111 ريض - حساب التكامل
الفصل الدراسي الأول 1445 هـ
حل الاختبار الفصلي الأول
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السؤال الأول (9 درجات) :

(1) استخدم مجموع ريمان لحساب التكامل المحدد $\int_0^4 (x^2+1)\,dx$. [3]

الحل : $[a,b]=[0,4]$ و $f(x)=x^2+1$.

$$\Delta_x = \frac{b-a}{n} = \frac{4-0}{n} = \frac{4}{n}$$

$$x_k = a + k\,\Delta_x = 0 + k\left(\frac{4}{n}\right) = \frac{4k}{n}$$

$$f(x_k) = \left(\frac{4k}{n}\right)^2 + 1 = \frac{16k^2}{n^2} + 1$$

$$R_n = \sum_{k=1}^{n} f(x_k)\,\Delta_x = \sum_{k=1}^{n}\left(\frac{16k^2}{n^2}+1\right)\left(\frac{4}{n}\right) = \sum_{k=1}^{n}\left(\frac{64k^2}{n^3}+\frac{4}{n}\right) = \sum_{k=1}^{n}\frac{64k^2}{n^3} + \sum_{k=1}^{n}\frac{4}{n}$$

$$= \frac{64}{n^3}\sum_{k=1}^{n}k^2 + \frac{4}{n}\sum_{k=1}^{n}1 = \frac{64}{n^3}\,\frac{n(n+1)(2n+1)}{6} + \frac{4}{n}(n) = \frac{32}{3}\left(\frac{(n+1)(2n+1)}{n^2}\right) + 4$$

$$\int_0^4 (x^2+1)\,dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\left[\frac{32}{3}\left(\frac{(n+1)(2n+1)}{n^2}\right)+4\right]$$

$$= \frac{32}{3}(2) + 4 = \frac{64}{3} + 4 = \frac{76}{3}$$

(2) جد $F'(x)$ إذا كانت $F(x) = \int_{\sin(x^2)}^{\pi^{2x}} \sqrt{2t^3+2}\,dt$. [2]

الحل :

$$F'(x) = \frac{d}{dx}\int_{\sin(x^2)}^{\pi^{2x}} \sqrt{2t^3+2}\,dt$$

$$= \sqrt{2\left(\pi^{2x}\right)^3+2}\,\left(\pi^{2x}\,(2)\ln\pi\right) - \sqrt{2\left(\sin(x^2)\right)^3+2}\,\left(\cos(x^2)\,(2x)\right)$$

$$= 2\pi^{2x}\ln\pi\sqrt{2\pi^{6x}+2} - 2x\,\cos(x^2)\sqrt{2\sin^3(x^2)+2}$$

احسب $\frac{dy}{dx}$ فيمايلي :

(3) $y = \tan^{-1}(3x)\ \log_5|1 - sec(3x)|$. [2]

الحل :

$$\frac{dy}{dx} = \left(\frac{1}{1+(3x)^2}\,(3)\right)\ \log_5|1 - sec(3x)| + \tan^{-1}(3x)\ \left(\frac{-\sec(3x)\tan(3x)\ (3)}{1-\sec(3x)}\,\frac{1}{\ln 5}\right)$$

$$= \frac{3\log_5|1-\sec(3x)|}{1+9x^2} - \frac{3\sec(3x)\tan(3x)\ \tan^{-1}(3x)}{\ln 5\,(1-\sec(3x))}$$

(4) $y = (\cot x)^{\sin x} + 4^x$. [2]

الحل :

لتكن $y = f(x) + g(x)$ حيث $f(x) = (\cot x)^{\sin x}$ و $g(x) = 4^x$.

عندئذ $\frac{dy}{dx} = y' = f'(x) + g'(x)$

أولاً - $g'(x) = 4^x\ (1)\ln 4 = 4^x \ln 4$

ثانياً - حساب $f'(x)$

$$f(x) = (\cot x)^{\sin x} \implies \ln|f(x)| = \ln\left|(\cot x)^{\sin x}\right| = \sin x\ \ln|\cot x|$$

بإشتقاق الطرفين

$$\frac{f'(x)}{f(x)} = \cos x\ \ln|\cot x| + \sin x\left(\frac{-\csc^2 x}{\cot x}\right)$$

$$f'(x) = f(x)\left[\cos x\ \ln|\cot x| - \frac{\sin x\csc^2 x}{\cot x}\right] = (\cot x)^{\sin x}\ \ [\cos x\ \ln|\cot x| - \sec x]$$

أي أن $\frac{dy}{dx} = (\cot x)^{\sin x}\ \ [\cos x\ \ln|\cot x| - \sec x] + 4^x \ln 4$

السؤال الثاني (16 درجة) : أحسب التكاملات التالية

(1) $\int \left(\sqrt{x}\, e^{x^2}\right)^2\ dx$. [2]

الحل :

$$\int \left(\sqrt{x}\, e^{x^2}\right)^2\ dx = \int \left(\sqrt{x}\right)^2\left(e^{x^2}\right)^2\ dx = \int x\, e^{2x^2}\ dx$$

$$= \frac{1}{4}\int e^{2x^2}(4x)\ dx = \frac{1}{4}e^{2x^2} + c$$

باستخدام القانون

$$\int e^{f(x)}\, f'(x)\, dx = e^{f(x)} + c$$

(2) $\int x\,\sqrt{x+1}\, dx$. [2]

