King Saud University
College of Sciences
Department of Mathematics

MATH 111
INTEGRAL CALCULUS

CLASS NOTES
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Chapter 1

DEFINITE INTEGRAL

1.1 Summation and its properties

The summation of the real numbers $a_1, a_2, \cdots, a_n$ is denoted by $\sum_{k=1}^{n} a_k$.

Therefore, $\sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$.

Properties of Summation:
If $a_k, b_k \in \mathbb{R}$ for every $1 \leq k \leq n$ and $m \in \mathbb{R}$ then :

1. $\sum_{k=1}^{n} 1 = n$
2. $\sum_{k=1}^{n} m\ a_k = m\ \sum_{k=1}^{n} a_k$
3. $\sum_{k=1}^{n} (a_k \pm b_k) = \sum_{k=1}^{n} a_k \pm \sum_{k=1}^{n} b_k$

Important Summations:

1. $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$
2. $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$
3. $\sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2.$

Example: Simplify the summation $\sum_{k=1}^{n}(k-1)^2$.

Solution:

$\sum_{k=1}^{n}(k-1)^2 = \sum_{k=1}^{n}(k^2-2k+1)$

$= \sum_{k=1}^{n}k^2 - 2\sum_{k=1}^{n}k + \sum_{k=1}^{n}1$

$= \frac{n(n+1)(2n+1)}{6} - 2\frac{n(n+1)}{2} + n.$

EXERCISES (1.1)

1. Calculate the following summations:

$$(i)\ \sum_{k=1}^{20}(3k+2)$$

$$(ii)\ \sum_{k=1}^{100}(k+1)^2$$

$$(iii)\ \sum_{k=1}^{30}(k^3+k^2-k)$$

2. Evaluate the following limits:

$$(i)\ \lim_{n\to\infty}\frac{1}{n^2}\ \sum_{k=1}^{n}(2k-5)$$

$$(ii)\ \lim_{n\to\infty}\frac{1}{n^3}\ \sum_{k=1}^{n}(k^2-k+1)$$

$$(iii)\ \lim_{n\to\infty}\frac{1}{n^4}\ \sum_{k=1}^{n}(2k^3+4)$$

$$(iv)\ \lim_{n\to\infty}\frac{1}{n^3}\ \sum_{k=1}^{n}(ak+b)\quad a,b\in\mathbb{R}^*$$

$$(v)\ \lim_{n\to\infty}\frac{1}{n^4}\ \sum_{k=1}^{n}(ak^2+bk)\quad a,b\in\mathbb{R}^*$$

3. Find the value of a that satisfies the following equations:

$$(i)\ \sum_{k=1}^{10}(ak-10)=120$$

$$(ii)\ \sum_{k=1}^{5}(ak^2+2)=120$$

$$(iii)\ \sum_{k=5}^{15}(ak+5)=275$$

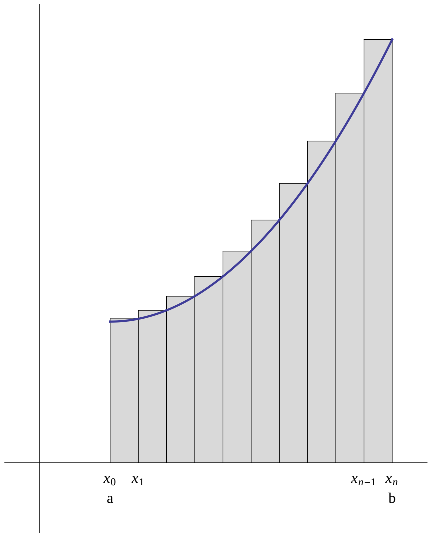
1.2 Definite Integral

Uniform partition of an interval:
To divide a closed interval $[a, b]$ into n sub-intervals all having the same length $\Delta_x = \dfrac{b-a}{n}$, put $x_0 = a$, $x_1 = a + \Delta_x$, $x_k = a + k\Delta_x$ and $x_n = a + n\,\Delta_x = b$. The set $P = \{x_0 = a, x_1, \cdots, x_k, \cdots, x_n = b\}$ is called the uniform partition of the interval $[a, b]$ into n sub-intervals.

Example (1): Divide the interval $[1, 3]$ into 5 uniform sub-intervals.
Solution: $\Delta_x = \dfrac{3-1}{5} = \dfrac{2}{5} = 0.4$
$x_0 = 1$, $x_1 = 1 + 0.4 = 1.4$, $x_2 = 1 + 2(0.4) = 1.8$
$x_3 = 1 + 3(0.4) = 2.2$, $x_4 = 1 + 4(0.4) = 2.6$, $x_5 = 1 + 5(0.4) = 3$
The set $P = \{1, 1.4, 1.8, 2.2, 2.6, 3\}$ is the required uniform partition of the interval $[1, 3]$.

Riemann Sum:

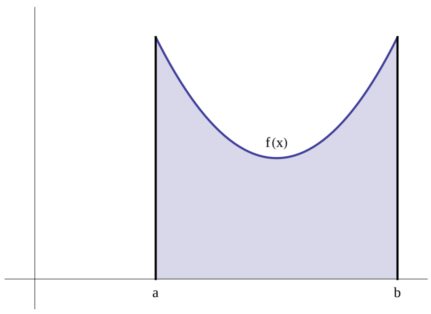
If f is a continuous function on the interval $[a, b]$ and $P = \{x_0 = a, x_1, \cdots, x_n = b\}$ is a uniform partition of $[a, b]$, then the Riemann sum of the function f on the interval $[a, b]$ with respect to the uniform partition P is defined as : $R_n = \sum_{k=1}^{n} f(x_k)\,\Delta_x$,
where $\Delta_x = \dfrac{b-a}{n}$
and $x_k = a + k\Delta_x = a + k\left(\dfrac{b-a}{n}\right)$.



Definite integral: The definite integral of the continuous function f on the interval $[a, b]$ is denoted by $\int_a^b f(x)\,dx$, and it is defined as :

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\,\Delta_x.$$

So $\int_a^b f(x)\,dx$ represents the area of the plane region bounded by the graph of the function f, x-axis and the two straight lines $x = a$ and $x = b$.



Example (2): Use Riemann sum to evaluate the definite integral $\int_1^2 (6x-5)\ dx$.

Solution: $f(x) = 6x - 5$ and $[a, b] = [1, 2]$.

$\Delta_x = \frac{2-1}{n} = \frac{1}{n}$.

$x_k = a + k\Delta_x = 1 + k\left(\frac{1}{n}\right) = 1 + \frac{k}{n}$.

$f(x_k) = f\left(1+\frac{k}{n}\right) = 6\left(1+\frac{k}{n}\right) - 5 = 6 + \frac{6k}{n} - 5 = \frac{6k}{n} + 1$.

$R_n = \sum_{k=1}^{n} f(x_k)\,\Delta_x = \sum_{k=1}^{n}\left(\frac{6k}{n}+1\right)\ \frac{1}{n} = \sum_{k=1}^{n}\left(\frac{6k}{n^2}+\frac{1}{n}\right) = \sum_{k=1}^{n}\frac{6k}{n^2} + \sum_{k=1}^{n}\frac{1}{n}$

$= \frac{6}{n^2}\sum_{k=1}^{n} k + \frac{1}{n}\sum_{k=1}^{n} 1 = \frac{6}{n^2}\ \frac{n(n+1)}{2} + \frac{1}{n}\ (n) = 3\ \frac{n+1}{n} + 1$.

$\int_1^2 (6x-5)\ dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\left(3\ \frac{n+1}{n} + 1\right) = 3(1) + 1 = 4$.

Example (3): Use Riemann sum to evaluate the definite integral $\int_0^2 x^2\ dx$.

Solution: $f(x) = x^2$ and $[a, b] = [0, 2]$.

$\Delta_x = \frac{2-0}{n} = \frac{2}{n}$.

$x_k = a + k\Delta_x = 0 + \frac{2k}{n} = \frac{2k}{n}$.

$f(x_k) = f\left(\frac{2k}{n}\right) = \left(\frac{2k}{n}\right)^2 = \frac{4k^2}{n^2}$.

$R_n = \sum_{k=1}^{n} f(x_k)\,\Delta_x = \sum_{k=1}^{n}\left(\frac{4k^2}{n^2}\right)\frac{2}{n} = \sum_{k=1}^{n}\frac{8k^2}{n^3} = \frac{8}{n^3}\sum_{k=1}^{n} k^2$

$= \frac{8}{n^3}\ \frac{n(n+1)(2n+1)}{6} = \frac{8}{6}\ \frac{n(n+1)(2n+1)}{n^3} = \frac{4}{3}\ \frac{(n+1)(2n+1)}{n^2}$

$\int_0^2 x^2\ dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\left(\frac{4}{3}\ \frac{(n+1)(2n+1)}{n^2}\right) = \frac{4}{3}(2) = \frac{8}{3}$

EXERCISES (1.2)

Use Riemann sum to evaluate the following definite integrals:

1. $\int_0^2 (3x - 2)\ dx$
2. $\int_1^3 (5x - 6)\ dx$
3. $\int_{-1}^4 (2x + 1)\ dx$
4. $\int_0^4 (x^2 + 1)\ dx$
5. $\int_2^4 (x^2 - x)\ dx$
6. $\int_0^3 (x^3 - 1)\ dx$
7. $\int_1^4 (x^3 + x)\ dx$

1.3 Properties of definite integral

If f and g are two continuous functions on the interval $[a, b]$ and $k \in \mathbb{R}$, then:

1. $\displaystyle\int_a^b [f(x) \pm g(x)]\ dx = \int_a^b f(x)\ dx \pm \int_a^b g(x)\ dx$.

2. $\displaystyle\int_a^b k\ f(x)\ dx = k\ \int_a^b f(x)\ dx$.

3. If $a < c < b$ then:

   $$\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx\ .$$

4. $\displaystyle\int_a^a f(x)\ dx = 0$.

5. $\displaystyle\int_a^b f(x)\ dx = -\int_b^a f(x)\ dx$.

6. If $f(x) \geq 0$ for every $x \in [a, b]$ then: $\displaystyle\int_a^b f(x)\ dx \geq 0$.

7. If $f(x) \geq g(x)$ for every $x \in [a, b]$ then: $\displaystyle\int_a^b f(x)\ dx \geq \int_a^b g(x)\ dx$.

EXERCISES (1.3)

1. If $\int_0^7 f(x)\ dx = 10$ and $\int_4^7 2f(x)\ dx = 6$, calculate $\int_0^4 f(x)\ dx$.

2. If $\int_a^b f(x)\ dx = 3$ and $\int_a^b g(x)\ dx = 2$, calculate $\int_a^b [5f(x) - 3g(x)]\ dx$.

3. Prove that :

   (*i*) $\int_0^2 \frac{1}{x^2+1}\ dx \leq 2$.

   (*ii*) $\int_0^4 \frac{x}{x^3+2}\ dx \leq \int_0^4 x\ dx$.

   (*iii*) $0 \leq \int_0^1 \frac{x}{\sqrt{x+3}}\ dx \leq 1$.

4. Prove that $\left|\int_a^b f(x)\ dx\right| \leq \int_a^b |f(x)|\ dx$.

5. Use quesion (4) to prove that :

   (*i*) $\left|\int_0^{2\pi} \cos x^2\ dx\right| \leq 2\pi$.

   (*ii*) $\left|\int_a^b \sin x\ dx\right| \leq (b-a)$.

Chapter 2

INDEFINITE INTEGRAL

2.1 Anti-derivative of a function

Definition of an anti-derivative of a function:
The function G is called an anti-derivative of the function f on the interval $[a, b]$ if $G'(x) = f(x)$ for every $x \in [a, b]$.

Example: Find the anti-derivative function of each of the following functions:

1. $f(x) = 2x$
2. $f(x) = \cos x$
3. $f(x) = \sec^2 x$
4. $f(x) = \sec x \tan x$

Solution:

1. $G(x) = x^2 + c$
2. $G(x) = \sin x + c$
3. $G(x) = \tan x + c$
4. $G(x) = \sec x + c$

   Where c is a constant.

Remark: If G_1 and G_2 are two different ant-derivatives of the function f on $[a, b]$, then $G_1(x) - G_2(x) = c$ for every $x \in [a, b]$, where c is a constant.

EXERCISES (2.1)

1. Find the ant-derivative function of each of the following functions:

   (i) $f(x) = 3x^2$

   (ii) $f(x) = 7x^6$

   (iii) $f(x) = 2\cos 2x$

   (iv) $f(x) = 5\sec^2 5x$

   (v) $f(x) = 3\sin 3x$

   (vi) $f(x) = 4\sec 4x \; \tan 4x$

   (vii) $f(x) = -6\csc 6x \; \cot 6x$

   $(viii)$ $f(x) = 2\csc^2 2x$

   (ix) $f(x) = \dfrac{1}{2\sqrt{x}}$

2. Show that $G(x) = \dfrac{x^{n+1}}{n+1} + c$ is the ant-derivative of $f(x) = x^n$ where $n \neq -1$ and c is a constant.

3. Show that $G(x) = \dfrac{\sin(ax)}{a} + c$ is the anti-derivative of $f(x) = \cos(ax)$ where $a \neq 0$ and c is a constant.

2.2 The Fundamental Theorem of Calculus

The mean value theorem for definite integrals:
If f is a continuous function on $[a,b]$, then there exists $c \in (a,b)$ such that

$(b-a)f(c) = \int_a^b f(x)\ dx$

Proof:
Since the function f is continuous on the closed and bounded interval $[a,b]$, then it attains its maximum and minimum values on $[a,b]$, hence there exist $x_1, x_2 \in [a,b]$ such $f(x_1) \le f(x) \le f(x_2)$ for every $x \in [a,b]$.

$\int_a^b f(x_1)\ dx \le \int_a^b f(x)\ dx \le \int_a^b f(x_2)\ dx$

$(b-a)\ f(x_1) \le \int_a^b f(x)\ dx \le (b-a)\ f(x_2)$

$f(x_1) \le \dfrac{\int_a^b f(x)\ dx}{b-a} \le\ f(x_2)$

From the intermediate value theorem of continuous functions there exists c between x_1 and x_2 (Hence $c \in (a,b)$) such that

$$f(c) = \frac{\int_a^b f(x)\ dx}{b-a} \implies (b-a)f(c) = \int_a^b f(x)\ dx.$$

Example (1): Find the value of c that satisfies the mean value theorem for definite integrals for the function $f(x) = x^2$ on the interval $[0,2]$.

Solution:

First: calculating the definite integral $\int_0^2 x^2\ dx$.

This definite integral was calculated using Riemann sum and it is equal $\frac{8}{3}$ (see Example (2) in section 1.2).

Second: Finding the value of c, from the relation $(b-a)\ f(c) = \int_a^b f(x)\ dx$.

