MATH 111 - Integral Calculus
Second Semester - 1446 H
Solution of the Final Exam
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Question (1): [7 marks]

1. Find the value of c that satisfies the mean value theorem of the definite integral for the function $f(x) = \sqrt[3]{x-1}$ on the interval $[1, 9]$. [3]

   Solution : Using the formula $(b-a)\ f(c) = \displaystyle\int_a^b f(x)\ dx$.

   $$(9-1)\ \sqrt[3]{c-1} = \int_1^9 \sqrt[3]{x-1}\ dx = \int_1^9 (x-1)^{\frac{1}{3}}\ dx$$

   $$8\sqrt[3]{c-1} = \left[\frac{(x-1)^{\frac{4}{3}}}{\frac{4}{3}}\right]_1^9 = \left[\frac{3}{4}\ (x-1)^{\frac{4}{3}}\right]_1^9$$

   $$8\sqrt[3]{c-1} = \frac{3}{4}(9-1)^{\frac{4}{3}} - \frac{3}{4}(1-1)^{\frac{4}{3}} = \frac{3}{4}(8)^{\frac{4}{3}} - 0 = \frac{3}{4}\left(2^4\right) = \frac{3\times 16}{4} = 12$$

   $$\Longrightarrow\ \sqrt[3]{c-1} = \frac{12}{8} = \frac{3}{2}\ \Longrightarrow\ c-1 = \frac{27}{8}\ \Longrightarrow\ c = \frac{27}{8} + 1 = \frac{35}{8} \in (1, 9).$$

   The desired value is $c = \dfrac{35}{8}$.

2. Find $F'(x)$, if $F(x) = \displaystyle\int_{\cos x}^{2^{3x}} \sqrt{1+t^3}\ dt$. [2]

   Solution :

   $$F'(x) = \frac{d}{dx}\int_{\cos x}^{2^{3x}} \sqrt{1+t^3}\ dt$$

   $$= \sqrt{1+\left(2^{3x}\right)^3}\ \left(2^{3x}(3)\ \ln 2\right) - \sqrt{1+(\cos x)^3}\ (-\sin x)$$

   $$= 3\ln 2\left(2^{3x}\right)\sqrt{1+2^{9x}} + \sin x\ \sqrt[3]{1+\cos^3 x}\ .$$

3. Find y' if $y = \tanh\left(\sqrt{x}\right) + \sinh^{-1}\left(\dfrac{2}{\sqrt{x}}\right)$. [2]

   Solution :

   $$y' = sech^2\left(\sqrt{x}\right)\ \left(\frac{1}{2\sqrt{x}}\right) + \frac{1}{\sqrt{1+\left(\frac{2}{\sqrt{x}}\right)^2}}\ \left(2\ \frac{-1}{2}\ x^{-\frac{3}{2}}\right)$$

   $$= \frac{sech^2\left(\sqrt{x}\right)}{2\sqrt{x}} - \frac{1}{x^{\frac{3}{2}}\ \sqrt{1+\frac{4}{x}}}\ .$$

Question (2): [14 marks]
Evaluate the following integrals :

1. $\displaystyle\int \frac{1}{\sqrt{-x^2-2x+8}}\,dx$. [3]

   Solution : By completing the square.

   $-x^2-2x+8 = -(x^2+2x)+8 = -(x^2+2x+1)+8+1$

   $= -(x+1)^2+9 = (3)^2-(x+1)^2$.

   $$\int \frac{1}{\sqrt{-x^2-2x+8}}\,dx = \int \frac{1}{\sqrt{(3)^2-(x+1)^2}}\,dx = \sin^{-1}\left(\frac{x+1}{3}\right)+c\ .$$

2. $\displaystyle\int x\ \tanh^{-1}x\ dx$. [3]

   Solution : Using integration by parts .