الحل : بوضع $x = u - 1 \implies u = x + 1$ و $dx = du$.

$$\int x\,\sqrt{x+1}\, dx = \int (u-1)\sqrt{u}\, du = \int (u-1)u^{\frac{1}{2}}\, du$$

$$= \int \left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right) du = \frac{u^{\frac{5}{2}}}{\frac{5}{2}} - \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{2}{5}\ (x+1)^{\frac{5}{2}} - \frac{2}{3}\ (x+1)^{\frac{3}{2}} + c$$

(3) $\int_0^1 x\, 5^{2-x^2}\, dx$. [2]

الحل :

$$\int_0^1 x\, 5^{2-x^2}\, dx = \frac{1}{-2}\int_0^1 5^{2-x^2}(-2x)\, dx = \frac{1}{-2}\left[\frac{5^{2-x^2}}{\ln 5}\right]_0^1$$

$$= \frac{1}{-2}\left[\frac{5^{2-(1)^2}}{\ln 5} - \frac{5^{2-(0)^2}}{\ln 5}\right] = \frac{1}{-2}\left[\frac{5^1}{\ln 5} - \frac{5^2}{\ln 5}\right] = \frac{1}{-2}\left(\frac{-20}{\ln 5}\right) = \frac{10}{\ln 5}$$

باستخدام القانون $\int a^{f(x)}\, f'(x)\, dx = \frac{a^{f(x)}}{\ln a} + c$.

(4) $\int \frac{2-x}{\sqrt{1-x^2}}\, dx$. [2]

الحل :

$$\int \frac{2-x}{\sqrt{1-x^2}}\, dx = \int \left(\frac{2}{\sqrt{1-x^2}} + \frac{-x}{\sqrt{1-x^2}}\right) dx$$

$$= \int \frac{2}{\sqrt{1-x^2}}\, dx + \int \left(1-x^2\right)^{-\frac{1}{2}}(-x)\, dx$$

$$= 2\int \frac{1}{\sqrt{1-x^2}}\, dx + \frac{1}{2}\int \left(1-x^2\right)^{-\frac{1}{2}}(-2x)\, dx$$

$$= 2\sin^{-1} x + \frac{1}{2}\,\frac{\left(1-x^2\right)^{\frac{1}{2}}}{\frac{1}{2}} + c = 2\sin^{-1} x + \sqrt{1-x^2} + c$$

(5) $\int \frac{\tan\left(\ln(x^2)\right)}{x}\,dx$. [2]

الحل :

$$\int \frac{\tan\left(\ln(x^2)\right)}{x}\,dx = \int \tan\left(2\ln|x|\right)\ \frac{1}{x}\,dx$$

$$= \frac{1}{2}\int \tan\left(2\ln|x|\right)\ \frac{2}{x}\,dx = \frac{1}{2}\ln\left|\sec\left(2\ln|x|\right)\right| + c$$

باستخدام القانون

$$\int \tan\left(f(x)\right)\ f'(x)\,dx = \ln\left|\sec\left(f(x)\right)\right| + c$$

(6) $\int \frac{\sec\left(\sqrt{x}\right)\tan\left(\sqrt{x}\right)}{\sqrt{x}}\,dx$. [2]

الحل :

$$\int \frac{\sec\left(\sqrt{x}\right)\tan\left(\sqrt{x}\right)}{\sqrt{x}}\,dx = \int \sec\left(\sqrt{x}\right)\tan\left(\sqrt{x}\right)\ \frac{1}{\sqrt{x}}\,dx$$

$$= 2\int \sec\left(\sqrt{x}\right)\tan\left(\sqrt{x}\right)\ \frac{1}{2\sqrt{x}}\,dx = 2\sec\left(\sqrt{x}\right) + c$$

باستخدام القانون

$$\int \sec\left(f(x)\right)\tan\left(f(x)\right)\ f'(x)\,dx = \sec\left(f(x)\right) + c$$

(7) $\int \frac{\left(\tan^{-1}x\right)^2}{x^2+1}\,dx$. [2]

الحل :

$$\int \frac{\left(\tan^{-1}x\right)^2}{x^2+1}\,dx = \int \left(\tan^{-1}x\right)^2\ \frac{1}{x^2+1}\,dx = \frac{\left(\tan^{-1}x\right)^3}{3} + c$$

باستخدام القانون

$$\int [f(x)]^n\ f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1} + c$$ ، حيث $n \neq -1$.

(8) $\int \frac{\sin(2x)\ \cos(2x)}{\sin^2(2x)}\,dx$. [2]

الحل :

$$\int \frac{\sin(2x)\ \cos(2x)}{\sin^2(2x)}\,dx = \int \frac{\sin(2x)\ \cos(2x)}{\sin(2x)\ \sin(2x)}\,dx = \int \frac{\cos(2x)}{\sin(2x)}\,dx$$

$$= \frac{1}{2}\int \frac{\cos(2x)\ (2)}{\sin(2x)}\,dx = \ln|\sin(2x)| + c$$

باستخدام القانون

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$$