MATH 111 - Integral Calculus
Second Semester - 1446 H
Solution of the First Exam
Dr Tariq A. Alfadhel

Question (1): [9 marks]

1. Use Riemann Sum to evaluate the definite integral $\int_0^4 (2x^2+1)\ dx$. [3]

   Solution : $[a,b] = [0,4]$, $f(x) = 2x^2+1$.

   $$\Delta_x = \frac{b-a}{n} = \frac{4-0}{n} = \frac{4}{n}$$

   $$x_k = a + k\ \Delta_x = 0 + k\left(\frac{4}{n}\right) = \frac{4k}{n}$$

   $$f(x_k) = 2\left(\frac{4k}{n}\right)^2 + 1 = 2\left(\frac{16k^2}{n^2}\right) + 1 = \frac{32k^2}{n^2} + 1$$

   $$R_n = \sum_{k=1}^{n} f(x_k)\ \Delta_x = \sum_{k=1}^{n}\left(\frac{32k^2}{n^2}+1\right)\ \left(\frac{4}{n}\right)$$

   $$= \sum_{k=1}^{n}\left(\frac{128k^2}{n^3}+\frac{4}{n}\right) = \sum_{k=1}^{n}\frac{128k^2}{n^3} + \sum_{k=1}^{n}\frac{4}{n}$$

   $$= \frac{128}{n^3}\sum_{k=1}^{n}k^2 + \frac{4}{n}\sum_{k=1}^{n}1 = \frac{128}{n^3}\ \frac{n(n+1)(2n+1)}{6} + \frac{4}{n}\ (n)$$

   $$= \frac{64}{3}\left(\frac{(n+1)(2n+1)}{n^2}\right) + 4$$

   $$\int_0^4 (2x^2+1)\ dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\left[\frac{64}{3}\left(\frac{(n+1)(2n+1)}{n^2}\right)+4\right]$$

   $$= \frac{64}{3}\ (2) + 4 = \frac{128}{3} + 4 = \frac{140}{3}\ .$$

2. Find $F'(x)$, if $F(x) = \int_{\tan(\sqrt[3]{x})}^{3^{5x}} \sqrt{t^2+1}\ dt$. [2]

   Solution :

   $$F'(x) = \frac{d}{dx}\int_{\tan(\sqrt[3]{x})}^{3^{5x}} \sqrt{t^2+1}\ dt$$

   $$= \sqrt{\left(3^{5x}\right)^2+1}\ \left(3^{5x}(5)\ \ln 3\right) - \sqrt{\left(\tan\left(\sqrt[3]{x}\right)\right)^2+1}\ \left(\sec^2\left(\sqrt[3]{x}\right)\ \left(\frac{1}{3}\ x^{-\frac{2}{3}}\right)\right)$$

   $$= 5\ 3^{5x}\ \ln 3\ \sqrt{3^{10x}+1} - \frac{\sec^2\left(\sqrt[3]{x}\right)\ \sqrt{\tan^2\left(\sqrt[3]{x}\right)+1}}{3\ x^{\frac{2}{3}}}\ .$$

Find $\frac{dy}{dx}$ of the following :

3. $y = \left[\csc^{-1}\left(\frac{3}{x}\right)\right] \ \log|\sec(2x)+\tan(2x)|$. [2]

Solution :

$$\frac{dy}{dx} = \left(\frac{-1}{\left(\frac{3}{x}\right)\sqrt{\left(\frac{3}{x}\right)^2-1}}\ \left(\frac{-3}{x^2}\right)\right) \ \log|\sec(2x)+\tan(2x)|$$

$$+\left[\csc^{-1}\left(\frac{3}{x}\right)\right] \ \left(\frac{\sec(2x)\tan(2x)\ (2)+\sec^2(2x)\ (2)}{\sec(2x)+\tan(2x)}\ \frac{1}{\ln 10}\right)$$

$$= \frac{3\ \log|\sec(2x)+\tan(2x)|}{3x\sqrt{\frac{9}{x^2}-1}} + \frac{2\sec(2x)\csc^{-1}\left(\frac{3}{x}\right)}{\ln 10} \ .$$

4. $y = (\cos x)^{\csc x} + 3^{2x^2}$. [2]

Solution :

Let $y = f(x) + g(x)$, where $f(x) = (\cos x)^{\csc x}$ and $g(x) = 3^{2x^2}$.

Then $\frac{dy}{dx} = y' = f'(x) + g'(x)$

First : $g'(x) = 3^{2x^2}\ (4x)\ln 3 = 4x\ 3^{x^2}\ \ln 3$

Second : Finding $f'(x)$

$f(x) = (\cos x)^{\csc x} \implies \ln|f(x)| = \ln|(\cos x)^{\csc x}| = \csc x\ \ln|\cos x|$

Differentiate both sides.

$$\frac{f'(x)}{f(x)} = (-\csc x\ \cot x)\ \ln|\cos x| + \csc x\left(\frac{-\sin x}{\cos x}\right)$$

$$f'(x) = f(x)\left[-\csc x\ \cot x\ \ln|\cos x| - \frac{\csc x\ \sin x}{\cos x}\right]$$

$$= (\cos x)^{\csc x}\ \left[-\csc x\ \cos x\ \ln|\cos x| - \frac{1}{\cos x}\right]$$

Therefore, $\frac{dy}{dx} = (\cos x)^{\csc x}\ \ [-\csc x\ \cos x\ \ln|\cos x| - \sec x] + 4x\ 3^{x^2}\ \ln 3$

Question (2): [16 marks]
Evaluate the following integrals :

1. $\int\left(x^{\frac{2}{3}}\ \sqrt[3]{\sec(x^3)}\right)^3\ dx$. [2]