   $u = \tanh^{-1}x \qquad dv = x\ dx$

   $du = \dfrac{1}{1-x^2}\,dx \qquad v = \dfrac{x^2}{2}$

   $$\int x\ \tanh^{-1}x\ dx = \frac{x^2}{2}\ \tanh^{-1}x - \int \frac{x^2}{2}\ \frac{1}{1-x^2}\ dx$$

   $$= \frac{x^2}{2}\ \tanh^{-1}x + \frac{1}{2}\int \frac{-x^2}{1-x^2}\ dx = \frac{x^2}{2}\ \tanh^{-1}x + \frac{1}{2}\int \frac{(1-x^2)-1}{1-x^2}\ dx\ .$$

   $$= \frac{x^2}{2}\ \tanh^{-1}x + \frac{1}{2}\int \left[\frac{1-x^2}{1-x^2} - \frac{1}{1-x^2}\right]\ dx$$

   $$= \frac{x^2}{2}\ \tanh^{-1}x + \frac{1}{2}\int \left[1 - \frac{1}{1-x^2}\right]\ dx = \frac{x^2}{2}\ \tanh^{-1}x + \frac{1}{2}\left(x - \tanh^{-1}x\right) + c$$

   $$= \frac{x^2}{2}\ \tanh^{-1}x + \frac{x}{2} - \frac{1}{2}\ \tanh^{-1}x + c$$

3. $\displaystyle\int \frac{3x^2+x+8}{x^3+4x}\,dx$. [3]

   Solution : Using the method of partial fractions.

   $$\frac{3x^2+x+8}{x^3+4x} = \frac{3x^2+x+8}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

   $$\frac{3x^2+x+8}{x(x^2+4)} = \frac{A\ (x^2+4)}{x\ (x^2+4)} + \frac{(Bx+C)\ x}{(x^2+4)\ x}$$

   $3x^2+x+8 = A\ (x^2+4) + (Bx+C)\ x = Ax^2+4A+Bx^2+Cx$

   $3x^2+x+8 = (A+B)x^2+Cx+4A$

By comparing the coefficients of the two polynomials in each side :

$$\begin{array}{lll} A+B=3 & \longrightarrow & (1) \\ C=1 & \longrightarrow & (2) \\ 4A=8 & \longrightarrow & (3) \end{array}$$

From equation (3) : $4A = 8 \implies A = \frac{8}{4} = 2$.

From Equation (1) : $2 + B = 3 \implies B = 1$.

$$\int \frac{3x^2+x+8}{x^3+4x}\,dx = \int \left(\frac{2}{x} + \frac{x+1}{x^2+4}\right)\,dx$$

$$= \int \frac{2}{x}\,dx + \int \frac{x}{x^2+4}\,dx + \int \frac{1}{x^2+4}\,dx$$

$$= 2\int \frac{1}{x}\,dx + \frac{1}{2}\int \frac{2x}{x^2+4}\,dx + \int \frac{1}{(x)^2+(2)^2}\,dx$$

$$= 2\ln|x| + \frac{1}{2}\ln\left(x^2+4\right) + \frac{1}{2}\ \tan^{-1}\left(\frac{x}{2}\right) + c\ .$$

4. $\int \frac{1}{(9+x^2)^{\frac{3}{2}}}\,dx$. [3]

Solution : Using trigonometric substitutions.

Put $x = 3\tan\theta \implies \tan\theta = \frac{x}{3}$.

$dx = 3\sec^2\theta\ d\theta$.

$$\left(9+x^2\right)^{\frac{3}{2}} = \left(9+9\tan^2\theta\right)^{\frac{3}{2}} = \left[9\left(1+\tan^2\theta\right)\right]^{\frac{3}{2}} = \left[3^2\ \sec^2\theta\right]^{\frac{3}{2}}$$

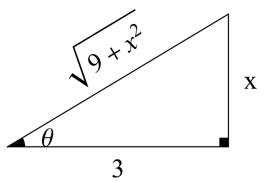
$$= \left(3^2\right)^{\frac{3}{2}}\ \left(\sec^2\theta\right)^{\frac{3}{2}} = 3^3\ \sec^3\theta$$

$$\int \frac{1}{(9+x^2)^{\frac{3}{2}}}\,dx = \int \frac{3\sec^2\theta}{3^3\ \sec^3\theta}\,d\theta = \frac{3}{3^3}\int \frac{1}{\sec\theta}\,d\theta$$

$$= \frac{1}{9}\int \cos\theta\ d\theta = \frac{1}{9}\sin\theta + c$$

$\tan\theta = \frac{x}{3}$.