$(2-0)\ f(c) = \int_0^2 x^2\ dx \implies 2c^2 = \dfrac{8}{3}$

$\implies c^2 = \dfrac{8}{6} = \dfrac{4}{3} \implies c = \pm\sqrt{\dfrac{4}{3}} = \pm\dfrac{2}{\sqrt{3}}.$

Note that $c = \dfrac{2}{\sqrt{3}} \in (0,2)$, while $c = -\dfrac{2}{\sqrt{3}} \notin (0,2)$.

Therefore, The required value is $c = \dfrac{2}{\sqrt{3}}$.

The Fundamental Theorem of Calculus:
Let $f : [a,b] \to \mathbb{R}$ be a continuous function on $[a,b]$,

1. If the function $G : [a,b] \to \mathbb{R}$ is defined as follows:

   $G(x) = \int_a^x f(t)\ dt$, for every $x \in [a,b]$, then G is an anti-derivative of f on $[a,b]$.

Therefore, $\dfrac{d}{dx}\displaystyle\int_a^x f(t)\ dt = f(x)$ for every $x \in [a,b]$.

2. If F is an anti-derivative of f, then: $\displaystyle\int_a^b f(x)\ dx = [F(x)]_a^b = F(b) - F(a)$.

Proof:
(1) To prove that $\displaystyle\lim_{h\to 0}\frac{G(x+h)-G(x)}{h} = f(x)$.
Note that $G(x+h)-G(x) = \displaystyle\int_a^{x+h} f(t)\ dt - \int_a^x f(t)\ dt = \int_x^{x+h} f(t)\ dt$.
For every $h > 0$ and for every $x \in [a,b]$, Applying the mean value theorem for definite integrals on f on the interval $[x, x+h]$.
There exists $c_h \in (x, x+h)$ such that

$$\int_x^{x+h} f(t)\ dt = h\ f(c_h) \Longrightarrow \frac{\int_x^{x+h} f(t)\ dt}{h} = f(c_h)$$

Since $x < c_h < x+h$, then $c_h \to x$ when $h \to 0$.

Therefore, $\displaystyle\lim_{h\to 0}\frac{G(x+h)-G(x)}{h} = \lim_{h\to 0}\frac{\int_x^{x+h} f(t)\ dt}{h} = \lim_{c_h\to x} f(c_h) = f(x)$.
Note that $\displaystyle\lim_{c_h\to x} f(c_h) = f(x)$, since the function f is continuous on $[a,b]$ and $c_h \to x$.
Therefore, $\dfrac{d}{dx}\displaystyle\int_a^x f(t)\ dt = G'(x) = f(x)$ for every $x \in [a,b]$.
(2) Let F be an anti-derivative of f on $[a,b]$, then
$G(x) - F(x) = c$, for every $x \in [a,b]$ where c is a constant.
When $x = a$, then $G(a) = \displaystyle\int_a^a f(t)\ dt = 0$,
Therefore, $G(a) - F(a) = c \Longrightarrow c = -F(a)$.
Hence $G(x) - F(x) = -F(a)$ for every $x \in [a,b]$.
When $x = b$, then $G(b) = \displaystyle\int_a^b f(t)\ dt$,
Therefore, $\displaystyle\int_a^b f(x)\ dx = G(b) = F(b) - F(a) = [F(x)]_a^b$.

Example (2): Evaluate the following:

1. $\dfrac{d}{dx}\displaystyle\int_1^x \frac{1}{1+t^2}\ dt$.

2. $\dfrac{d}{dx}\displaystyle\int_0^x \sin t\ dt$.

Solution:

1. $\dfrac{d}{dx}\displaystyle\int_1^x \frac{1}{1+t^2}\ dt = \frac{1}{1+x^2}$.

2. $\dfrac{d}{dx}\displaystyle\int_0^x \sin t\ dt = \sin x$.

Example (3): Calculate the following:

1. $\int_1^2 2x \; dx$.
2. $\int_0^{\frac{\pi}{2}} \cos x \; dx$.

Solution:

1. $\int_1^2 2x \; dx = \left[x^2\right]_1^2 = (2)^2 - (1)^2 = 4 - 1 = 3$

   Note that $F(x) = x^2$ is an anti-derivative of $f(x) = 2x$.

2. $\int_0^{\frac{\pi}{2}} \cos x \; dx = [\sin x]_0^{\frac{\pi}{2}} = \sin\left(\frac{\pi}{2}\right) - \sin(0) = 1 - 0 = 1$

   Note that $F(x) = \sin x$ is an anti-derivative of $f(x) = \cos x$.

Theorem (1): If f is a continuous function on $[a, b]$ and g is differentiable and its range is contained in $[a, b]$ then:
$\frac{d}{dx}\int_a^{g(x)} f(t) \; dt = f\left(g(x)\right) \; g'(x).$

Example (4): Evaluate the following:

1. $\frac{d}{dx}\int_0^{\sqrt{x}} \frac{1}{1+t^2} \; dt.$
2. $\frac{d}{dx}\int_{-1}^{3x^2} \sin t \; dt.$

Solution:

1. $\frac{d}{dx}\int_0^{\sqrt{x}} \frac{1}{1+t^2} \; dt = \frac{1}{1+\left(\sqrt{x}\right)^2} \; \frac{1}{2\sqrt{x}} = \frac{1}{1+x} \; \frac{1}{2\sqrt{x}}.$
2. $\frac{d}{dx}\int_{-1}^{3x^2} \sin t \; dt = \sin(3x^2) \; (6x).$

Theorem (2): If f is a continuous function on $[a, b]$, and g, h are two differentiable function and their ranges are contained in $[a, b]$ then:
$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t) \; dt = f\left(h(x)\right) \; h'(x) - f\left(g(x)\right) \; g'(x)$

Example (5): Evaluate the following:

1. $\frac{d}{dx}\int_{\cos x}^{2x^2} \frac{1-t^2}{1+t^4} \; dt.$

2. $\dfrac{d}{dx}\displaystyle\int_{\frac{1}{x}}^{x^2} \sqrt{1+t^4}\ dt.$

Solution:

1. $\dfrac{d}{dx}\displaystyle\int_{\cos x}^{2x^2} \frac{1-t^2}{1+t^4}\ dt = \frac{1-\left(2x^2\right)^2}{1+\left(2x^2\right)^4}\ (4x) - \frac{1-(\cos x)^2}{1+(\cos x)^4}\ (-\sin x).$

2. $\dfrac{d}{dx}\displaystyle\int_{\frac{1}{x}}^{x^2} \sqrt{1+t^4}\ dt = \sqrt{1+(x^2)^4}\ (2x) - \sqrt{1+\left(\frac{1}{x}\right)^4}\ \left(-\frac{1}{x^2}\right).$

EXERCISES (2.2)

1. Find $F'(x)$ of the following:

   (*i*) $F(x) = \displaystyle\int_{-1}^{x} \sqrt{2+\sin t}\ dt$

   (*ii*) $F(x) = \displaystyle\int_{2}^{x} \frac{1}{\sqrt{1+t^2}}\ dt$

   (*iii*) $F(x) = \displaystyle\int_{0}^{\sqrt{x}} \frac{1}{1+t^2}\ dt$

   (*iv*) $F(x) = \displaystyle\int_{-3}^{\sin x} \frac{t}{1+t^4}\ dt$

   (*v*) $F(x) = \displaystyle\int_{-1}^{x^2} \sqrt{3+\cos t}\ dt + \int_{x^2}^{3} \sqrt{3+\cos t}\ dt$

   (*vi*) $F(x) = \displaystyle\int_{\sin x}^{x^3} \frac{t}{2+t^2}\ dt$

   (*vii*) $F(x) = \displaystyle\int_{\sqrt{x}}^{x^2+1} \frac{t^2}{1+t^2}\ dt$

2. If $F(x) = \displaystyle\int_{3x}^{3x^2+1} \frac{t}{4+t^2}\ dt$, calculate $F'(0)$.

3. If $F(x) = \displaystyle\int_{\cos x}^{\sin x} \frac{1}{\sqrt{t^2+1}}\ dt$, calculate $F'\left(\frac{\pi}{2}\right)$.

2.3 Indefinite Integral

Definition: The indefinite integral of the function f is denoted by $\int f(x)\ dx$ and it is defined as $\int f(x)\ dx = G(x) + c$, where G is the anti-derivative of f and c is a constant.

Basic rules of integration:

1. $\int 1\ dx = x + c.$
2. $\int x^n\ dx = \frac{x^{n+1}}{n+1} + c$, where $n \neq -1$.
3. $\int \cos x\ dx = \sin x + c.$
4. $\int \sin x\ dx = -\cos x + c.$
5. $\int \sec^2 x\ dx = \tan x + c.$
6. $\int \csc^2 x\ dx = -\cot x + c.$
7. $\int \sec x \tan x\ dx = \sec x + c.$
8. $\int \csc x \cot x\ dx = -\csc x + c.$

Properties of indefinite integral:

1. $\int m\ f(x)\ dx = m \int f(x)\ dx$, where $m \in \mathbb{R}$
2. $\int [f(x) \pm g(x)]\ dx = \int f(x)\ dx \pm \int g(x)\ dx$

Example: Evaluate the following indefinite integrals:

1. $\int \left(7x^2 + 5\sqrt{x}\right)\ dx.$
2. $\int \left(\frac{5}{x^4} - \frac{2}{\sqrt[3]{x}}\right)\ dx.$
3. $\int \left(-4\cos x + 8\sec^2 x\right)\ dx.$
4. $\int (3 + 4\sec x \tan x)\ dx.$

Solution:

1. $\int \left(7x^2 + 5\sqrt{x}\right)\ dx = \int 7x^2\ dx + \int 5\sqrt{x}\ dx$

$= 7\int x^2\ dx + 5\int x^{\frac{1}{2}}\ dx = 7\ \frac{x^3}{3} + 5\ \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c.$

2. $\int \left(\frac{5}{x^4} - \frac{2}{\sqrt[3]{x}}\right)\ dx = \int \frac{5}{x^4}\ dx - \int \frac{2}{\sqrt[3]{x}}\ dx = 5\int \frac{1}{x^4}\ dx - 2\int \frac{1}{x^{\frac{1}{3}}}\ dx$

$= 5\int x^{-4}\ dx - 2\int x^{-\frac{1}{3}} = 5\ \frac{x^{-3}}{-3} - 2\ \frac{x^{\frac{2}{3}}}{\frac{2}{3}} + c.$

3. $\int \left(-4\cos x + 8\sec^2 x\right)\ dx = \int -4\cos x\ dx + \int 8\sec^2 x\ dx$

$= -4\int \cos x\ dx + 8\int \sec^2 x\ dx = -4\sin x + 8\tan x + c.$

4. $\int (3 + 4\sec x\tan x)\ dx = \int 3\ dx + \int 4\sec x\tan x\ dx$

$= 3\int 1\ dx + 4\int \sec x\tan x\ dx = 3x + 4\sec x + c.$

EXERCISES (2.3)

Evaluate the following integrals:

1. $\displaystyle\int \left(\frac{-2}{x^2} - \frac{5}{x^{-4}}\right)\, dx.$

2. $\displaystyle\int \left(\frac{4}{\sqrt{x}} + \frac{1}{x^{\frac{2}{3}}}\right)\, dx.$

3. $\displaystyle\int (7\cos x + 2\sin x)\, dx.$

4. $\displaystyle\int \left(4\sec^2 x - \frac{8\sec x\tan x}{3}\right)\, dx.$

5. $\displaystyle\int \left(\frac{\csc^2 x}{5} + 3\csc x\cot x\right)\, dx.$

6. $\displaystyle\int \frac{x^2 - 1}{\sqrt[3]{x}}\, dx.$

7. $\displaystyle\int \frac{x^2 + x - 2}{\sqrt[4]{x}}\, dx.$

8. $\displaystyle\int \frac{(x+1)^2}{x^{\frac{3}{2}}}\, dx.$

2.4 Integration by substitution

Theorem: If g is a differentiable function on $[a, b]$, f is a continuous function on an interval J that contains the range of g and F is an anti-derivative of f on J, then: $\int f(g(x))\, g'(x)\ dx = F(g(x)) + c$, for every$x \in [a, b]$.

Example (1): Evaluate the integral $\int (x^2+1)^{11} x\ dx$.

First solution: Put $u = x^2 + 1$, then $du = 2x\ dx \Longrightarrow \frac{1}{2}\ du = x\ dx$.

$\int (x^2+1)^{11} x\ dx = \int u^{11}\ \frac{1}{2}\ du = \frac{1}{2} \int u^{11}\ du$

$= \frac{1}{2}\ \frac{u^{12}}{12} + c = \frac{1}{2}\ \frac{(x^2+1)^{12}}{12} + c.$

Second solution: Using the formula $\int [f(x)]^n\ f'(x)\ dx = \frac{[f(x)]^{n+1}}{n+1} + c$, where $n \neq -1$.

$\int (x^2+1)^{11} x\ dx = \frac{1}{2} \int (x^2+1)^{11} (2x)\ dx = \frac{1}{2}\ \frac{(x^2+1)^{12}}{12} + c.$

Generalization of basic rules of integration:

1. $\int x^n\ dx = \frac{x^{n+1}}{n+1} + c$, where $n \neq -1$.

   $\int [f(x)]^n\ f'(x)\ dx = \frac{[f(x)]^{n+1}}{n+1} + c$, where $n \neq -1$.

