 = A_1\ x(1+x^2) + A_2(1+x^2) + (Bx+C)x^2$$

$$2x+1 = A_1x + A_1x^3 + A_2 + A_2x^2 + Bx^3 + Cx^2$$

$$2x+1 = (A_1+B)\,x^3 + (A_2+C)\,x^2 + A_1x + A_2$$

By comparing the coefficients of the two polynomials in each side :

$$\begin{array}{lll} A_1 + B = 0 & \longrightarrow & (1) \\ A_2 + C = 0 & \longrightarrow & (2) \\ A_1 = 2 & \longrightarrow & (3) \\ A_2 = 1 & \longrightarrow & (4) \end{array}$$

From equations (1) and (3) : $2 + B = 0 \implies B = -2$.

From equations (2) and (4) : $1 + C = 0 \implies C = -1$.

$$\int \frac{2x+1}{x^2(1+x^2)}\ dx = \int \left(\frac{2}{x} + \frac{1}{x^2} + \frac{-2x-1}{1+x^2}\right)\ dx$$

$$= \int \frac{2}{x}\ dx + \int \frac{1}{x^2}\ dx + \int \frac{-2x-1}{1+x^2}\ dx$$

$$= 2\int \frac{1}{x}\ dx + \int x^{-2}\ dx - \int \frac{2x}{1+x^2}\ dx - \int \frac{1}{1+x^2}\ dx$$

$$= 2\ln|x| + \frac{x^{-1}}{-1} - \ln(1+x^2) - \tan^{-1}x + c\ .$$

9. $\displaystyle\int \frac{1}{x^{\frac{1}{2}} + x^{\frac{2}{3}}}\ dx$.

Solution :

Using the substitution $x = u^6$, then $u = x^{\frac{1}{6}}$.

$dx = 6u^5\ du$.

$$\int \frac{1}{x^{\frac{1}{2}} + x^{\frac{2}{3}}}\ dx = \int \frac{6u^5}{(u^6)^{\frac{1}{2}} + (u^6)^{\frac{2}{3}}}\ du = 6\int \frac{u^5}{u^3+u^4}\ du$$

$$= 6\int \frac{u^5}{u^3\ (u+1)}\ du = 6\int \frac{u^2}{u+1}\ du$$

Using long division of polynomials :

$$6\int \frac{u^2}{u+1}\ du = 6\int \left(u - 1 + \frac{1}{u+1}\right)\ du = 6\left(\frac{u^2}{2} - u + \ln|u+1|\right) + c$$

$$= 3u^2 - 6u + 6\ln|u+1| + c = 3x^{\frac{1}{3}} - 6x^{\frac{1}{6}} + 6\ln\left|x^{\frac{1}{6}} + 1\right| + c$$