From the triangle :
$\sin\theta = \frac{x}{\sqrt{9+x^2}}$



$$\int \frac{1}{(9+x^2)^{\frac{3}{2}}}\,dx = \frac{1}{9}\ \frac{x}{\sqrt{9+x^2}} + c\ .$$

5. $\displaystyle\int \frac{1}{4\ (2^{-x}) + 2^x}\ dx$. [2]

Solution : .

$$\int \frac{1}{4\ (2^{-x}) + 2^x}\ dx = \int \frac{2^x}{2^x\ \left[4\ (2^{-x}) + 2^x\right]}\ dx\ .$$

$$= \int \frac{2^x}{4 + 2^{2x}}\ dx = \frac{1}{\ln 2} \int \frac{2^x\ \ln 2}{(2)^2 + (2^x)^2}\ dx$$

$$= \frac{1}{\ln 2}\left(\frac{1}{2}\tan^{-1}\left(\frac{2^x}{2}\right)\right) + c = \frac{1}{2\ \ln 2}\tan^{-1}\left(2^{x-1}\right) + c$$

Question (3): [19 marks]

1. Evaluate the limit $\displaystyle\lim_{x\to\infty} \frac{x - x\ e^x}{x + 2x\ e^x}$. [2]

   Solution :

   $$\lim_{x\to\infty} \frac{x - x\ e^x}{x + 2x\ e^x} = \lim_{x\to\infty} \frac{x\ (1 - e^x)}{x\ (1 + 2\ e^x)} = \lim_{x\to\infty} \frac{1 - e^x}{1 + 2\ e^x} \qquad \left(\frac{-\infty}{\infty}\right)$$

   Using L'Hôpital's rule.

   $$\lim_{x\to\infty} \frac{-e^x}{2\ e^x} = \lim_{x\to\infty} \frac{-1}{2} = -\frac{1}{2}$$

   Therefore, $\displaystyle\lim_{x\to\infty} \frac{x - x\ e^x}{x + 2x\ e^x} = -\frac{1}{2}$.

2. Discus whether the improper integral $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^2 + 4}\ dx$ converges or diverges. [3]

   Solution :

   $$\int_{-\infty}^{\infty} \frac{1}{x^2 + 4}\ dx = \int_{-\infty}^{0} \frac{1}{x^2 + 4}\ dx + \int_{0}^{\infty} \frac{1}{x^2 + 4}\ dx$$

   $$= \lim_{t\to-\infty} \int_t^0 \frac{1}{(x)^2 + (2)^2}\ dx + \lim_{s\to\infty} \int_0^s \frac{1}{(x)^2 + (2)^2}\ dx$$

   $$= \lim_{t\to-\infty} \left[\frac{1}{2}\ \tan^{-1}\left(\frac{x}{2}\right)\right]_t^0 + \lim_{s\to\infty} \left[\frac{1}{2}\ \tan^{-1}\left(\frac{x}{2}\right)\right]_0^s$$

   $$= \lim_{t\to-\infty} \left[\frac{1}{2}\ \tan^{-1}\left(\frac{0}{2}\right) - \frac{1}{2}\ \tan^{-1}\left(\frac{t}{2}\right)\right]$$

   $$+ \lim_{s\to\infty} \left[\frac{1}{2}\ \tan^{-1}\left(\frac{s}{2}\right) - \frac{1}{2}\ \tan^{-1}\left(\frac{0}{2}\right)\right]$$