Solution :

$$\int \left(x^{\frac{2}{3}} \ \sqrt[3]{\sec(x^3)}\right)^3 \ dx = \int \left(x^{\frac{2}{3}}\right)^3 \ \left(\sqrt[3]{\sec(x^3)}\right)^3 \ dx = \int x^2 \ \sec(x^3) \ dx$$

$$= \frac{1}{3} \int \sec(x^3) \ (3x^2) \ dx = \frac{1}{3} \ \ln\left|\sec(x^3) + \tan(x^3)\right| + c \ .$$

2. $\displaystyle\int \frac{2}{\sqrt{e^{4x} - 4}} \ dx$. [2]

Solution : .

$$\int \frac{2}{\sqrt{e^{4x} - 4}} \ dx = \int \frac{2}{\sqrt{\left(e^{2x}\right)^2 - (2)^2}} \ dx = \int \frac{e^{2x} \ (2)}{e^{2x} \ \sqrt{\left(e^{2x}\right)^2 - (2)^2}} \ dx$$

$$= \frac{1}{2} \ \sec^{-1}\left(\frac{e^{2x}}{2}\right) + c \ .$$

3. $\displaystyle\int_0^1 10^{2\log x} \ 3^{3x^3} \ dx$. [2]

Solution :

$$\int_0^1 10^{2\log x} \ 3^{3x^3} \ dx = \int_0^1 10^{\log x^2} \ 3^{3x^3} \ dx = \int_0^1 x^2 \ 3^{3x^3} \ dx$$

$$= \frac{1}{9} \int_0^1 3^{3x^3} (9x^2) \ dx = \frac{1}{9} \left[\frac{3^{3x^3}}{\ln 3}\right]_0^1 = \frac{1}{9} \left[\frac{3^{3(1)^3}}{\ln 3} - \frac{3^{3(0)^3}}{\ln 3}\right]$$

$$= \frac{1}{9} \left[\frac{3^3}{\ln 3} - \frac{3^0}{\ln 3}\right] = \frac{1}{9} \ \left(\frac{27 - 1}{\ln 3}\right) = \frac{26}{9 \ln 3} \ .$$

4. $\displaystyle\int \frac{x^2}{4 + x^2} \ dx$. [2]

Solution :

$$\int \frac{x^2}{4 + x^2} \ dx = \int \frac{\left(x^2 + 4\right) - 4}{x^2 + 4} \ dx = \int \left[\frac{x^2 + 4}{x^2 + 4} - \frac{4}{x^2 + 4}\right] \ dx$$

$$= \int \left[1 - \frac{4}{x^2 + 4}\right] \ dx = \int 1 \ dx - 4 \int \frac{1}{(x)^2 + (2)^2} \ dx$$

$$= x - 4 \left(\frac{1}{2} \ \tan^{-1}\left(\frac{x}{2}\right)\right) + c = x - 2\tan^{-1}\left(\frac{x}{2}\right) + c$$

5. $\displaystyle\int \frac{\cos\left(\ln(x^2)\right)}{x}\ dx$. [2]

Solution :

$$\int \frac{\cos\left(\ln(x^2)\right)}{x}\ dx = \int \cos\left(2\ln|x|\right)\ \frac{1}{x}\ dx$$

$$= \frac{1}{2}\int \cos\left(2\ln|x|\right)\ \frac{2}{x}\ dx = \frac{1}{2}\sin\left(2\ln|x|\right) + c$$

6. $\displaystyle\int \frac{\cot\left(\sqrt[3]{x}\right)}{x^{\frac{2}{3}}}\ dx$. [2]

Solution :

$$\int \frac{\cot\left(\sqrt[3]{x}\right)}{x^{\frac{2}{3}}}\ dx = \int \cot\left(x^{\frac{1}{3}}\right)\ x^{-\frac{2}{3}}\ dx$$

$$= 3\int \cot\left(x^{\frac{1}{3}}\right)\left(\frac{1}{3}\ x^{-\frac{2}{3}}\right)\ dx = 3\ln\left|\sin\left(\sqrt[3]{x}\right)\right| + c$$

7. $\displaystyle\int \frac{\left(1+\sec^{-1}x\right)^5}{\sqrt{x^4-x^2}}\ dx$. [2]

Solution :

$$\int \frac{\left(1+\sec^{-1}x\right)^5}{\sqrt{x^4-x^2}}\ dx = \int \frac{\left(1+\sec^{-1}x\right)^5}{\sqrt{x^2\left(x^2-1\right)}}\ dx$$

$$= \int \frac{\left(1+\sec^{-1}x\right)^5}{|x|\sqrt{x^2-1}}\ dx = \int \left(1+\sec^{-1}x\right)^5\left(\frac{1}{x\sqrt{x^2-1}}\right)\ dx$$

$$= \frac{\left(1+\sec^{-1}x\right)^6}{6} + c\ ,\ \text{where } x > 0\ .$$

8. $\displaystyle\int \left(\sec x\ \tan x\right)\ \left(\sec^2 x\right)^{-1}\ dx$. [2]

First solution :

$$\int \left(\sec x\ \tan x\right)\ \left(\sec^2 x\right)^{-1}\ dx = \int \left(\sec x\right)^{-2}\left(\sec x\ \tan x\right)\ dx$$

$$= \frac{\left(\sec x\right)^{-1}}{-1} + c = -\cos x + c$$

Second solution :

$$\int \left(\sec x\ \tan x\right)\ \left(\sec^2 x\right)^{-1}\ dx = \int \frac{\sec x\ \tan x}{\sec^2 x}\ dx = \int \frac{\tan x}{\sec x}\ dx$$

$$= \int \sin x\ dx = -\cos x + c\ .$$