2. $\int \cos x\ dx = \sin x + c.$

   $\int \cos(f(x))\ \ f'(x)\ dx = \sin(f(x)) + c.$

3. $\int \sin x\ dx = -\cos x + c.$

   $\int \sin(f(x))\ \ f'(x)\ dx = -\cos(f(x)) + c.$

4. $\int \sec^2 x\ dx = \tan x + c.$

   $\int \sec^2(f(x))\ \ f'(x)\ dx = \tan(f(x)) + c.$

5. $\int \csc^2 x\ dx = -\cot x + c.$

   $\int \csc^2(f(x))\ \ f'(x)\ dx = -\cot(f(x)) + c.$

6. $\int \sec x \tan x\ dx = \sec x + c.$

$$\int \sec(f(x)) \tan(f(x)) \ f'(x) \ dx = \sec(f(x)) + c.$$

7. $\displaystyle\int \csc x \cot x \ dx = -\csc x + c.$

$$\int \csc(f(x)) \cot(f(x)) \ f'(x) \ dx = -\csc(f(x)) + c.$$

Example (2): Evaluate the following integrals:

1. $\displaystyle\int \sqrt{x^2 + 2x}(x + 1) \ dx.$

   Solution: $\displaystyle\int \sqrt{x^2 + 2x} \ (x + 1) \ dx = \int \left(x^2 + 2x\right)^{\frac{1}{2}} (x + 1) \ dx$

   $\displaystyle= \frac{1}{2} \int \left(x^2 + 2x\right)^{\frac{1}{2}} \ [2(x + 1)] \ dx = \frac{1}{2} \int \left(x^2 + 2x\right)^{\frac{1}{2}} (2x + 2) \ dx$

   $\displaystyle= \frac{1}{2} \frac{(x^2 + 2x)^{\frac{3}{2}}}{\frac{3}{2}} + c.$

2. $\displaystyle\int \frac{x^3 + x}{\sqrt[3]{x^4 + 2x^2 + 3}} \ dx.$

   Solution: $\displaystyle\int \frac{x^3 + x}{\sqrt[3]{x^4 + 2x^2 + 3}} \ dx = \int \frac{x^3 + x}{(x^4 + 2x^2 + 3)^{\frac{1}{3}}} \ dx$

   $\displaystyle= \int \left(x^4 + 2x^2 + 3\right)^{-\frac{1}{3}} (x^3 + x) \ dx = \frac{1}{4} \int \left(x^4 + 2x^2 + 3\right)^{-\frac{1}{3}} (4x^3 + 4x) \ dx$

   $\displaystyle= \frac{1}{4} \ \frac{\left(x^4 + 2x^2 + 3\right)^{\frac{2}{3}}}{\frac{2}{3}} + c.$

3. $\displaystyle\int \cos(5x + 7) \ dx.$

   Solution: $\displaystyle\int \cos(5x + 7) \ dx = \frac{1}{5} \int \cos(5x + 7) \ 5 \ dx$

   $\displaystyle= \frac{1}{5} \sin(5x + 7) + c.$

4. $\displaystyle\int x \sec^2(x^2 + 2) \ dx.$

   Solution: $\displaystyle\int x \sec^2(x^2 + 2) \ dx = \frac{1}{2} \int \sec^2(x^2 + 2) \ (2x) \ dx$

   $\displaystyle= \frac{1}{2} \tan(x^2 + 2) + c.$

5. $\displaystyle\int \sec(6x - 2) \tan(6x - 2) \ dx.$

Solution: $\displaystyle\int \sec(6x-2)\tan(6x-2)\ dx = \frac{1}{6}\int \sec(6x-2)\tan(6x-2)\ (6)\ dx$

$\displaystyle = \frac{1}{6}\sec(6x-2) + c.$

Example (3): Find the value of c that satisfies the mean value theorem for definite integrals for the function $f(x) = 1 + x^2$ on the interval $[-1, 2]$.

Solution: Using the relation $\displaystyle(b-a)f(c) = \int_a^b f(x)\ dx.$

Where $f(x) = 1 + x^2$ and $[a, b] = [-1, 2]$.

$\displaystyle(2-(-1))\ (1+c^2) = \int_{-1}^{2} (1+x^2)\ dx = \left[x + \frac{x^3}{3}\right]_{-1}^{2}$

$\displaystyle 3(1+c^2) = \left(2 + \frac{8}{3}\right) - \left(-1 - \frac{1}{3}\right)$

$\displaystyle 3 + 3c^2 = 2 + \frac{8}{3} + 1 + \frac{1}{3} = 6$

$3c^2 = 3 \implies c^2 = 1 \implies c = \pm 1$

Note that $c = 1 \in (-1, 2)$ is the required value.

While $c = -1 \notin (-1, 2)$, hence $c = -1$ does not satisfy the mean value theorem for definite integrals.

Example (4): Find the value of c that satisfies the mean value theorem for definite integrals for the function $f(x) = \sqrt{x+1}$ on the interval $[-1, 8]$.

Solution: Using the relation $\displaystyle(b-a)f(c) = \int_a^b f(x)\ dx.$

Where $f(x) = \sqrt{x+1}$ and $[a, b] = [-1, 8]$.

$\displaystyle(8-(-1))\ \sqrt{c+1} = \int_{-1}^{8} \sqrt{x+1}\ dx = \int_{-1}^{8} (x+1)^{\frac{1}{2}}\ dx$

$\displaystyle 9\sqrt{c+1} = \left[\frac{(x+1)^{\frac{3}{2}}}{\frac{3}{2}}\right]_{-1}^{8} = \frac{2}{3}\left[(x+1)^{\frac{3}{2}}\right]_{-1}^{8}$

$\displaystyle 9\sqrt{c+1} = \frac{2}{3}\left[(8+1)^{\frac{3}{2}} - (-1+1)^{\frac{3}{2}}\right] = \frac{2}{3}\ [27 - 0] = 18$

$9\sqrt{c+1} = 18 \implies \sqrt{c+1} = 2 \implies c + 1 = 4 \implies c = 3 \in (-1, 8).$

$c = 3$ is the required value.

EXERCISES 2.4

(1) Evaluate the following integrals:

1. $\int 3x(x^2+1)^5 \; dx.$

2. $\int (x^3+3x)^{11}(x^2+1) \; dx.$

3. $\int x^3\sqrt{x^4+1} \; dx.$

4. $\int \frac{3x}{(1+x^2)^{\frac{2}{3}}} \; dx.$

5. $\int \frac{3x^5}{\sqrt[3]{x^6+1}} \; dx.$

6. $\int \cos(5x-2) \; dx.$

7. $\int x\sin(x^2+1) \; dx.$

8. $\int x^2\sec^2(x^3-2) \; dx.$

9. $\int \frac{\csc^2\left(\sqrt{x}\right)}{\sqrt{x}} \; dx.$

10. $\int \frac{\sec\left(\frac{1}{x}\right)\tan\left(\frac{1}{x}\right)}{x^2} \; dx.$

11. $\int \csc(3x+1)\cot(3x+1) \; dx.$

12. $\int \left(x^2+2\tan x\right)^5 \; \left(x+\sec^2 x\right) \; dx.$

13. $\int \frac{(2+\sin x)^4}{\sec x} \; dx.$

(2) Find the value of c that satisfies the mean value theorem for definite integrals for the following functions on the given intervals:

(i) $f(x)=2x-1$ on the interval $[0,2]$.

(ii) $f(x)=\sqrt[3]{x+1}$ on the interval $[-1,7]$.

(iii) $f(x)=x^2$ on the interval $[-2,0]$.

(iv) $f(x)=4x-x^2$ on the interval $[0,3]$.

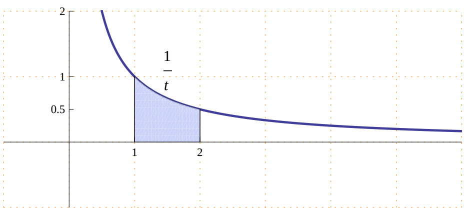
(v) $f(x)=\cos x$ on the interval $[0,2\pi]$.

Chapter 3

LOGARITHMIC AND EXPONENTIAL FUNCTIONS

3.1 Natural Logarithmic function

Definition: The natural logarithmic function is denoted by $\ln(x)$ and it is defined as $\ln(x) = \int_1^x \frac{1}{t}\, dt$ for every $x \in (0, \infty)$.

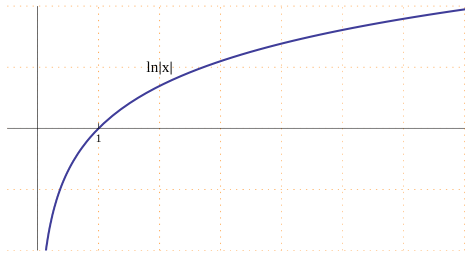


Remarks (1):

1. The domain of the natural logarithmic function is $(0, \infty)$.
2. $\ln(1) = 0$
3. $\ln(x) > 0$ for every $x > 1$.
4. $\ln(x) < 0$ for every $0 < x < 1$.
5. $\frac{d}{dx} \ln(x) = \frac{d}{dx} \int_1^x \frac{1}{t}\, dt = \frac{1}{x} > 0$ for every $x \in (0, \infty)$.

   Therefore, the natural logarithmic function is increasing on its domain.

Sketching the graph of the natural logarithmic function:



Remarks (2):

1. $\lim\limits_{x\to\infty} \ln(x) = \infty$
2. $\lim\limits_{x\to 0^+} \ln(x) = -\infty$

Some properties of the natural logarithmic function:
If $x, y > 0$ and $r \in \mathbb{Q}$ then:

1. $\ln(xy) = \ln(x) + \ln(y)$
2. $\ln\left(\dfrac{x}{y}\right) = \ln(x) - \ln(y)$
3. $\ln(x^r) = r\ln(x)$

Proof:
(1) $\ln(xy) - \ln x = \int_1^{xy} \frac{1}{t}\, dt - \int_1^{x} \frac{1}{t}\, dt = \int_x^{xy} \frac{1}{t}\, dt.$
Put $u = \dfrac{t}{x} \implies t = xu$
$dt = x\ du.$
When $t = x \implies u = 1.$
When $t = xy \implies u = y.$
$\ln(xy) - \ln x = \int_x^{xy} \frac{1}{t}\, dt = \int_1^{y} \frac{1}{xu} x\, du = \int_1^{y} \frac{1}{u}\, du = \ln y.$
$\ln(xy) = \ln x + \ln y.$
(2) $\ln(x) = \ln\left(y\ \dfrac{x}{y}\right) = \ln(y) + \ln\left(\dfrac{x}{y}\right).$
$\ln\left(\dfrac{x}{y}\right) = \ln(x) - \ln(y).$

The derivative of the natural logarithmic function:
$\dfrac{d}{dx}\ln(x) = \dfrac{1}{x}$
$\dfrac{d}{dx}\ln|f(x)| = \dfrac{f'(x)}{f(x)}$

Example (1): Find the derivative of the following:

1. $y = \sqrt{x}\ \ln|x|$.

   Solution: $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x}}\ \ln|x| + \sqrt{x}\ \dfrac{1}{x}$.

2. $f(x) = \ln|x^2 + 3x - 1|$.

   Solution: $f'(x) = \dfrac{2x+3}{x^2+3x-1}$.

3. $f(x) = \ln|\sin x + 5|$.

   Solution: $f'(x) = \dfrac{\cos x}{\sin x + 5}$.

4. $f(x) = \dfrac{(x^2+1)^5(x-8)^3}{(x^3-1)^{\frac{3}{2}}}$.

   Solution: $\ln|f(x)| = \ln\left|\dfrac{(x^2+1)^5(x-8)^3}{(x^3-1)^{\frac{3}{2}}}\right|$

   $= \ln\left|(x^2+1)^5(x-8)^3\right| - \ln\left|(x^3-1)^{\frac{3}{2}}\right|$

   $= \ln\left|(x^2+1)^5\right| + \ln\left|(x-8)^3\right| - \ln\left|(x^3-1)^{\frac{3}{2}}\right|$

   $= 5\ln|x^2+1| + 3\ln|x-8| - \dfrac{3}{2}\ln|x^3-1|$.

   Differentiating both sides:

   $\dfrac{f'(x)}{f(x)} = 5\ \dfrac{2x}{x^2+1} + 3\ \dfrac{1}{x-8} - \dfrac{3}{2}\ \dfrac{3x^2}{x^3-1}$

   $f'(x) = f(x)\ \left[\dfrac{10x}{x^2+1} + \dfrac{3}{x-8} - \dfrac{3}{2}\ \dfrac{3x^2}{x^3-1}\right]$

   $f'(x) = \left(\dfrac{(x^2+1)^5(x-8)^3}{(x^3-1)^{\frac{3}{2}}}\right)\ \left[\dfrac{10x}{x^2+1} + \dfrac{3}{x-8} - \dfrac{3}{2}\ \dfrac{3x^2}{x^3-1}\right]$

5. $f(x) = (\sin x)^x$.

   Solution: $\ln|f(x)| = \ln|(\sin x)^x| = x\ \ln|\sin x|$.

   Differentiating both sides:

   $\dfrac{f'(x)}{f(x)} = (1)\ \ln|\sin x| + x\ \dfrac{\cos x}{\sin x}$

   $f'(x) = f(x)\ \left[\ln|\sin x| + x\ \dfrac{\cos x}{\sin x}\right]$

   $f'(x) = (\sin x)^x\ \left[\ln|\sin x| + x\ \dfrac{\cos x}{\sin x}\right]$.

Important integral:

$\int \frac{1}{x}\, dx = \ln|x| + c.$

$\int \frac{f'(x)}{f(x)}\, dx = \ln|f(x)| + c.$

Example (2): Evaluate the following integrals:

1. $\int \frac{x^2+1}{x^3+3x+8}\, dx.$

   Solution:

   $$\int \frac{x^2+1}{x^3+3x+8}\, dx = \frac{1}{3}\int \frac{3x^2+3}{x^3+3x+8}\, dx = \frac{1}{3}\ln|x^3+3x+8| + c.$$

2. $\int \frac{x+\cos x}{x^2+2\sin x}\, dx.$

   Solution:

   $$\int \frac{x+\cos x}{x^2+2\sin x}\, dx = \frac{1}{2}\int \frac{2x+2\cos x}{x^2+2\sin x}\, dx = \frac{1}{2}\ln|x^2+2\sin x| + c.$$

3. $\int \frac{x^2+x+1}{x}\, dx.$

   Solution:

   $$\int \frac{x^2+x+1}{x}\, dx = \int \left(\frac{x^2}{x} + \frac{x}{x} + \frac{1}{x}\right)\, dx$$

   $$= \int \left(x + 1 + \frac{1}{x}\right)\, dx = \frac{x^2}{2} + x + \ln|x| + c.$$

4. $\int \frac{1}{x\sqrt{\ln x}}\, dx.$

   Solution:

   $$\int \frac{1}{x\sqrt{\ln x}\, dx}\, dx = \int \frac{1}{x\ (\ln x)^{\frac{1}{2}}}\, dx$$

   $$= \int (\ln x)^{-\frac{1}{2}}\, \frac{1}{x}\, dx = \frac{(\ln x)^{\frac{1}{2}}}{\frac{1}{2}} + c.$$

5. $\int \frac{1}{x\ln x}\, dx.$

   Solution:

   $$\int \frac{1}{x\ln x}\, dx = \int \frac{\frac{1}{x}}{\ln x}\, dx = \ln|\ln x| + c.$$

6. $\int \frac{x}{x+1}\, dx.$

Solution:

$$\int \frac{x}{x+1}\ dx = \int \frac{(x+1)-1}{x+1}\ dx = \int \left(\frac{x+1}{x+1} - \frac{1}{x+1}\right)\ dx$$

$$= \int \left(1 - \frac{1}{x+1}\right)\ dx = x - \ln|x+1| + c.$$

Integrals involving trigonometric functions:

