MATH 111 - Integral Calculus
Second Semester - 1446 H
Solution of the Second Exam
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Question (1): [4 marks]

Find $\frac{dy}{dx}$ of the following :

1. $y = \coth\left(\sqrt[3]{x}\right) + csch^{-1}\left(\frac{2}{x}\right)$. [2]

   Solution :

   $$\frac{dy}{dx} = -csch^2\left(\sqrt[3]{x}\right)\ \left(\frac{1}{3}x^{-\frac{2}{3}}\right) + \frac{-1}{\frac{2}{x}\sqrt{\left(\frac{2}{x}\right)^2+1}}\ \left(\frac{-2}{x^2}\right)$$

   $$= \frac{-csch^2\left(\sqrt[3]{x}\right)}{3x^{\frac{2}{3}}} + \frac{1}{x\sqrt{\frac{4}{x^2}+1}}\ .$$

2. $y = \coth^{-1}\left(\frac{x^3}{3}\right) + \cosh^{-1}(3x)$. [2]

   Solution :

   $$\frac{dy}{dx} = \frac{1}{1-\left(\frac{x^3}{3}\right)^2}\ \left(x^2\right) + \frac{1}{\sqrt{(3x)^2-1}}\ (3)$$

   $$= \frac{x^2}{1-\frac{x^6}{9}} + \frac{3}{\sqrt{9x^2-1}}\ .$$

Question (2): [21 marks]

Evaluate the following integrals :

1. $\int x^{-2} csch^2\left(\frac{1}{x}\right)\ dx$. [2]

   Solution :

   $$\int x^{-2} csch^2\left(\frac{1}{x}\right)\ dx = \int csch^2\left(\frac{1}{x}\right)\ \left(\frac{1}{x^2}\right)\ dx$$

   $$= \int -csch^2\left(\frac{1}{x}\right)\ \left(\frac{-1}{x^2}\right)\ dx = \coth\left(\frac{1}{x}\right) + c\ .$$

2. $\int \frac{1}{\sqrt{x^2+2x+10}}\ dx$. [2]

   Solution : By completing the square.

$x^2 + 2x + 10 = (x^2 + 2x + 1) + 10 - 1 = (x+1)^2 + (3)^2$.

$$\int \frac{1}{\sqrt{x^2 + 2x + 10}}\ dx = \int \frac{1}{\sqrt{(x+1)^2 + (3)^2}}\ dx$$

$$= \sinh^{-1}\left(\frac{x+1}{3}\right) + c \ .$$

3. $\int (2x-1)\ \cosh x\ dx$. [2]

Solution : Using integration by parts.

$$\begin{array}{ll} u = 2x - 1 & dv = \cosh x\ dx \\ du = 2\ dx & v = \sinh x \end{array}$$

$$\int (2x-1)\ \cosh x\ dx = (2x-1)\ \sinh x - \int 2 \sinh x\ dx$$

$$= (2x-1)\ \sinh x - 2 \int \sinh x\ dx = (2x-1)\ \sinh x - 2\cosh x + c \ .$$

4. $\int \cot^4 x\ \csc^4 x\ dx$. [2]

Solution :

Using the substitution $u = \cot x$.

Hence $du = -\csc^2 x\ dx \implies (-1)\ du = \csc^2 x\ dx$.

$$\int \cot^4 x\ \csc^4 x\ dx = \int \cot^4 x\ \csc^2 x\ \csc^2 x\ dx$$

$$= \int \cot^4 x\ \left(1 + \cot^2 x\right)\ \csc^2 x\ dx = \int u^4\ (1 + u^2)\ (-1)\ du$$

$$= -\int \left(u^4 + u^6\right)\ du = -\left[\frac{u^5}{5} + \frac{u^7}{7}\right] + c = -\frac{\cot^5 x}{5} - \frac{\cot^7 x}{7} + c$$

5. $\int e^{\frac{x}{2}}\ \sinh(3x)\ dx$. [2]

Solution :

$$\int e^{\frac{x}{2}}\ \sinh(3x)\ dx = \int e^{\frac{x}{2}} \left(\frac{e^{3x} - e^{-3x}}{2}\right)\ dx = \int \left(\frac{e^{\frac{7x}{2}} - e^{-\frac{5x}{2}}}{2}\right)\ dx$$

$$\int \left(\frac{e^{\frac{7x}{2}}}{2} - \frac{e^{-\frac{5x}{2}}}{2}\right)\ dx = \frac{1}{2}\ \frac{2}{7} \int e^{\frac{7x}{2}} \left(\frac{7}{2}\right)\ dx - \frac{1}{2}\ \frac{2}{-5} \int e^{-\frac{5x}{2}} \left(-\frac{5}{2}\right)\ dx$$

$$= \frac{1}{7}\ e^{\frac{7x}{2}} + \frac{1}{5}\ e^{-\frac{5x}{2}} + c = \frac{e^{\frac{7x}{2}}}{7} + \frac{e^{-\frac{5x}{2}}}{5} + c \ .$$

6. $\int \frac{1}{(9-x^2)^{\frac{3}{2}}}\ dx$. [3]

Solution : Using trigonometric substitutions.

