111 ريض - حساب التكامل
الفصل الدراسي الأول 1445 هـ
حل الاختبار الفصلي الثاني
د. طارق عبدالرحمن مُحَمَّد الفاضل

السؤال الأول (4 درجات) : أحسب $\frac{dy}{dx}$ فيما يلي :

(1) $y = \cosh(3x^2) + sech^{-1}(2x)$. [2]

الحل :

$$\frac{dy}{dx} = \sinh(3x^2)\,(6x) + \frac{-1}{2x\,\sqrt{1-(2x)^2}}\,(2)$$
$$= 6x\sinh(3x^2) - \frac{1}{x\,\sqrt{1-4x^2}}$$

(2) $y = \coth^{-1}(3x) + \tanh^2\sqrt{2x}$. [2]

الحل :

$$\frac{dy}{dx} = \frac{-1}{1-(3x)^2}(3) + 2\left(\tanh\sqrt{2x}\right)^1\ sech^2\sqrt{2x}\,\frac{1}{2\sqrt{2x}}(2)$$
$$= \frac{-3}{1-9x^2} + \frac{2\tanh\sqrt{2x}\,sech^2\sqrt{2x}}{\sqrt{2x}}$$

السؤال الثاني (21 درجة): أحسب التكاملات التالية

(1) $\int e^{-x}\cosh x\,dx$. [2]

الحل :

$$\int e^{-x}\cosh x\,dx = \int e^{-x}\left(\frac{e^x+e^{-x}}{2}\right)dx = \int\left(\frac{e^0+e^{-2x}}{2}\right)dx$$
$$= \int\frac{1}{2}\,dx + \int\frac{e^{-2x}}{2}\,dx = \frac{1}{2}\int 1\,dx + \frac{1}{2}\,\frac{1}{-2}\int e^{-2x}\,(-2)\,dx$$
$$= \frac{1}{2}\,x - \frac{1}{4}\,e^{-2x} + c = \frac{x}{2} - \frac{e^{-2x}}{4} + c$$

باستخدام القانون

$$\int e^{f(x)}\,f'(x)\,dx = e^{f(x)} + c$$

(2) $\int \frac{dx}{\sqrt{x}\,\sqrt{1+x}}$. [2]

الحل :

$$\int \frac{dx}{\sqrt{x}\,\sqrt{1+x}} = \int \frac{1}{\sqrt{x}\,\sqrt{(1)^2 + (\sqrt{x})^2}}\,dx$$

$$= 2\int \frac{\frac{1}{2\sqrt{x}}}{\sqrt{(1)^2 + (\sqrt{x})^2}}\,dx = 2\sinh^{-1}\left(\sqrt{x}\right) + c$$

باستخدام القانون

$\int \frac{f'(x)}{\sqrt{[f(x)]^2 + a^2}}\,dx = \sinh^{-1}\left(\frac{f(x)}{a}\right) + c$ ، حيث $a > 0$.

حل آخر : باستخدام التعويض $u = \sqrt{x+1} \implies u^2 = x+1 \implies u^2 - 1 = x$

عندئذ $2u\,du = dx$ و $\sqrt{x} = \sqrt{u^2 - 1}$

$$\int \frac{dx}{\sqrt{x}\,\sqrt{1+x}} = \int \frac{2u}{\sqrt{u^2-1}\,u}\,du = 2\int \frac{1}{\sqrt{u^2-1}}\,du$$

$$= 2\cosh^{-1}(u) + c = 2\cosh^{-1}\left(\sqrt{x+1}\right) + c$$

باستخدام القانون

$\int \frac{f'(x)}{\sqrt{[f(x)]^2 - a^2}}\,dx = \cosh^{-1}\left(\frac{f(x)}{a}\right) + c$ ، حيث $|f(x)| > a$ و $a > 0$.