1. $\displaystyle\int \tan x\ dx = \ln|\sec x| + c.$

   $\displaystyle\int \tan(f(x))\ f'(x)\ dx = \ln|\sec(f(x))| + c.$

2. $\displaystyle\int \cot x\ dx = \ln|\sin x| + c.$

   $\displaystyle\int \cot(f(x))\ f'(x)\ dx = \ln|\sin(f(x))| + c.$

3. $\displaystyle\int \sec x\ dx = \ln|\sec x + \tan x| + c.$

   $\displaystyle\int \sec(f(x))\ f'(x)\ dx = \ln|\sec(f(x)) + \tan(f(x))| + c.$

4. $\displaystyle\int \csc x\ dx = \ln|\csc x - \cot x| + c.$

   $\displaystyle\int \csc(f(x))\ f'(x)\ dx = \ln|\csc(f(x)) - \cot(f(x))| + c.$

Example (3): Evaluate the following integrals:

1. $\displaystyle\int \tan(3x)\ dx.$

   Solution:

   $$\int \tan(3x)\ dx = \frac{1}{3}\int \tan(3x)\ (3)\ dx = \frac{1}{3}\ \ln|\sec(3x)| + c.$$

2. $\displaystyle\int x\sec(x^2-3)\ dx.$

   Solution:

   $$\int x\sec(x^2-3)\ dx = \frac{1}{2}\int \sec(x^2-3)\ (2x)\ dx$$

   $$= \frac{1}{2}\ln\left|\sec(x^2-3) + \tan(x^2-3)\right| + c.$$

EXERCISES (3.1)

1. Find the derivatives of the following functions:

(1) $f(x) = \ln\left|x^4 + x^3 + 1\right|$ (2) $f(x) = \ln\left|x^2 + \cos 2x\right|$

(3) $f(x) = \sin x \ \ln|5x|$ (4) $f(x) = \tan(\ln|3x|)$

(5) $f(x) = [3x + \ln|\sin x|]^8$ (6) $f(x) = \ln\left|\dfrac{\sqrt{x^2+1}\ \sin^5 x}{(x^3+4)^2}\right|$

(7) $f(x) = \dfrac{(3x+1)^{\frac{3}{2}}\ (x^2-1)^{\frac{2}{3}}}{\sqrt[3]{x^2+2}}$ (8) $f(x) = (x^2+1)^x$

(9) $f(x) = x^{\sin x}$ (10) $f(x) = (\cos x)^{x^3-1}$

2. Evaluate the following integrals:

(1) $\displaystyle\int \frac{x^3+x+1}{x^4+2x^2+4x}\, dx$ (2) $\displaystyle\int \frac{x^2-\sin x}{x^3+3\cos x}\, dx$

(3) $\displaystyle\int \frac{x^2+3x-4}{x}\, dx$ (4) $\displaystyle\int \frac{\cos(\ln|x|)}{x}\, dx$

(5) $\displaystyle\int \frac{(8+\ln|x|)^5}{2x}\, dx$ (6) $\displaystyle\int \frac{1}{x\ \sqrt[3]{\ln|x|}}\, dx$

(7) $\displaystyle\int \frac{1}{2x\ \ln|x^3|}\, dx$ (8) $\displaystyle\int \frac{x}{x+2}\, dx$

(9) $\displaystyle\int x^2\ \tan\left(x^3\right)\, dx$ (10) $\displaystyle\int \frac{\sec(\sqrt{x})}{\sqrt{x}}\, dx$

3.2 The Natural Exponential function

Definition: The natural exponential function is the inverse function of the natural logarithmic function, and it is denoted by e^x, where e is an irrational number.

Remarks:

1. The domain of the natural exponential function is $\mathbb{R}$.

2. The range of the natural exponential function is $(0, \infty)$.

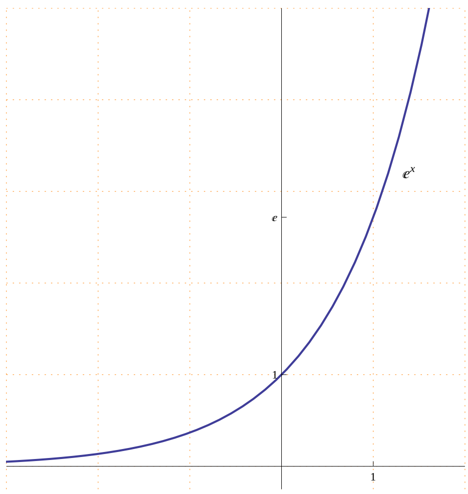
   Note that e^x is always positive on its domain.

3. $e^0 = 1$ and $e \approx 2.718128$.

4. $\ln(e^x) = x$ for every $x \in \mathbb{R}$, therefore, $\ln(e) = 1$.

   $e^{\ln(x)} = x$ for every $x \in (0, \infty)$.

5. Sketching the graph of the natural exponential function:



6. $\underset{x\to\infty}{lim}\ e^x = \infty$ and $\underset{x\to-\infty}{lim}\ e^x = 0$.

Some properties of the natural exponential function:
If $x, y \in \mathbb{R}$, then :

1. $e^x e^y = e^{x+y}$.

2. $\dfrac{e^x}{e^y} = e^{x-y}$.

3. $(e^x)^y = e^{xy}$.

Example(1): Find the value of x that satisfies the equation $e^{x-1} = 3$.
Solution: $e^{x-1} = 3 \Longrightarrow \ln\left(e^{x-1}\right) = \ln(3)$
$\Longrightarrow x - 1 = \ln(3) \Longrightarrow x = 1 + \ln(3)$.

Example(2): Find the value of x that satisfies the equation $\ln(x+2) = 5$.
Solution: $\ln(x+2) = 5 \Longrightarrow e^{\ln(x+2)} = e^5$
$\Longrightarrow x + 2 = e^5 \Longrightarrow x = e^5 - 2$.

The derivative of the natural exponential function:

$$\frac{d}{dx}\, e^x = e^x.$$

$$\frac{d}{dx}\, e^{f(x)} = e^{f(x)} f'(x).$$

Example(3): Find the derivatives of the following:

1. $f(x) = e^{x^2+x}$.

   Solution: $f'(x) = e^{x^2+x}(2x+1)$.

2. $f(x) = e^{\sin x} + \dfrac{1}{e^x}$.

   Solution: $f(x) = e^{\sin x} + e^{-x}$

   $f'(x) = e^{\sin x}\cos x + e^{-x}(-1)$.

3. $f(x) = \left(e^{5x} + x^2\right)^3$.

   Solution: $f'(x) = 3\ \left(e^{5x} + x^2\right)^2 \left(e^{5x}(5) + 2x\right)$.

4. $f(x) = \ln\left|e^{\tan x} + 4x^3\right|$.

   Solution: $f'(x) = \dfrac{e^{\tan x}\sec^2 x + 12x^2}{e^{\tan x} + 4x^3}$.

The integral of the natural exponential function:

$$\int e^x\ dx = e^x + c.$$

$$\int e^{f(x)} f'(x)\ dx = e^{f(x)} + c.$$

Example(4): Evaluate the following integrals:

1. $\displaystyle\int e^{7x+1}\ dx$.

   Solution: $\displaystyle\int e^{7x+1}\ dx = \frac{1}{7}\int e^{7x+1}(7)\ dx = \frac{1}{7}\ e^{7x+1} + c$.

2. $\displaystyle\int x\ e^{x^2-3}\ dx.$

Solution: $\displaystyle\int x\ e^{x^2-3}\ dx = \frac{1}{2}\int e^{x^2-3}(2x)\ dx = \frac{1}{2}\ e^{x^2-3} + c.$

3. $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\ dx.$

Solution: $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\ dx = \int e^{\sqrt{x}}\frac{1}{\sqrt{x}}\ dx$

$\displaystyle = 2\int e^{\sqrt{x}}\frac{1}{2\sqrt{x}}\ dx = 2\ e^{\sqrt{x}} + c.$

4. $\displaystyle\int \frac{e^{\sin x}}{\sec x}\ dx.$

Solution: $\displaystyle\int \frac{e^{\sin x}}{\sec x}\ dx = \int e^{\sin x}\frac{1}{\sec x}\ dx$

$\displaystyle = \int e^{\sin x}\cos x\ dx = e^{\sin x} + c.$

5. $\displaystyle\int \frac{e^{\tan x}}{\cos^2 x}\ dx.$

Solution: $\displaystyle\int \frac{e^{\tan x}}{\cos^2 x}\ dx = \int e^{\tan x}\frac{1}{\cos^2 x}\ dx$

$\displaystyle = \int e^{\tan x}\sec^2 x\ dx = e^{\tan x} + c.$

6. $\displaystyle\int_0^{\ln(5)} e^x\ dx.$

Solution: $\displaystyle\int_0^{\ln(5)} e^x\ dx = [e^x]_0^{\ln(5)} = e^{\ln(5)} - e^0 = 5 - 1 = 4.$

7. $\displaystyle\int e^x\cos\left(e^x\right)\ dx.$

Solution: $\displaystyle\int e^x\cos\left(e^x\right)\ dx = \int \cos\left(e^x\right)\ e^x\ dx = \sin\left(e^x\right) + c.$

8. $\displaystyle\int \frac{e^{5x}}{e^{5x}+4}\ dx.$

Solution: $\displaystyle\int \frac{e^{5x}}{e^{5x}+4}\ dx = \frac{1}{5}\int \frac{e^{5x}\ (5)}{e^{5x}+4}\ dx = \frac{1}{5}\ \ln\left|e^{5x}+4\right| + c.$

9. $\displaystyle\int \frac{e^{5x}}{\left(e^{5x}+4\right)^3}\ dx.$

Solution: $\displaystyle\int \frac{e^{5x}}{\left(e^{5x}+4\right)^3}\ dx = \int \left(e^{5x}+4\right)^{-3}\ e^{5x}\ dx$

$$= \frac{1}{5} \int \left(e^{5x} + 4\right)^{-3} \; e^{5x} \; (5) \; dx = \frac{1}{5} \; \frac{\left(e^{5x} + 4\right)^{-2}}{-2} + c.$$

10. $\displaystyle\int \frac{e^{5 \ln x}}{x^3} \; dx.$

Solution:

$$\int \frac{e^{5 \ln x}}{x^3} \; dx = \int \frac{e^{\ln x^5}}{x^3} \; dx = \int \frac{x^5}{x^3} \; dx = \int x^2 \; dx = \frac{x^3}{3} + c$$

EXERCISES (3.2)

1. Find the value of x that satisfies the following:
   (1) $e^{2x-1}=5$ (2) $e^{x^2-4}=1$
   (3) $\ln|x-1|=7$ (4) $\ln\left|x^3-1\right|=0$

2. Find the derivatives of t he following:
   (1) $f(x)=e^{x^3-x+2}$ (2) $f(x)=\dfrac{1}{e^{2x}}+e^{\cos 2x}$
   (3) $f(x)=\left(\ln|3x|+e^{4x+2}\right)^8$ (4) $f(x)=\ln\left|e^{2\tan x}+\sec 3x\right|$
   (5) $f(x)=e^{x^2}\ln\left|x^3-1\right|$ (6) $f(x)=\tan\left(e^{x^2+1}\right)$

3. Evaluate the following integrals:
   (1) $\displaystyle\int 5e^{8x+1}\ dx$ (2) $\displaystyle\int 7x^2\ e^{x^3-4}\ dx$
   (3) $\displaystyle\int \frac{e^{\frac{1}{x}}}{x^2}\ dx$ (4) $\displaystyle\int \frac{e^{\sqrt[3]{x}}}{x^{\frac{2}{3}}}\ dx$
   (5) $\displaystyle\int \frac{e^{\cos 3x}}{\csc 3x}\ dx$ (6) $\displaystyle\int \frac{e^{\cot 2x}}{\sin^2 2x}\ dx$
   (7) $\displaystyle\int_{\ln 2}^{\ln 5} 3e^x\ dx$ (8) $\displaystyle\int e^{7x}\ \csc^2\left(2+e^{7x}\right)\ dx$
   (9) $\displaystyle\int \frac{e^{3x}}{e^{3x}+2}\ dx$ (10) $\displaystyle\int \frac{e^{3x}}{(e^{3x}+2)^5}\ dx$
   (11) $\displaystyle\int e^x\left(4-\cos\left(e^x\right)\right)\ dx$ (12) $\displaystyle\int e^{7\ln|x+1|}\ dx$
   (13) $\displaystyle\int \frac{e^{2\ln|x|}}{x^3}\ dx$ (14) $\displaystyle\int e^{3x^2+\ln|x|}\ dx$

3.3 The general logarithmic and exponential functions

The general exponential function:
If $a > 0$ is a real number, then the general exponent function of base a is denoted by a^x, and it is defined as $a^x = e^{x \ln a}$.

Remarks (1):

1. The domain of the general exponential function is $\mathbb{R}$.
2. The range of the general exponential function is $(0, \infty)$.
3. $\lim\limits_{x \to \infty} a^x = \infty$, where $a > 1$.
4. $\lim\limits_{x \to -\infty} a^x = 0$, where $a > 1$.

Some properties of the general exponential function:
If $x, y \in \mathbb{R}$, then:

1. $a^x a^y = a^{x+y}$.
2. $\dfrac{a^x}{a^y} = a^{x-y}$.
3. $(a^x)^y = a^{xy}$.

The derivative of the general exponential function:

$$\frac{d}{dx}\, a^x = a^x \ln a.$$

$$\frac{d}{dx}\, a^{f(x)} = a^{f(x)} f'(x)\ \ln a.$$

Example (1): Find the derivatives of the following:

1. $f(x) = 3^{x^2+x}$.

   Solution: $f'(x) = 3^{x^2+x}(2x+1)\ \ln(3)$.

2. $f(x) = 5^{\sqrt{x}}$.

   Solution: $f'(x) = 5^{\sqrt{x}} \left(\dfrac{1}{2\sqrt{x}}\right)\ \ln(5)$.

3. $f(x) = \pi^{\frac{1}{x}}$.

   Solution: $f'(x) = \pi^{\frac{1}{x}} \left(\dfrac{-1}{x^2}\right)\ \ln(\pi)$.

4. $f(x) = 3^{\tan(x^2+1)}$.