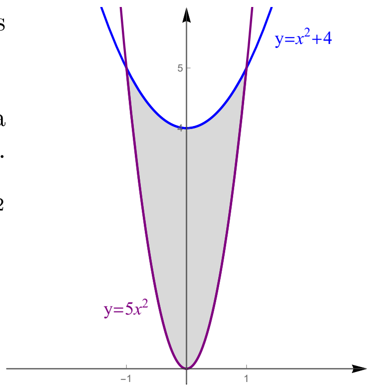
   $$= \left[\frac{1}{2}\ (0) - \frac{1}{2}\left(-\frac{\pi}{2}\right)\right] + \left[\frac{1}{2}\left(\frac{\pi}{2}\right) - \frac{1}{2}\ (0)\right] = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}.$$

Hence, the improper integral converges.

3. Sketch the region region bounded by the curves of $y = 5x^2$ and $y = 4+x^2$, and find its area. [3]

Solution :

$y = 5x^2$ represents a parabola opens upwards, and its vertex is $(0, 0)$.



$y = x^2 + 4$ represents a parabola opens upwards, and its vertex is $(0, 4)$.

Points of intersection of $y = 5x^2$ and $y = x^2 + 4$:
$5x^2 = x^2 + 4 \implies 4x^2 - 4 = 0$
$\implies x^2 - 1 = 0$
$\implies (x-1)(x+1) = 0$
$\implies x = -1 \ , \ x = 1.$

$$\mathbf{A} = \int_{-1}^{1} \left[(x^2+4) - (5x^2)\right] \ dx = \int_{-1}^{1} (-4x^2+4) \ dx$$

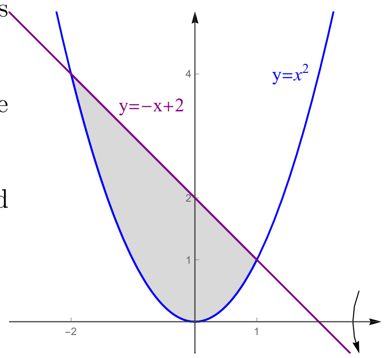
$$= \left[-4 \ \frac{x^3}{3} + 4x\right]_{-1}^{1} = \left[-4 \ \frac{(1)^3}{3} + 4 \ (1)\right) - \left(-4 \ \frac{(-1)^3}{3} + 4(-1)\right]$$

$$= -\frac{4}{3} + 4 - \left(\frac{4}{3} - 4\right) = -\frac{4}{3} + 4 - \frac{4}{3} + 4 = 8 - \frac{8}{3} = \frac{16}{3} \ .$$

4. Sketch the region bounded bu the curves $y = x^2$ and $y = -x + 2$, and find the volume of the solid generated by revolving this region about the $x-$axis. [3]

Solution :

$y = x^2$ represents a parabola opens upwards, and its vertex is $(0, 0)$.



$y = -x + 2$ represents a straight line passing through $(0, 2)$ with slope -1.

Points of intersection of $y = x^2$ and $y = -x + 2$:
$x^2 = -x + 2$
$\implies x^2 + x - 2 = 0$
$\implies (x+2)(x-1) = 0$
$\implies x = -2 \ , \ x = 1$

Using Washer Method :

$$\mathbf{V} = \pi \int_{-2}^{1} \left[(-x+2)^2 - \left(x^2\right)^2\right] \; dx$$

$$= \pi \int_{-2}^{1} \left[x^2 - 4x + 4 - x^4\right] \; dx = \pi \left[-\frac{x^5}{5} + \frac{x^3}{3} - 2x^2 + 4x\right]_{-2}^{2}$$

$$= \pi \left[\left(-\frac{(1)^5}{5} + \frac{(1)^3}{3} - 2(1)^2 + 4(1)\right) - \left(-\frac{(-2)^5}{5} + \frac{(-2)^3}{3} - 2(-2)^2 + 4(-2)\right)\right]$$

$$= \pi \left[\left(-\frac{1}{5} + \frac{1}{3} - 2 + 4\right) - \left(\frac{32}{5} - \frac{8}{3} - 8 - 8\right)\right]$$

$$= \pi \left(-\frac{33}{5} + \frac{9}{3} + 18\right) = \pi \left(-\frac{33}{5} + 21\right) = \frac{72\pi}{5}$$