(3) $\int_1^e x^3 \ln x\,dx$. [2]

الحل : باستخدام طريقة التكامل بالتجزئ

$$\begin{array}{ll} u = \ln x & dv = x^3\,dx \\ du = \frac{1}{x}\,dx & v = \frac{x^4}{4} \end{array}$$

$$\int_1^e x^3 \ln x\,dx = \left[\frac{x^4}{4}\,\ln x\right]_1^e - \int_1^e \frac{x^4}{4}\,\frac{1}{x}\,dx = \left[\frac{x^4}{4}\,\ln x\right]_1^e - \frac{1}{4}\int_1^e x^3\,dx$$

$$= \left[\frac{x^4}{4}\,\ln x\right]_1^e - \frac{1}{4}\left[\frac{x^4}{4}\right]_1^e = \left[\frac{e^4}{4}\ln(e) - \frac{1}{4}\ln(1)\right] - \frac{1}{4}\left[\frac{e^4}{4} - \frac{1}{4}\right]$$

$$= \frac{e^4}{4} - \frac{e^4}{16} + \frac{1}{16} = \frac{3e^4}{16} + \frac{1}{16}$$

$$[2]\ .\int \sin^5 x\ \cos^3 x\, dx \quad (4)$$

الحل :

باستخدام التعويض $u = \sin x$

عندئذ $du = \cos x\, dx$

$$\int \sin^5 x\ \cos^3 x\, dx = \int \sin^5 x\ \cos^2 x\ \cos x\, dx \int \sin^5 x\ \left(1 - \sin^2 x\right)\ \cos x\, dx$$

$$= \int u^5\ \left(1 - u^2\right)\ du = \int \left(u^5 - u^7\right)\ du = \frac{u^6}{6} - \frac{u^8}{8} + c = \frac{\sin^6 x}{6} - \frac{\sin^8 x}{8} + c$$

$$[2]\ .\int \cosh^{-1} x\, dx \quad (5)$$

الحل :

الحل : باستخدام طريقة التكامل بالتجزئ

$$\begin{array}{ll} u = \cosh^{-1} x & dv = 1\, dx \\ du = \dfrac{1}{\sqrt{x^2 - 1}}\, dx & v = x \end{array}$$

$$\int \cosh^{-1} x\, dx = x \cosh^{-1} x - \int \frac{1}{\sqrt{x^2 - 1}}\, x\, dx$$

$$= x \cosh^{-1} x - \frac{1}{2} \int \left(x^2 - 1\right)^{-\frac{1}{2}} (2x)\, dx$$

$$= x \cosh^{-1} x - \frac{1}{2} \frac{\left(x^2 - 1\right)^{\frac{1}{2}}}{\frac{1}{2}} + c = x \cosh^{-1} x - \sqrt{x^2 - 1} + c$$

$$[3]\ .\int \frac{dx}{(4 + x^2)^{\frac{3}{2}}} \quad (6)$$

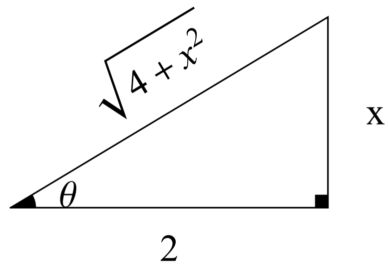
الحل : باستخدام التعويضات المثلثية

ضع $x = 2\tan\theta \implies \tan\theta = \dfrac{x}{2}$

$dx = 2\sec^2\theta\, d\theta$

$$\int \frac{dx}{(4 + x^2)^{\frac{3}{2}}} = \int \frac{2\sec^2\theta}{\left(4 + 4\tan^2\theta\right)^{\frac{3}{2}}}\, d\theta = \int \frac{2\sec^2\theta}{\left(4\left(1 + \tan^2\theta\right)\right)^{\frac{3}{2}}}\, d\theta = \int \frac{2\sec^2\theta}{\left(4\sec^2\theta\right)^{\frac{3}{2}}}\, d\theta$$

$$= \int \frac{2\sec^2\theta}{(4)^{\frac{3}{2}}\sec^3\theta}\,d\theta = \frac{2}{8}\int \frac{1}{\sec\theta}\,d\theta = \frac{1}{4}\int \cos\theta\,d\theta = \frac{1}{4}\sin\theta + c$$



من المثلث : نلاحظ أن $\sin\theta = \dfrac{x}{\sqrt{4+x^2}}$

$$\int \frac{dx}{(4+x^2)^{\frac{3}{2}}} = \frac{1}{4}\,\frac{x}{\sqrt{4+x^2}} + c$$

(7) $\displaystyle\int \frac{\sin x}{4-\cos^2 x}\,dx$. [3]

الحل :

$$\int \frac{\sin x}{4-\cos^2 x}\,dx = -\int \frac{-\sin x}{(2)^2-(\cos x)^2}\,dx = -\frac{1}{2}\tanh^{-1}\left(\frac{\cos x}{2}\right) + c$$

باستخدام القانون

$\displaystyle\int \frac{f'(x)}{a^2-[f(x)]^2}\,dx = \frac{1}{a}\tanh^{-1}\left(\frac{f(x)}{a}\right) + c$ حيث $|f(x)| < a$ و $a > 0$.