Solution: $f'(x) = 3^{\tan(x^2+1)}\ \sec^2(x^2+1)\ (2x)\ \ln 3.$

5. $f(x) = \left(4^{x^2} + 7^{3x+1}\right)^6.$

   Solution: $f'(x) = 6\left(4^{x^2} + 7^{3x+1}\right)^5 \left(4^{x^2}(2x)\ \ln(4) + 7^{3x+1}(3)\ln(7)\right).$

6. $f(x) = \ln\left|5^{x^2} + x^3\right|.$

   Solution: $f'(x) = \dfrac{5^{x^2}(2x)\ln(5) + 3x^2}{5^{x^2} + x^3}.$

The integration of the general exponential function:

$$\int a^x\ dx = \frac{a^x}{\ln a} + c.$$

$$\int a^{f(x)}\ f'(x)\ dx = \frac{a^{f(x)}}{\ln a} + c.$$

Example (2): Evaluate the following integrals:

1. $\displaystyle\int x^2 6^{x^3-2}\ dx.$

   Solution: $\displaystyle\int x^2 6^{x^3-2}\ dx = \frac{1}{3}\int 6^{x^3-2}(3x^2)\ dx = \frac{1}{3}\,\frac{6^{x^3-2}}{\ln 6} + c.$

2. $\displaystyle\int \frac{3^{\cot x}}{\sin^2 x}\ dx.$

   Solution: $\displaystyle\int \frac{3^{\cot x}}{\sin^2 x}\ dx = \int 3^{\cot x}\frac{1}{\sin^2 x}\ dx = \int 3^{\cot x}\csc^2 x\ dx$

   $\displaystyle = -\int 3^{\cot x}(-csc^2 x)\ dx = -\frac{3^{\cot x}}{\ln 3} + c.$

3. $\displaystyle\int \left(5^x + \frac{1}{2^x}\right)\ dx.$

   Solution: $\displaystyle\int \left(5^x + \frac{1}{2^x}\right)\ dx = \int \left(5^x + 2^{-x}\right)\ dx = \int 5^x\ dx + \int 2^{-x}\ dx$

   $\displaystyle = \int 5^x\ dx + \frac{1}{-1}\int 2^{-x}(-1)\ dx = \frac{5^x}{\ln 5} - \frac{2^{-x}}{\ln 2} + c.$

4. $\displaystyle\int (7^x + 4)^{10}\ 7^x\ dx.$

   Solution: $\displaystyle\int (7^x + 4)^{10}\ 7^x\ dx = \frac{1}{\ln 7}\int (7^x + 4)^{10}\,(7^x \ln 7)\ dx$

$$= \frac{1}{\ln 7} \frac{(7^x + 4)^{11}}{11} + c.$$

5. $\displaystyle\int \frac{2^x}{2^x + 1}\ dx.$

   Solution: $\displaystyle\int \frac{2^x}{2^x + 1}\ dx = \frac{1}{\ln 2} \int \frac{2^x \ln 2}{2^x + 1}\ dx = \frac{1}{\ln 2} \ln |2^x + 1| + c.$

6. $\displaystyle\int 3^x \left(1 + \cos(3^x)\right)\ dx.$

   Solution: $\displaystyle\int 3^x \left(1 + \cos(3^x)\right)\ dx = \int \left[3^x + \cos(3^x)\, 3^x\right]\ dx$

   $\displaystyle= \int 3^x\ dx + \int \cos(3^x)\, 3^x\ dx = \int 3^x\ dx + \frac{1}{\ln 3} \int \cos(3^x)\,(3^x \ln 3)\ dx$

   $\displaystyle= \frac{3^x}{\ln 3} + \frac{1}{\ln 3}\ \sin(3^x) + c.$

7. $\displaystyle\int 4^x\ 5^{4^x} dx.$

   Solution: $\displaystyle\int 4^x\ 5^{4^x} dx = \frac{1}{\ln 4} \int 5^{4^x}\ (4^x \ln 4)\ dx = \frac{1}{\ln 4} \frac{5^{4^x}}{\ln 5} + c.$

The general logarithmic function:
The general logarithmic function of base a is denoted by $\log_a x$, and it is the inverse function of the exponential function a^x, where $a > 0$ is real number.

Remarks (2):

1. $\log x = \log_{10} x$ and $\ln x = \log_e x$.
2. $\log_a x = \dfrac{\ln x}{\ln a}$.
3. $\log_a a = 1$.
4. $\log_a x = y \iff x = a^y$.

Some properties of the general logarithmic function:
If $x, y > 0$ and $r \in \mathbb{Q}$, then:

1. $\log_a(xy) = \log_a x + \log_a y$.
2. $\log_a \left(\dfrac{x}{y}\right) = \log_a x - \log_a y$.
3. $\log_a x^r = r \log_a x$.

Example (3): Find the value of x that satisfies the equation $\log(x+1) = 2$.
Solution: $\log(x+1) = 2 \implies 10^{\log(x+1)} = 10^2$
$\implies x + 1 = 100 \implies x = 100 - 1 = 99.$

Example (4): Find the value of x that satisfies the equation $3^{2x-1} = 7$.
Solution: $3^{2x-1} = 7 \implies \log_3 3^{2x-1} = \log_3 7$
$\implies 2x - 1 = \log_3 7 \implies 2x = 1 + \log_3 7 \implies x = \dfrac{1 + \log_3 7}{2}.$

The derivative of the general logarithmic function:

$$\frac{d}{dx} \log_a |x| = \frac{1}{\ln a} \frac{1}{x} = \frac{1}{x \ln a}.$$

$$\frac{d}{dx} \log_a |f(x)| = \frac{1}{\ln a} \frac{f'(x)}{f(x)} = \frac{f'(x)}{f(x) \ \ln a}.$$

Example (5): Find the derivatives of the following:

1. $f(x) = \log_5 \left|x^3 + \sin 5x\right|.$

   Solution: $f'(x) = \dfrac{1}{\ln 5} \dfrac{3x^2 + \cos 5x \ (5)}{x^3 + \sin 5x}.$

2. $f(x) = \left[\log_3 |x^2 - 1| + e^{3x}\right]^8.$

   Solution: $f'(x) = 8 \left[\log_3 |x^2 - 1| + e^{3x}\right]^7 \left(\dfrac{1}{\ln 3} \dfrac{2x}{x^2 - 1} + e^{3x}(3)\right).$

3. $f(x) = \sin\left(\log_7 |2x + 3|\right).$

   Solution: $f'(x) = \cos\left(\log_7 |2x + 3|\right) \ \left(\dfrac{1}{\ln 7} \dfrac{2}{2x + 3}\right).$

4. $f(x) = \sin(x^2) \ \log_7 |2x + 3|.$

   Solution: $f'(x) = \cos(x^2) \ (2x) \ \log_7 |2x + 3| + \sin(x^2) \ \dfrac{1}{\ln 7} \dfrac{2}{2x + 3}.$

EXERCISES (3.3)

1. Find the value of x that satisfies the following:
   (1) $5^{3x+4} = 30$ (2) $2^{x^2-5x+6} = 1$
   (3) $\log_5(x+4) = 2$ (4) $\log(2x-2) = 0$

2. Find the derivatives of the following:
   (1) $f(x) = 5^{\sin 2x}$ (2) $f(x) = \tan\left(7^{x^2}\right)$
   (3) $f(x) = \ln\left|3^{\cos x} + e^{5x}\right|$ (4) $f(x) = \left(\sec x^2 + \pi^{2x-1}\right)^5$
   (5) $f(x) = \log_5\left|x^4 - \csc 6x\right|$ (6) $f(x) = \sin\left(\log_3\left|x^2-1\right|\right)$
   (7) $f(x) = \sqrt{x}\ \log_6|2+\cot x|$ (8) $f(x) = \left(\log_2|3x-2| + \sqrt[4]{x}\right)^8$

3. Evaluate the following integrals:
   (1) $\displaystyle\int \frac{5^{\sqrt{x}}}{\sqrt{x}}\ dx$ (2) $\displaystyle\int \frac{\pi^{\sin 3x}}{\sec 3x}\ dx$
   (3) $\displaystyle\int x3^{x^2+1}\cos\left(3^{x^2+1}\right)\ dx$ (4) $\displaystyle\int \frac{\sin x\ 7^{\cos x}}{7^{\cos x}+3}\ dx$
   (5) $\displaystyle\int \frac{6^{3x}}{\left(6^{3x}+2\right)^9}\ dx$ (6) $\displaystyle\int 2^x\ 5^{2^x}\ dx$
   (7) $\displaystyle\int x\ 4^{x^2}\left(1+\sec\left(4^{x^2}\right)\right)\ dx$ (8) $\displaystyle\int \frac{\left(4+\log_2 x\right)^7}{3x}\ dx$
   (9) $\displaystyle\int \frac{\cos\left(\log 3x\right)}{2x}\ dx$ (10) $\displaystyle\int \frac{1}{x\ \log_3 x}\ dx$

3.4 Inverse trigonometric functions

The derivatives of the inverse trigonometric functions:

1. $\dfrac{d}{dx}\sin^{-1}x = \dfrac{1}{\sqrt{1-x^2}} \quad , \quad |x| < 1.$

   $\dfrac{d}{dx}\sin^{-1}(f(x)) = \dfrac{f'(x)}{\sqrt{1-[f(x)]^2}} \quad , \quad |f(x)| < 1.$

2. $\dfrac{d}{dx}\cos^{-1}x = -\dfrac{1}{\sqrt{1-x^2}} \quad , \quad |x| < 1.$

   $\dfrac{d}{dx}\cos^{-1}(f(x)) = -\dfrac{f'(x)}{\sqrt{1-[f(x)]^2}} \quad , \quad |f(x)| < 1.$

3. $\dfrac{d}{dx}\tan^{-1}x = \dfrac{1}{1+x^2}.$

   $\dfrac{d}{dx}\tan^{-1}(f(x)) = \dfrac{f'(x)}{1+[f(x)]^2}.$

4. $\dfrac{d}{dx}\cot^{-1}x = -\dfrac{1}{1+x^2}.$

   $\dfrac{d}{dx}\cot^{-1}(f(x)) = -\dfrac{f'(x)}{1+[f(x)]^2}.$

5. $\dfrac{d}{dx}\sec^{-1}x = \dfrac{1}{x\sqrt{x^2-1}} \quad , \quad |x| > 1.$

   $\dfrac{d}{dx}\sec^{-1}(f(x)) = \dfrac{f'(x)}{f(x)\,\sqrt{[f(x)]^2-1}} \quad , \quad |f(x)| > 1.$

6. $\dfrac{d}{dx}\csc^{-1}x = -\dfrac{1}{x\sqrt{x^2-1}} \quad , \quad |x| > 1.$

   $\dfrac{d}{dx}\csc^{-1}(f(x)) = -\dfrac{f'(x)}{f(x)\,\sqrt{[f(x)]^2-1}} \quad , \quad |f(x)| > 1.$

Example (1): Find the derivatives of the following:

1. $f(x) = \sin^{-1}\left(\sqrt{x}\right).$

   Solution: $f'(x) = \dfrac{1}{\sqrt{1-(\sqrt{x})^2}}\,\dfrac{1}{2\sqrt{x}} = \dfrac{1}{2\sqrt{x}\sqrt{1-x}}.$

2. $f(x) = \tan^{-1}(2x^2+3).$

   Solution: $f'(x) = \dfrac{1}{1+(2x^2+3)^2}\,(4x) = \dfrac{4x}{1+(2x^2+3)^2}.$

3. $f(x) = \sec^{-1}(3 + \sin 3x)$.

   Solution: $f'(x) = \dfrac{1}{(3 + \sin 3x)\ \sqrt{(3 + \sin 3x)^2 - 1}}\ (\cos 3x)\ (3)$.

4. $f(x) = e^{\cos^{-1}(4x+1)}$.

   Solution: $f'(x) = e^{\cos^{-1}(4x+1)}\ \dfrac{-1}{\sqrt{1 - (4x+1)^2}}\ (4)$.

5. $f(x) = \ln \left| e^{5x} + \sec^{-1}(3x) \right|$.

   Solution: $f'(x) = \dfrac{e^{5x}\ (5) + \frac{1}{3x\sqrt{(3x)^2 - 1}}\ (3)}{e^{5x} + \sec^{-1}(3x)}$.

6. $f(x) = \left(5^x + \tan^{-1}(2x+1)\right)^5$.

   Solution: $f'(x) = 5\left(5^x + \tan^{-1}(2x+1)\right)^4\ \left(5^x \ln 5 + \dfrac{1}{1 + (2x+1)^2}\ (2)\right)$

Integrals of specific quadratic forms:

1. $\displaystyle\int \frac{1}{\sqrt{a^2 - x^2}}\ dx = \sin^{-1}\left(\frac{x}{a}\right) + c \quad , \quad |x| < a.$

   $\displaystyle\int \frac{f'(x)}{\sqrt{a^2 - [f(x)]^2}}\ dx = \sin^{-1}\left(\frac{f(x)}{a}\right) + c \quad , \quad |f(x)| < a.$

2. $\displaystyle\int \frac{1}{a^2 + x^2}\ dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c.$

   $\displaystyle\int \frac{f'(x)}{a^2 + [f(x)]^2}\ dx = \frac{1}{a}\tan^{-1}\left(\frac{f(x)}{a}\right) + c.$

3. $\displaystyle\int \frac{1}{x\sqrt{x^2 - a^2}}\ dx = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + c \quad , \quad |x| > a.$

   $\displaystyle\int \frac{f'(x)}{f(x)\ \sqrt{[f(x)]^2 - a^2}}\ dx = \frac{1}{a}\sec^{-1}\left(\frac{f(x)}{a}\right) + c \quad , \quad |f(x)| > a.$

Example (2): Evaluate the following integrals:

1. $\displaystyle\int \frac{x^2}{9 + x^6}\ dx.$

   Solution: $\displaystyle\int \frac{x^2}{9 + x^6}\ dx = \int \frac{x^2}{3^2 + (x^3)^2}\ dx$

   $\displaystyle = \frac{1}{3}\int \frac{3x^2}{3^2 + (x^3)^2}\ dx = \frac{1}{3}\ \frac{1}{3}\ \tan^{-1}\left(\frac{x^3}{3}\right) + c.$

2. $\displaystyle\int \frac{x}{\sqrt{16-x^4}}\ dx.$

Solution: $\displaystyle\int \frac{x}{\sqrt{16-x^4}}\ dx = \int \frac{x}{\sqrt{4^2-(x^2)^2}}\ dx$

$\displaystyle= \frac{1}{2}\int \frac{2x}{\sqrt{4^2-(x^2)^2}}\ dx = \frac{1}{2}\ \sin^{-1}\left(\frac{x^2}{4}\right) + c.$

3. $\displaystyle\int \frac{x}{\sqrt{16-x^2}}\ dx.$

Solution: $\displaystyle\int \frac{x}{\sqrt{16-x^2}}\ dx = \int \left(16-x^2\right)^{-\frac{1}{2}}\ x\ dx$

$\displaystyle= \frac{1}{-2}\int \left(16-x^2\right)^{-\frac{1}{2}}\ (-2x)\ dx = \frac{1}{-2}\ \frac{\left(16-x^2\right)^{\frac{1}{2}}}{\frac{1}{2}} + c.$

4. $\displaystyle\int \frac{1}{x\sqrt{1-(\ln x)^2}}\ dx.$

Solution: $\displaystyle\int \frac{1}{x\sqrt{1-(\ln x)^2}}\ dx = \int \frac{\left(\frac{1}{x}\right)}{\sqrt{1^2-(\ln x)^2}}\ dx$