5. Find the arc length of the function $y = \dfrac{e^x + e^x}{2}$ on the interval $[0, \ln 3]$. [3]

Solution :

$y = \dfrac{e^x + e^x}{2} = \cosh x$

$y' = \sinh x$.

$$\mathbf{L} = \int_{0}^{\ln 3} \sqrt{1 + (\sinh x)^2} \; dx = \int_{0}^{\ln 3} \sqrt{1 + \sinh^2 x} \; dx$$

$$= \int_{0}^{\ln 3} \sqrt{\cosh^2 x} \; dx = \int_{0}^{\ln 3} |\cosh x| \; dx = \int_{0}^{\ln 3} \cosh x \; dx$$

$$= [\sinh]_0^{\ln 3} = \sinh(\ln 3) - \sinh(0)$$

$$= \frac{e^{\ln 3} - e^{-\ln 3}}{2} - 0 = \frac{3 - \frac{1}{3}}{2} = \frac{4}{3} \; .$$

6. Convert the polar equation $\dfrac{1}{r} = \dfrac{\sin\theta + 5\cos\theta}{2}$ into a Cartesian equation.[1]

Solution :

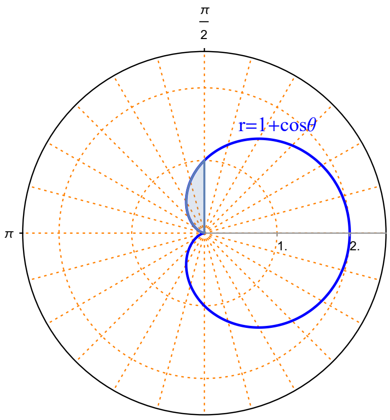
$$\frac{1}{r} = \frac{\sin\theta + 5\cos\theta}{2} \implies r(\sin\theta + 5\cos\theta) = 2$$

$$\implies r\sin\theta + 5(r\cos\theta) = 2 \implies y + 5x = 2 \implies y = -5x + 2 \; .$$

7. Sketch the region inside the graph of the polar equation $r = 1 + \cos\theta$ that is in the second quadrant. Then compute its area. [4]

Solution :

$r = 1 + \cos\theta$ represents a cardioid, symmetric with respect to the polar axis.



$$\mathbf{A} = \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} (1+\cos\theta)^2 \; d\theta$$
$$= \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} \left(1 + 2\cos\theta + \cos^2\theta\right) \; d\theta$$
$$= \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} \left[1 + 2\cos\theta + \left(\frac{1+\cos 2\theta}{2}\right)\right] \; d\theta$$
$$= \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} \left[\frac{3}{2} + 2\cos\theta + \frac{1}{2} \; \cos 2\theta\right] \; d\theta$$

$$\mathbf{A} = \frac{1}{2}\left[\frac{3}{2}\;\theta + 2\sin\theta + \frac{1}{4}\sin(2\theta)\right]_{\frac{\pi}{2}}^{\pi}$$

$$= \frac{1}{2}\left[\frac{3}{2}\;(\pi) + 2\sin(\pi) + \frac{1}{4}\sin(2\pi) - \left(\frac{3}{2}\;\left(\frac{\pi}{2}\right) + 2\sin\left(\frac{\pi}{2}\right) + \frac{1}{4}\sin(\pi)\right)\right]$$

$$= \frac{1}{2}\left[\left(\frac{3\pi}{2} + 0 + 0\right) - \left(\frac{3\pi}{4} + 2 + 0\right)\right]$$

$$= \frac{1}{2}\left[\frac{3\pi}{2} - \frac{3\pi}{4} - 2\right] = \frac{1}{2}\left[\frac{3\pi}{4} - 2\right] = \frac{3\pi}{8} - 1 \; .$$