(8) $\displaystyle\int \frac{x+3}{x^3+9x}\,dx$. [3]

الحل : باستخدام طريقة الكسور الجزئية

$$\frac{x+3}{x^3+9x} = \frac{x+3}{x(x^2+9)} = \frac{A}{x} + \frac{Bx+C}{x^2+9}$$

$$\frac{x+3}{x(x^2+9)} = \frac{A(x^2+9)}{x(x^2+9)} + \frac{x\,(Bx+C)}{x(x^2+9)}$$

$$x+3 = A(x^2+9) + x(Bx+C)$$

$$x+3 = Ax^2+9A+Bx^2+Cx = (A+B)x^2+Cx+9A$$

بمقارنة معاملات كثيرتي الحدود في الطرفين نحصل على :

$$\begin{array}{lll} A+B=0 & \longrightarrow & (1) \\ C=1 & \longrightarrow & (2) \\ 9A=3 & \longrightarrow & (3) \end{array}$$

من المعادلة (3) نحصل على : $9A = 3 \implies A = \frac{1}{3}$.

من المعادلة (1) نحصل على : $\frac{1}{3} + B = 0 \implies B = -\frac{1}{3}$.

$$\int \frac{x+3}{x^3+9x}\,dx = \int \left(\frac{\frac{1}{3}}{x} + \frac{-\frac{1}{3}x+1}{x^2+9} \right) dx$$

$$= \int \frac{\frac{1}{3}}{x}\,dx + \int \frac{-\frac{1}{3}x}{x^2+9}\,dx + \int \frac{1}{x^2+9}\,dx$$

$$= \frac{1}{3}\int \frac{1}{x}\,dx - \frac{1}{3}\,\frac{1}{2}\int \frac{2x}{x^2+9}\,dx + \int \frac{1}{x^2+(3)^2}\,dx$$

$$= \frac{1}{3}\ln|x| - \frac{1}{6}\ln(x^2+9) + \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + c$$

(9) $\int \frac{dx}{\sqrt[4]{x}+\sqrt{x}}$. [2]

الحل : $\int \frac{dx}{\sqrt[4]{x}+\sqrt{x}} = \int \frac{1}{x^{\frac{1}{4}}+x^{\frac{1}{2}}}\,dx$

باستخدام التعويض $x = u^4$ ، أي أن $u = x^{\frac{1}{4}}$

$$dx = 4u^3\,du$$

$$\int \frac{dx}{x^{\frac{1}{4}}+x^{\frac{1}{2}}} = \int \frac{4u^3}{(u^4)^{\frac{1}{4}}+(u^4)^{\frac{1}{2}}}\,du = \int \frac{4u^3}{u+u^2}\,du$$

$$= \int \frac{4u^3}{u(1+u)}\,du = 4\int \frac{u^2}{u+1}\,du$$

باستخدام القسمة المطولة لكثيرات الحدود

$$4\int \frac{u^2}{u+1}\,du = 4\int \left(u - 1 + \frac{1}{u+1}\right) du$$

$$= 4\left(\frac{u^2}{2} - u + \ln|u+1|\right) + c = 2u^2 - 4u + 4\ln|u-1| + c$$

$$\int \frac{dx}{x^{\frac{1}{4}}+x^{\frac{1}{2}}} = 2\left(x^{\frac{1}{4}}\right)^2 - 4x^{\frac{1}{4}} + 4\ln\left|x^{\frac{1}{4}}+1\right| + c = 2x^{\frac{1}{2}} - 4x^{\frac{1}{4}} + 4\ln\left|x^{\frac{1}{4}}+1\right| + c$$