$\displaystyle= \sin^{-1}\left(\frac{\ln x}{1}\right) + c = \sin^{-1}(\ln x) + c.$

5. $\displaystyle\int \frac{e^{2x}}{25+e^{4x}}\ dx.$

Solution: $\displaystyle\int \frac{e^{2x}}{25+e^{4x}}\ dx = \int \frac{e^{2x}}{5^2+\left(e^{2x}\right)^2}\ dx$

$\displaystyle= \frac{1}{2}\int \frac{e^{2x}\ (2)}{5^2+\left(e^{2x}\right)^2}\ dx = \frac{1}{2}\ \frac{1}{5}\ \tan^{-1}\left(\frac{e^{2x}}{5}\right) + c.$

6. $\displaystyle\int \frac{\sin x}{\sqrt{25-\cos^2 x}}\ dx.$

Solution: $\displaystyle\int \frac{\sin x}{\sqrt{25-\cos^2 x}}\ dx = \int \frac{\sin x}{\sqrt{5^2-(\cos x)^2}}\ dx$

$\displaystyle= \frac{1}{-1}\int \frac{-\sin x}{\sqrt{5^2-(\cos x)^2}}\ dx = -\sin^{-1}\left(\frac{\cos x}{5}\right) + c.$

7. $\displaystyle\int \frac{1}{x^2+6x+25}\ dx.$

Solution: $\displaystyle\int \frac{1}{x^2+6x+25}\ dx = \int \frac{1}{(x^2+6x+9)+16}\ dx$

$\displaystyle= \int \frac{1}{(x+3)^2+4^2}\ dx = \frac{1}{4}\ \tan^{-1}\left(\frac{x+3}{4}\right) + c.$

8. $\displaystyle\int \frac{1}{(x-1)\sqrt{x^2-2x-3}}\ dx.$

Solution: $\displaystyle\int \frac{1}{(x-1)\sqrt{x^2-2x-3}}\ dx = \int \frac{1}{(x-1)\sqrt{(x^2-2x+1)-4}}\ dx$

$\displaystyle= \int \frac{1}{(x-1)\sqrt{(x-1)^2-2^2}}\ dx = \frac{1}{2}\ \sec^{-1}\left(\frac{x-1}{2}\right) + c.$

9. $\displaystyle\int \frac{x+1}{x^2+1}\ dx.$

Solution: $\displaystyle\int \frac{x+1}{x^2+1}\ dx = \int \frac{x}{x^2+1}\ dx + \int \frac{1}{x^2+1}\ dx$

$\displaystyle= \frac{1}{2}\int \frac{2x}{x^2+1}\ dx + \int \frac{1}{x^2+1}\ dx = \frac{1}{2}\ \ln(x^2+1) + \tan^{-1} x + c.$

10. $\displaystyle\int \frac{x+2}{\sqrt{16-x^2}}\ dx.$

Solution: $\displaystyle\int \frac{x+2}{\sqrt{16-x^2}}\ dx = \int \frac{x}{\sqrt{16-x^2}}\ dx + \int \frac{2}{\sqrt{16-x^2}}\ dx$

$\displaystyle= \frac{1}{-2}\int \left(16-x^2\right)^{-\frac{1}{2}}(-2x)\ dx + 2\int \frac{1}{\sqrt{4^2-x^2}}\ dx$

$\displaystyle= \frac{1}{-2}\ \frac{\left(16-x^2\right)^{\frac{1}{2}}}{\frac{1}{2}} + 2\sin^{-1}\left(\frac{x}{4}\right) + c.$

11. $\displaystyle\int \frac{x+\tan^{-1} x}{x^2+1}\ dx.$

Solution: $\displaystyle\int \frac{x+\tan^{-1} x}{x^2+1}\ dx = \int \frac{x}{x^2+1}\ dx + \int \frac{\tan^{-1} x}{x^2+1}\ dx$

$\displaystyle= \frac{1}{2}\int \frac{2x}{x^2+1}\ dx + \int \left(\tan^{-1} x\right)^1\ \frac{1}{x^2+1}\ dx$

$\displaystyle= \frac{1}{2}\ \ln(x^2+1) + \frac{\left(\tan^{-1} x\right)^2}{2} + c.$

12. $\displaystyle\int \frac{1}{\sqrt{e^{2x}-36}}\ dx.$

Solution: $\displaystyle\int \frac{1}{\sqrt{e^{2x}-36}}\ dx = \int \frac{1}{\sqrt{\left(e^x\right)^2-6^2}}\ dx$

$\displaystyle= \int \frac{e^x}{e^x\sqrt{\left(e^x\right)^2-6^2}}\ dx = \frac{1}{6}\ \sec^{-1}\left(\frac{e^x}{6}\right) + c.$

EXERCISES (3.4)

1. Find the derivatives of the following:

(1) $f(x) = \sqrt{x}\ \sin^{-1}(5x)$ (2) $f(x) = \ln|3x-1|\ \cos^{-1}\left(\sqrt{x}\right)$

(3) $f(x) = \ln\left|x^3 + \tan^{-1}(2x)\right|$ (4) $f(x) = \cot^{-1}\left(e^{3x}\right) + 5^{\sqrt[3]{x}}$

(5) $f(x) = e^{\sec^{-1}(4x)}$ (6) $f(x) = \left(\csc^{-1}\left(\frac{1}{x}\right) + x^4\right)^7$

2. Evaluate the following integrals:

(1) $\displaystyle\int \frac{x}{\sqrt{25-9x^4}}\,dx$ (2) $\displaystyle\int \frac{5x^2}{x^6+36}\,dx$

(3) $\displaystyle\int \frac{1}{x\,\sqrt{x^8-16}}\,dx$ (4) $\displaystyle\int \frac{3x}{\sqrt{4-(x^2+1)^2}}\,dx$

(5) $\displaystyle\int \frac{\cos x}{1+\sin^2 x}\,dx$ (6) $\displaystyle\int \frac{7}{25+(x-1)^2}\,dx$

(7) $\displaystyle\int \frac{1}{\sqrt{e^{4x}-16}}\,dx$ (8) $\displaystyle\int \frac{1}{x\,\sqrt{x^3-36}}\,dx$

(9) $\displaystyle\int \frac{2}{\sqrt{-x^2+2x+8}}\,dx$ (10) $\displaystyle\int \frac{3}{x^2+4x+13}\,dx$

(11) $\displaystyle\int \frac{5}{(x-2)\,\sqrt{x^2-4x}}\,dx$ (12) $\displaystyle\int \frac{\sin^{-1} x}{\sqrt{1-x^2}}\,dx$

(13) $\displaystyle\int \frac{x-4}{\sqrt{9-x^2}}\,dx$ (14) $\displaystyle\int \frac{x+5}{x^2+36}\,dx$

(15) $\displaystyle\int \frac{\left(1+\tan^{-1} x\right)^2}{1+x^2}\,dx$ (16) $\displaystyle\int \frac{3-\sin^{-1} x}{\sqrt{1-x^2}}\,dx$

Chapter 4

HYPERBOLIC FUNCTIONS AND THEIR INVERSES

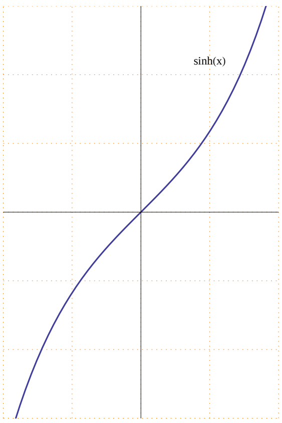
4.1 Hyperbolic functions

1. **Hyperbolic sine function:**

   It is denoted by $\sinh x$, and it is defined as $\sinh x = \dfrac{e^x - e^{-x}}{2}$.

   Remarks(1):

   - The domain of the hyperbolic sine function is $\mathbb{R}$, and its range is $\mathbb{R}$.
   - it is an odd function (symmetric with respect to the origin), and $\sinh(0) = 0$.
   - Sketching the graph of the hyperbolic sine function:

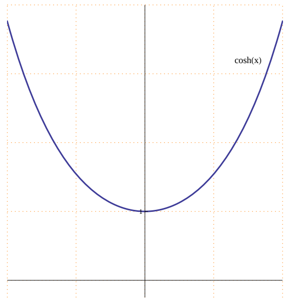


2. **Hyperbolic cosine function:**

It is denoted by $\cosh x$, and it is defined as $\cosh x = \dfrac{e^x + e^{-x}}{2}$.

Remarks(2):

- The domain of the hyperbolic cosine function is $\mathbb{R}$, and its range is $[1, \infty)$.
- it is an even function (symmetric with respect to the y-axis), and $\cosh(0) = 1$.
- Sketching the graph of the hyperbolic cosine function:



The rest of the hyperbolic functions are defined as:

3. **Hyperbolic tangent function:**

$\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$, for every $x \in \mathbb{R}$.

4. **Hyperbolic cotangent function:**

$\coth x = \dfrac{1}{\tanh x} = \dfrac{\cosh x}{\sinh x} = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$, for every $x \in \mathbb{R} - \{0\}$.

5. **Hyperbolic secant function:**

$sechx = \dfrac{1}{\cosh x} = \dfrac{2}{e^x + e^{-x}}$, for every $x \in \mathbb{R}$.

6. **Hyperbolic cosecant function:**

$cschx = \dfrac{1}{\sinh x} = \dfrac{2}{e^x - e^{-x}}$, fo revery $x \in \mathbb{R} - \{0\}$.

Important Identities:

1. $\cosh^2 x - \sinh^2 x = 1$, for every $x \in \mathbb{R}$.
2. $1 - \tanh^2 x = sech^2 x$, fro every $x \in \mathbb{R}$.
3. $\coth^2 x - 1 = csch^2 x$, for every $x \in \mathbb{R} - \{0\}$.

The derivatives of the hyperbolic functions:

1. $\dfrac{d}{dx} \sinh x = \cosh x.$

   $\dfrac{d}{dx} \sinh(f(x)) = \cosh(f(x))\ f'(x).$

2. $\dfrac{d}{dx} \cosh x = \sinh x.$

   $\dfrac{d}{dx} \cosh(f(x)) = \sinh(f(x))\ f'(x).$

3. $\dfrac{d}{dx} \tanh x = sech^2 x.$

   $\dfrac{d}{dx} \tanh(f(x)) = sech^2(f(x))\ f'(x).$

4. $\dfrac{d}{dx} \coth x = -csch^2 x.$

   $\dfrac{d}{dx} \coth(f(x)) = -csch^2(f(x))\ f'(x).$

5. $\dfrac{d}{dx} sechx = -sechx\ \tanh x.$

   $\dfrac{d}{dx} sech(f(x)) = -sech(f(x))\ \tanh(f(x))\ f'(x).$

6. $\dfrac{d}{dx} cschx = -cschx\ \coth x.$

   $\dfrac{d}{dx} csch(f(x)) = -csch(f(x))\ \coth(f(x))\ f'(x).$

Proof:

(1) $\dfrac{d}{dx} \sinh x = \dfrac{d}{dx}\left(\dfrac{e^x - e^{-x}}{2}\right) = \dfrac{e^x - e^{-x}(-1)}{2} = \dfrac{e^x + e^{-x}}{2} = \cosh x.$

(2) $\dfrac{d}{dx} \cosh x = \dfrac{d}{dx}\left(\dfrac{e^x + e^{-x}}{2}\right) = \dfrac{e^x + e^{-x}(-1)}{2} = \dfrac{e^x - e^{-x}}{2} = \sinh x.$

Example (1): Find the derivatives of the following:

1. $f(x) = sech\left(1 + \sqrt{x}\right).$

   Solution: $f'(x) = -sech\left(1 + \sqrt{x}\right)\ \tanh\left(1 + \sqrt{x}\right)\ \left(\dfrac{1}{2\sqrt{x}}\right)$

2. $f(x) = e^{\sinh 4x}$.

   Solution: $f'(x) = e^{\sinh 4x}\ \cosh 4x\ (4)$.

3. $f(x) = \ln\left|\cosh(1 - x^2)\right|$.

   Solution: $f'(x) = \dfrac{\sinh(1 - x^2)\ (-2x)}{\cosh(1 - x^2)}$.

4. $f(x) = \tanh(5^x)$.

   Solution: $f'(x) = sech^2(5^x)\ 5^x\ \ln 5$.

5. $f(x) = \left(\coth(3x) + e^{6x}\right)^4$.

   Solution: $f'(x) = 4\left(\coth(3x) + e^{6x}\right)^3\left(-csch^2(3x)\ (3) + e^{6x}\ (6)\right)$.

6. $f(x) = x^{cschx}$.

   Solution: $\ln|f(x)| = \ln\left|x^{cschx}\right| = cschx\ \ln|x|$.

   Differentiating both sides

   $\dfrac{f'(x)}{f(x)} = (-cschx\ \coth x)\ \ln|x| + cschx\ \dfrac{1}{x}$

   $f'(x) = f(x)\ \left[(-cschx\ \coth x)\ \ln|x| + cschx\ \frac{1}{x}\right]$

   $f'(x) = x^{cschx}\ \left[(-cschx\ \coth x)\ \ln|x| + cschx\ \frac{1}{x}\right]$.