Put $x = 3\sin\theta \implies \sin\theta = \frac{x}{3}$

$dx = 3\cos\theta\ d\theta$

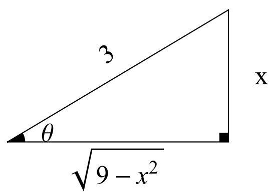
$\left(9-x^2\right)^{\frac{3}{2}} = \left(9-9\sin^2\theta\right)^{\frac{3}{2}} = \left[9\left(1-\sin^2\theta\right)\right]^{\frac{3}{2}}$

$= \left(9\cos^2\theta\right)^{\frac{3}{2}} = (9)^{\frac{3}{2}}\ \left(\cos^2\theta\right)^{\frac{3}{2}} = \left(3^2\right)^{\frac{3}{2}}\ \left(\cos^2\theta\right)^{\frac{3}{2}} = 3^3\ \cos^3\theta$

$\int \frac{1}{(9-x^2)^{\frac{3}{2}}}\ dx = \int \frac{3\cos\theta}{3^3\ \cos^3\theta}\ d\theta = \frac{1}{3^2}\ \int \frac{1}{\cos^2\theta}\ d\theta$

$= \frac{1}{9}\int \sec^2\theta\ d\theta = \frac{1}{9}\ \tan\theta + c$

$\sin\theta = \frac{x}{3}$.



From the triangle :
$\tan\theta = \frac{x}{\sqrt{9-x^2}}$

$\int \frac{1}{(9-x^2)^{\frac{3}{2}}}\ dx = \frac{1}{9}\ \frac{x}{\sqrt{9-x^2}} + c$

7. $\int \frac{1}{\sqrt{e^{6x}+1}}\ dx$. [2]

Solution :

$\int \frac{1}{\sqrt{e^{6x}+1}}\ dx = \int \frac{1}{\sqrt{\left(e^{3x}\right)^2+(1)^2}}\ dx$

$= \int \frac{e^{3x}}{e^{3x}\sqrt{\left(e^{3x}\right)^2+(1)^2}}\ dx = \frac{1}{3}\int \frac{e^{3x}\ (3)}{e^{3x}\sqrt{\left(e^{3x}\right)^2+(1)^2}}\ dx$

$= \frac{1}{3}\left(-csch^{-1}\left(e^{3x}\right)\right) + c = -\frac{1}{3}\ csch^{-1}\left(e^{3x}\right) + c$

8. $\int \tan^{-1}x\ dx$. [3]

Solution : Using integration by parts.

$u = \tan^{-1}x \qquad dv = 1\ dx$
$du = \frac{1}{x^2+1}\ dx \qquad v = x$

$$\int \tan^{-1} x \, dx = x \tan^{-1} x - \int x \, \frac{1}{x^2+1} \, dx$$

$$= x \tan^{-1} x - \frac{1}{2} \int \frac{2x}{x^2+1} \, dx = x \tan^{-1} x - \frac{1}{2} \ln(x^2+1) + c \, .$$

9. $\displaystyle\int \frac{x^2+x+9}{x^3+9x} \, dx$. [3]

Solution : Using the method of partial fractions.

$$\frac{x^2+x+9}{x^3+9x} = \frac{x^2+x+9}{x(x^2+9)} = \frac{A}{x} + \frac{Bx+C}{x^2+9}$$

$$\frac{x^2+x+9}{x(x^2+9)} = \frac{A(x^2+9)}{x(x^2+9)} + \frac{x\,(Bx+C)}{x(x^2+9)}$$

$$x^2+x+9 = A(x^2+9) + x(Bx+C)$$

$$x^2+x+9 = Ax^2 + 9A + Bx^2 + Cx = (A+B)x^2 + Cx + 9A$$

By comparing the coefficients of the two polynomials in each side :

$$\begin{array}{lcl} A+B=1 & \longrightarrow & (1) \\ C=1 & \longrightarrow & (2) \\ 9A=9 & \longrightarrow & (3) \end{array}$$

From equation (3) : $9A = 9 \implies A = \frac{9}{9} = 1$.

From equation (1) : $1 + B = 1 \implies B = 0$.

$$\int \frac{x^2+x+9}{x^3+9x} \, dx = \int \left(\frac{1}{x} + \frac{1}{x^2+9} \right) \, dx$$

$$= \int \frac{1}{x} \, dx + \int \frac{1}{x^2+(3)^2} \, dx$$

$$= \ln|x| + \frac{1}{3} \tan^{-1}\left(\frac{x}{3}\right) + c \, .$$