Integrals of hyperbolic functions:

1. $\displaystyle\int \cosh x\ dx = \sinh x + c.$

   $\displaystyle\int \cosh(f(x))\ f'(x)\ dx = \sinh(f(x)) + c.$

2. $\displaystyle\int \sinh x\ dx = \cosh x + c.$

   $\displaystyle\int \sinh(f(x))\ f'(x)\ dx = \cosh(f(x)) + c.$

3. $\displaystyle\int sech^2 x\ dx = \tanh x + c.$

$$\int sech^2\left(f(x)\right)\ f'(x)\ dx = \tanh\left(f(x)\right) + c.$$

4. $\int csch^2 x\ dx = -\coth x + c.$

$$\int csch^2\left(f(x)\right)\ f'(x)\ dx = -\coth\left(f(x)\right) + c.$$

5. $\int sechx \tanh x\ dx = -sechx + c.$

$$\int sech\left(f(x)\right)\ \ \tanh\left(f(x)\right)\ f'(x)\ dx = -sech\left(f(x)\right) + c.$$

6. $\int cschx \coth x\ dx = -cschx + c.$

$$\int csch\left(f(x)\right)\ \ \coth\left(f(x)\right)\ f'(x)\ dx = -csch\left(f(x)\right) + c.$$

7. $\int \tanh x\ dx = \ln|\cosh x| + c.$

$$\int \tanh\left(f(x)\right)\ f'(x)\ dx = \ln|\cosh\left(f(x)\right)| + c.$$

8. $\int \coth x\ dx = \ln|\sinh x| + c.$

$$\int \coth\left(f(x)\right)\ f'(x)\ dx = \ln|\sinh\left(f(x)\right)| + c.$$

Example (2): Evaluate the following integrals:

1. $\int x^2 \cosh(x^3)\ dx.$

   Solution: $\int x^2 \cosh(x^3)\ dx = \frac{1}{3}\int \cosh(x^3)\ (3x^2)\ dx = \frac{1}{3}\ \sinh(x^3) + c.$

2. $\int e^x \tanh\left(e^x\right)\ dx.$

   Solution: $\int e^x \tanh\left(e^x\right)\ dx = \int \tanh\left(e^x\right)\ e^x\ dx = \ln\left|\cosh\left(e^x\right)\right| + c.$

3. $\int \frac{sech^2\left(\sqrt{x}\right)}{\sqrt{x}}\ dx.$

   Solution: $\int \frac{sech^2\left(\sqrt{x}\right)}{\sqrt{x}}\ dx = \int sech^2\left(\sqrt{x}\right)\ \frac{1}{\sqrt{x}}\ dx$

   $= 2\int sech^2\left(\sqrt{x}\right)\ \frac{1}{2\sqrt{x}}\ dx = 2\tanh\left(\sqrt{x}\right) + c.$

4. $\displaystyle\int \frac{csch\left(\frac{1}{x}\right)\ \coth\left(\frac{1}{x}\right)}{x^2}\ dx.$

Solution: $\displaystyle\int \frac{csch\left(\frac{1}{x}\right)\ \coth\left(\frac{1}{x}\right)}{x^2}\ dx = \int csch\left(\frac{1}{x}\right)\ \coth\left(\frac{1}{x}\right)\ \left(\frac{1}{x^2}\right)\ dx$

$\displaystyle= \int -csch\left(\frac{1}{x}\right)\ \coth\left(\frac{1}{x}\right)\ \left(\frac{-1}{x^2}\right)\ dx = csch\left(\frac{1}{x}\right) + c.$

5. $\displaystyle\int e^{\tanh x}\ sech^2 x\ dx.$

Solution: $\displaystyle\int e^{\tanh x}\ sech^2 x\ dx = e^{\tanh x} + c.$

6. $\displaystyle\int \frac{\sinh x}{1+\cosh x}\ dx.$

Solution: $\displaystyle\int \frac{\sinh x}{1+\cosh x}\ dx = \ln|1+\cosh x| + c.$

7. $\displaystyle\int \frac{\sinh x}{1+\cosh^2 x}\ dx.$

Solution: $\displaystyle\int \frac{\sinh x}{1+\cosh^2 x}\ dx = \int \frac{\sinh x}{1+(\cosh x)^2}\ dx = \tan^{-1}(\cosh x) + c.$

8. $\displaystyle\int \frac{1}{sechx\sqrt{4-\sinh^2 x}}\ dx.$

Solution: $\displaystyle\int \frac{1}{sechx\sqrt{4-\sinh^2 x}}\ dx = \int \frac{\cosh x}{\sqrt{2^2-(\sinh x)^2}}\ dx$

$\displaystyle= \sin^{-1}\left(\frac{\sinh x}{2}\right) + c.$

9. $\displaystyle\int \frac{\coth x}{\sqrt{\sinh^2 x - 4}}\ dx.$

Solution: $\displaystyle\int \frac{\coth x}{\sqrt{\sinh^2 x - 4}}\ dx = \int \frac{\cosh x}{\sinh x\sqrt{(\sinh x)^2-(2)^2}}\ dx$

$\displaystyle= \frac{1}{2}\sec^{-1}\left(\frac{\sinh x}{2}\right) + c.$

EXERCISES (4.1)

1. Find the derivatives of the following:

(1) $f(x) = \sinh\left(e^{2x}\right)$ (2) $f(x) = \cosh(\ln|x|)$

(3) $f(x) = \tanh\left(\sin^{-1}x\right)$ (4) $f(x) = sech\left(\dfrac{1}{x}\right)$

(5) $f(x) = csch\left(x^2\right)\ \cos 3x$ (6) $f(x) = \left(\tanh\sqrt{x} + \coth\sqrt[3]{x}\right)^6$

(7) $f(x) = \ln\left|cosh\left(x^2\right) - 4^x\right|$ (8) $f(x) = e^{\sinh^2 x}$

2. Evaluate the following integrals:

(1) $\displaystyle\int e^{4x}\ \sinh\left(5 - e^{4x}\right)\ dx$ (2) $\displaystyle\int \frac{\cosh(\ln|x|)}{5x}\ dx$

(3) $\displaystyle\int \frac{sech^2\left(\frac{1}{x^2}\right)}{x^3}\ dx$ (4) $\displaystyle\int \frac{sech\sqrt{x}\ \tanh\sqrt{x}}{\sqrt{x}}\ dx$

(5) $\displaystyle\int \frac{\coth(\sin x)}{\sec x}\ dx$ (6) $\displaystyle\int \frac{e^{\cosh 3x}}{csch 3x}\ dx$

(7) $\displaystyle\int \frac{5^{sechx}\sinh x}{\cosh^2 x}\ dx$ (8) $\displaystyle\int \frac{sech 4x\ \tanh 4x}{2 + sech 4x}\ dx$

(9) $\displaystyle\int \frac{csch^2 x}{4 + \coth^2 x}\ dx$ (10) $\displaystyle\int \frac{3}{cschx\ \sqrt{9 - \cosh^2 x}}\ dx$

4.2 The inverse hyperbolic functions

1. **The inverse hyperbolic sine function:**

   It is denoted by $\sinh^{-1} x$, and it is defined as:

   $\sinh^{-1} : \mathbb{R} \longrightarrow \mathbb{R}$

   $\sinh y = x \iff y = \sinh^{-1} x.$

2. **The inverse hyperbolic cosine function:**

   It is denoted by $\cosh^{-1} x$, and it is defined as:

   $\cosh^{-1} : [1, \infty) \longrightarrow [0, \infty)$

   $\cosh y = x \iff y = \cosh^{-1} x.$

3. **The inverse hyperbolic tangent function:**

   It is denoted by $\tanh^{-1} x$, and it is defined as:

   $\tanh^{-1} : (-1, 1) \longrightarrow \mathbb{R}$

   $\tanh y = x \iff y = \tanh^{-1} x.$

4. **The inverse hyperbolic cotangent function:**

   It is denoted by $\coth^{-1} x$, and it is defined as:

   $\coth^{-1} : \mathbb{R} - [-1, 1] \longrightarrow \mathbb{R} - \{0\}$

   $\coth y = x \iff y = \coth^{-1} x.$

5. **The inverse hyperbolic secant function:**

   It is denoted by $sech^{-1}x$, and it is defined as:

   $sech^{-1} : (0, 1] \longrightarrow [0, \infty)$

   $sechy = x \iff y = sech^{-1}x.$

6. **The inverse hyperbolic cosecant function:**

   It is denoted by $csch^{-1}x$, and it is defined as:

   $csch^{-1} : \mathbb{R} - \{0\} \longrightarrow \mathbb{R} - \{0\}$

   $cschy = x \iff y = csch^{-1}x.$

Logarithmic forms of the inverse hyperbolic functions:

1. $\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$, for every $x \in \mathbb{R}$.

2. $\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right)$, for every $x \in [1, \infty)$.

3. $\tanh^{-1} x = \dfrac{1}{2}\ln\left|\dfrac{1+x}{1-x}\right|$, for every $x \in (-1, 1)$.

4. $\coth^{-1} x = \dfrac{1}{2}\ln\left|\dfrac{x+1}{x-1}\right|$, for every $x \in \mathbb{R} - [-1, 1]$.

5. $sech^{-1} x = \ln\left(\dfrac{1+\sqrt{1-x^2}}{x}\right)$, for every $x \in (0, 1]$.

6. $csch^{-1} x = \ln\left(\dfrac{1}{x} + \sqrt{1 + \dfrac{1}{x^2}}\right)$, for every $x \in \mathbb{R}^*$.

Proof: Only (1) will be proved, the rest can be proved in a similar way.

1. $y = \sinh^{-1} x \implies x = \sinh y = \dfrac{e^y - e^{-y}}{2}$

   $e^y - 2x - e^{-y} = 0$

   $e^{2y} - 2xe^y - 1 = 0$

   $(e^y)^2 - 2x\,(e^y) - 1 = 0$

   $e^y = \dfrac{2x \pm \sqrt{(2x)^2 - 4(1)(-1)}}{2} = \dfrac{2x \pm \sqrt{4x^2+4}}{2}$

   $= \dfrac{2x \pm 2\sqrt{x^2+1}}{2} = x \pm \sqrt{x^2+1}$

   Note that $\sqrt{x^2+1} > x$, therefore, $x - \sqrt{x^2+1} < 0$, and since $e^y > 0$ always, then: $e^y = x + \sqrt{x^2+1} \implies y = \ln\left|x + \sqrt{x^2+1}\right|$

   $\sinh^{-1} x = \ln\left|x + \sqrt{x^2+1}\right|$

The derivatives of the inverse hyperbolic functions:

1. $\dfrac{d}{dx}\ \sinh^{-1} x = \dfrac{1}{\sqrt{1+x^2}}$.

   $\dfrac{d}{dx}\ \sinh^{-1}(f(x)) = \dfrac{f'(x)}{\sqrt{1+[f(x)]^2}}$.

2. $\dfrac{d}{dx}\ \cosh^{-1} x = \dfrac{1}{\sqrt{x^2-1}}$, for every $x \in (1, \infty)$.

   $\dfrac{d}{dx}\ \cosh^{-1}(f(x)) = \dfrac{f'(x)}{\sqrt{[f(x)]^2-1}}$, for every $f(x) \in (1, \infty)$.

3. $\dfrac{d}{dx}\ \tanh^{-1} x = \dfrac{1}{1-x^2}$, for every $x \in (-1, 1)$.

   $\dfrac{d}{dx}\ \tanh^{-1}(f(x)) = \dfrac{f'(x)}{1-[f(x)]^2}$, for every $f(x) \in (-1, 1)$.

4. $\dfrac{d}{dx}\ \coth^{-1} x = \dfrac{1}{1-x^2}$, for every $x \in \mathbb{R} - [-1, 1]$.

   $\dfrac{d}{dx}\ \coth^{-1}(f(x)) = \dfrac{f'(x)}{1-[f(x)]^2}$, for every $f(x) \in \mathbb{R} - [-1, 1]$.

5. $\dfrac{d}{dx}\ sech^{-1} x = -\dfrac{1}{x\sqrt{1-x^2}}$, for every $x \in (0, 1)$.

   $\dfrac{d}{dx}\ sech^{-1}(f(x)) = -\dfrac{f'(x)}{f(x)\sqrt{1-[f(x)]^2}}$, for every $f(x) \in (0, 1)$.

6. $\dfrac{d}{dx}\ csch^{-1} x = -\dfrac{1}{|x|\sqrt{1+x^2}}$, for every $x \in \mathbb{R} - \{0\}$.

   $\dfrac{d}{dx}\ csch^{-1}(f(x)) = -\dfrac{f'(x)}{|f(x)|\sqrt{1+[f(x)]^2}}$, for every $f(x) \in \mathbb{R} - \{0\}$.

Proof:

1. $y = \sinh^{-1} x \implies \sinh y = x$.

   Using Implicit differentiation

   $$\cosh y\ y' = 1 \implies y' = \frac{1}{\cosh y} = \frac{1}{\sqrt{1+\sinh^2 y}} = \frac{1}{\sqrt{1+x^2}}.$$

(2) $y = \cosh^{-1} x \implies \cosh y = x$.

   Using Implicit differentiation

   $$\sinh y\ y' = 1 \implies y' = \frac{1}{\sinh y} = \frac{1}{\sqrt{\cosh^2 y - 1}} = \frac{1}{\sqrt{x^2-1}}.$$

Example (1): Find the derivatives of the following:

1. $f(x) = \sinh^{-1}(5x-1)$.

   Solution: $f'(x) = \dfrac{1}{\sqrt{1+(5x-1)^2}}\ (5)$.

2. $f(x) = \tanh^{-1}\left(\sqrt{x}\right)$.

   Solution: $f'(x) = \dfrac{1}{1-\left(\sqrt{x}\right)^2}\ \dfrac{1}{2\sqrt{x}}$

3. $f(x) = csch^{-1}\left(e^{3x}\right)$.

   Solution: $f'(x) = -\dfrac{1}{e^{3x}\sqrt{1+\left(e^{3x}\right)^2}}\ \left(e^{3x}\ (3)\right) = -\dfrac{3}{\sqrt{1+e^{6x}}}$.

4. $f(x) = 6^{\cosh^{-1}(2x)}$.

   Solution: $f'(x) = 6^{\cosh^{-1}(2x)}\ \dfrac{1}{\sqrt{(2x)^2-1}}\ (2)\ \ln 6$.

5. $f(x) = \ln|x^2 + \coth^{-1}(2x)|$.

   Solution: $f'(x) = \dfrac{2x + \frac{1}{1-(2x)^2}\ (2)}{x^2+\coth^{-1}(2x)}$.

6. $f(x) = \left[sech^{-1}(3x) + 7^x\right]^5$.

   Solution: $f'(x) = 5\left[sech^{-1}(3x)+7^x\right]^4\left[\dfrac{-1}{3x\sqrt{1-(3x)^2}}\ (3) + 7^x\ln 7\right]$.

7. $f(x) = \sqrt{x}\ \sinh^{-1}(5x)$.

   Solution: $f'(x) = \dfrac{1}{2\sqrt{x}}\ \sinh^{-1}(5x) + \sqrt{x}\ \dfrac{1}{\sqrt{1+(5x)^2}}\ (5)$.

8. $f(x) = \dfrac{\cosh^{-1}(x^2)}{e^{3x}}$.

   Solution: $f'(x) = \dfrac{\frac{1}{\sqrt{(x^2)^2-1}}\ (2x)\ e^{3x} - \cosh^{-1}(x^2)\ e^{3x}\ (3)}{\left(e^{3x}\right)^2}$

Integrals of specific quadratic forms:

1. $\displaystyle\int \frac{1}{\sqrt{a^2+x^2}}\ dx = \sinh^{-1}\left(\frac{x}{a}\right) + c.$

   $\displaystyle\int \frac{f'(x)}{\sqrt{a^2+[f(x)]^2}}\ dx = \sinh^{-1}\left(\frac{f(x)}{a}\right) + c.$

2. $\displaystyle\int \frac{1}{\sqrt{x^2-a^2}}\ dx = \cosh^{-1}\left(\frac{x}{a}\right) + c$, where $x > a$.

   $\displaystyle\int \frac{f'(x)}{\sqrt{[f(x)]^2-a^2}}\ dx = \cosh^{-1}\left(\frac{f(x)}{a}\right) + c$, where $f(x) > a$.

3. $\displaystyle\int \frac{1}{a^2-x^2}\,dx = \frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right) + c$, where $|x| < a$.

$\displaystyle\int \frac{f'(x)}{a^2-[f(x)]^2}\,dx = \frac{1}{a}\tanh^{-1}\left(\frac{f(x)}{a}\right) + c$, where $|f(x)| < a$.

4. $\displaystyle\int \frac{1}{a^2-x^2}\,dx = \frac{1}{a}\coth^{-1}\left(\frac{x}{a}\right) + c$, where $|x| > a$.

$\displaystyle\int \frac{f'(x)}{a^2-[f(x)]^2}\,dx = \frac{1}{a}\coth^{-1}\left(\frac{f(x)}{a}\right) + c$, where $|f(x)| > a$.

5. $\displaystyle\int \frac{1}{x\sqrt{a^2-x^2}}\,dx = -\frac{1}{a}\,sech^{-1}\left(\frac{|x|}{a}\right) + c$, where $|x| < a$.

$\displaystyle\int \frac{f'(x)}{f(x)\sqrt{a^2-[f(x)]^2}}\,dx = -\frac{1}{a}\,sech^{-1}\left(\frac{|f(x)|}{a}\right) + c$, where $|f(x)| < a$.

6. $\displaystyle\int \frac{1}{x\sqrt{a^2+x^2}}\,dx = -\frac{1}{a}\,csch^{-1}\left(\frac{|x|}{a}\right) + c$, where $|x| \neq 0$.

$\displaystyle\int \frac{f'(x)}{f(x)\sqrt{a^2+[f(x)]^2}}\,dx = -\frac{1}{a}\,csch^{-1}\left(\frac{|f(x)|}{a}\right) + c$, where $|f(x)| \neq 0$.

Example (2): Evaluate the following integrals:

1. $\displaystyle\int \frac{x}{\sqrt{x^4-16}}\,dx.$

   Solution: $\displaystyle\int \frac{x}{\sqrt{x^4-16}}\,dx = \frac{1}{2}\int \frac{2x}{\sqrt{(x^2)^2-4^2}}\,dx = \frac{1}{2}\cosh^{-1}\left(\frac{x^2}{4}\right)+c.$

2. $\displaystyle\int \frac{e^x}{25-e^{2x}}\,dx.$

   Solution: $\displaystyle\int \frac{e^x}{25-e^{2x}}\,dx = \int \frac{e^x}{5^2-(e^x)^2}\,dx = \frac{1}{5}\tanh^{-1}\left(\frac{e^x}{5}\right)+c$

   where $|e^x| < 5$.

3. $\displaystyle\int \frac{1}{\sqrt{4x^2+25}}\,dx.$

   Solution: $\displaystyle\int \frac{1}{\sqrt{4x^2+25}}\,dx = \frac{1}{2}\int \frac{2}{\sqrt{(2x)^2+5^2}}\,dx$

   $\displaystyle= \frac{1}{2}\sinh^{-1}\left(\frac{2x}{5}\right)+c.$

4. $\displaystyle\int \frac{1}{\sqrt{x}\,\sqrt{4+x}}\,dx.$

Solution: $\int \frac{1}{\sqrt{x}\,\sqrt{4+x}}\,dx = 2\int \frac{\frac{1}{2\sqrt{x}}}{\sqrt{2^2+(\sqrt{x})^2}}\,dx$

$= 2\sinh^{-1}\left(\frac{\sqrt{x}}{2}\right) + c.$

5. $\int \frac{1}{x\sqrt{16-x^4}}\,dx.$

Solution: $\int \frac{1}{x\sqrt{16-x^4}}\,dx = \int \frac{x}{x^2\sqrt{4^2-(x^2)^2}}\,dx$

$= \frac{1}{2}\int \frac{2x}{x^2\sqrt{4^2-(x^2)^2}}\,dx = \frac{1}{2}\,\frac{-1}{4}\,sech^{-1}\left(\frac{x^2}{4}\right) + c.$

6. $\int \frac{1}{\sqrt{1+e^{2x}}}\,dx.$

Solution: $\int \frac{1}{\sqrt{1+e^{2x}}}\,dx = \int \frac{e^x}{e^x\sqrt{1^2+(e^x)^2}}\,dx = -csch^{-1}(e^x) + c.$

7. $\int \frac{1}{\sqrt{x^2+2x-8}}\,dx.$

Solution: $\int \frac{1}{\sqrt{x^2+2x-8}}\,dx = \int \frac{1}{\sqrt{(x^2+2x+1)-9}}\,dx$

$= \int \frac{1}{\sqrt{(x+1)^2-3^2}}\,dx = \cosh^{-1}\left(\frac{x+1}{3}\right) + c.$

8. $\int \frac{1}{(x-1)\,\sqrt{-x^2+2x+3}}\,dx.$

Solution: $\int \frac{1}{(x-1)\,\sqrt{-x^2+2x+3}}\,dx = \int \frac{1}{(x-1)\,\sqrt{3-(x^2-2x)}}\,dx$

$= \int \frac{1}{(x-1)\,\sqrt{4-(x^2-2x+1)}}\,dx = \int \frac{1}{(x-1)\,\sqrt{2^2-(x-1)^2}}\,dx$

$= -\frac{1}{2}\,sech^{-1}\left(\frac{x-1}{2}\right) + c.$

EXERCISES (4.2)

1. Find the derivatives of the following:

(1) $f(x) = \sinh^{-1}(3^x)\ \sqrt{x}$ (2) $f(x) = \cosh^{-1}\left(\sqrt{x}\right)\ \ln\left|x^2 - 1\right|$

(3) $f(x) = \left(e^{4x} + \tanh^{-1} 3x\right)^5$ (4) $f(x) = \ln\left|\coth^{-1} 2x + \sinh 2x\right|$

(5) $f(x) = sech^{-1}\left(\dfrac{1}{x}\right) + e^{\cosh x}$ (6) $f(x) = csch^{-1}(\cos x) + \ln|\sinh x|$

2. Evaluate the following integrals:

(1) $\displaystyle\int \frac{5x^2}{\sqrt{9 + 16x^6}}\, dx$ (2) $\displaystyle\int \frac{3x}{\sqrt{25x^4 + 16}}\, dx$

(3) $\displaystyle\int \frac{x^3}{36 - 9x^8}\, dx$ (4) $\displaystyle\int \frac{3x}{x^2\ \sqrt{9 - 16x^4}}\, dx$

(5) $\displaystyle\int \frac{5x}{x^2\ \sqrt{4 - 9x^4}}\, dx$ (6) $\displaystyle\int \frac{\sinh x}{4 - 25\cosh^2 x}\, dx$

(7) $\displaystyle\int \frac{x - 2}{\sqrt{25 + x^2}}\, dx$ (8) $\displaystyle\int \frac{x + 1}{16 - x^2}\, dx$

(9) $\displaystyle\int \frac{\cosh^{-1} x}{\sqrt{x^2 - 1}}\, dx$ (10) $\displaystyle\int \frac{2}{\sqrt{9 - e^{6x}}}\, dx$

(11) $\displaystyle\int \frac{5}{x\ \sqrt{25 + 4x^4}}\, dx$ (12) $\displaystyle\int \frac{1}{x\ \sqrt{25 - x^3}}\, dx$

(13) $\displaystyle\int \frac{7}{\sqrt{x^2 + 6x}}\, dx$ (14) $\displaystyle\int \frac{5}{\sqrt{x^2 + 2x + 10}}\, dx$

(15) $\displaystyle\int \frac{2}{(x + 3)\ \sqrt{x^2 + 6x + 25}}\, dx$ (16) $\displaystyle\int \frac{3}{(x + 1)\ \sqrt{3 - x^2 - 2x}}\, dx$

Chapter 5

TECHNIQUES OF INTEGRATION

5.1 Integration by Parts

Theorem: If $u = f(x)$, $v = g(x)$, and f', g' are both continuous functions then:

$$\int u\ dv = u\ v - \int v\ du$$

Example: Evaluate the following integrals:

1. $\int x \cos x\ dx$.

   Solution:

   $u = x \qquad dv = \cos x\ dx$
   $du = dx \qquad v = \sin x$

   $$\int x \cos x\ dx = x \sin x - \int \sin x\ dx$$

   $= x \sin x - (-\cos x) + c = x \sin x + \cos x + c.$

2. $\int x^2\ e^x\ dx$.

   Solution:

   $u = x^2 \qquad dv = e^x\ dx$
   $du = 2x\ dx \qquad v = e^x$

   $$\int x^2\ e^x\ dx = x^2 e^x - \int 2x\ e^x\ dx = x^2 e^x - 2 \int x\ e^x\ dx$$

   Using integration by parts again:

$u = x \qquad dv = e^x\ dx$
$du = dx \qquad v = e^x$

$$\int x^2\ e^x\ dx = x^2 e^x - 2\left[x\ e^x - \int e^x\ dx\right]$$

$$= x^2 e^x - 2x\ e^x + 2\int e^x\ dx = x^2 e^x - 2x\ e^x + e^x = (x^2 - 2x + 2)e^x + c.$$

3. $\displaystyle\int x^2\ \ln|x|\ dx.$

Solution:

$u = \ln|x| \qquad v = x^2\ dx$
$du = \dfrac{1}{x}\ dx \qquad v = \dfrac{x^3}{3}$

$$\int x^2\ \ln|x|\ dx = \frac{x^3}{3}\ \ln|x| - \int \frac{x^3}{3}\ \frac{1}{x}\ dx$$

$$= \frac{x^3}{3}\ \ln|x| - \frac{1}{3}\int x^2\ dx = \frac{x^3}{3}\ \ln|x| - \frac{1}{3}\ \frac{x^3}{3} + c.$$

4. $\displaystyle\int \ln(1+x^2)\ dx.$

Solution:

$u = \ln(1+x^2) \qquad dv = dx$
$du = \dfrac{2x}{1+x^2}\ dx \qquad v = x$

$$\int \ln(1+x^2)\ dx = x\ln(1+x^2) - \int x\ \frac{2x}{1+x^2}\ dx$$

$$= x\ln(1+x^2) - \int \frac{2x^2}{1+x^2}\ dx = x\ln(1+x^2) - \int \frac{(2x^2+2)-2}{1+x^2}\ dx$$

$$= x\ln(1+x^2) - \int \frac{2(x^2+1)}{1+x^2}\ dx + \int \frac{2}{1+x^2}\ dx$$

$$= x\ln(1+x^2) - \int 2\ dx - 2\int \frac{1}{1+x^2}\ dx = x\ln(1+x^2) - 2x + 2\tan^{-1}x + c.$$

5. $\displaystyle\int \tan^{-1} x\ dx.$

Solution:

$u = \tan^{-1} x \qquad dv = dx$
$du = \dfrac{1}{1+x^2} \qquad v = x$

$$\int \tan^{-1} x\ dx = x\tan^{-1} x - \int \frac{1}{1+x^2}\ x\ dx$$

$$= x\tan^{-1}x - \frac{1}{2}\int \frac{2x}{1+x^2}\ dx = x\tan^{-1}x - \frac{1}{2}\ln(1+x^2) + c.$$

6. $\displaystyle\int x\ \sec^{-1}x\ dx.$

Solution:

$u = \sec^{-1}x \qquad dv = dx$

$du = \dfrac{1}{x\sqrt{x^2-1}}\ dx \qquad v = \dfrac{x^2}{2}$

$$\int x\ \sec^{-1}x\ dx = \frac{x^2}{2}\sec^{-1}x - \int \frac{x^2}{2}\ \frac{1}{x\sqrt{1-x^2}}\ dx$$

$$= \frac{x^2}{2}\sec^{-1}x - \frac{1}{2}\int \left(x^2-1\right)^{-\frac{1}{2}}\ x\ dx$$

$$= \frac{x^2}{2}\sec^{-1}x - \frac{1}{2}\ \frac{1}{2}\int \left(x^2-1\right)^{-\frac{1}{2}}\ (2x)\ dx$$

$$= \frac{x^2}{2}\sec^{-1}x - \frac{1}{4}\frac{\left(x^2-1\right)^{\frac{1}{2}}}{\frac{1}{2}} + c = \frac{x^2}{2}\sec^{-1}x - \frac{1}{2}\sqrt{x^2-1} + c.$$

7. $\displaystyle\int e^x\ \cos x\ dx.$

Solution:

$u = \cos x \qquad dv = e^x\ dx$

$du = -\sin x\ dx \qquad v = e^x$

$$\int e^x\ \cos x\ dx = e^x\cos x - \int -\sin x\ e^x\ dx = e^x\cos x + \int \sin x\ e^x\ dx$$

Using integration by parts again:

$u = \sin x \qquad dv = e^x\ dx$

$du = \cos x\ dx \qquad v = e^x$

$$\int e^x\ \cos x\ dx = e^x\cos x + \int \sin x\ e^x\ dx$$

$$\int e^x\ \cos x\ dx = e^x\cos x + e^x\sin x - \int e^x\cos x\ dx$$

$$2\int e^x\ \cos x\ dx = e^x\cos x + e^x\sin x$$

$$\int e^x\ \cos x\ dx = \frac{1}{2}\left[e^x\cos x + e^x\sin x\right] + c.$$

EXERCISES (5.1)

Evaluate the following integrals:

(1) $\int (2x+1)\ \cosh x\ dx$ (2) $\int (x^2-1)\ \sin 2x\ dx$

(3) $\int x^3\ e^x\ dx$ (4) $\int (x+1)\ \sec^2 x\ dx$

(5) $\int \ln|x|\ dx$ (6) $\int x^5\ \ln|x|\ dx$

(7) $\int x^{-4}\ \ln|x|\ dx$ (8) $\int \sin^{-1} x\ dx$

(9) $\int \sec^{-1} x\ dx$ (10) $\int x\ csch^{-1} x\ dx$

(11) $\int e^{2x}\ \sin 3x\ dx$ (12) $\int e^{5x}\ \cos 2x\ dx$

(13) $\int e^{3x}\ \sinh x\ dx$ (14) $\int e^{x}\ \cosh 5x\ dx$

(15) $\int e^{\sqrt{x}}\ dx$ (16) $\int (\ln|x|)^2\ dx$

(17) $\int x^3\ \sqrt{x^2-1}\ dx$ (18) $\int x^7\ \sqrt{x^4+2}\ dx$

5.2 Trigonometric Integrals

First- Integrals of the form $\int \sin^n x \ dx$ and $\int \cos^n x \ dx$.

1. If n is an odd number,then the integral can be solved by substitution.

- Use the identity $\sin^2 x = 1 - \cos^2 x$ and the substitution $u = \cos x$ to solve the integral $\int \sin^n x \ dx$.
- Use the identity $\cos^2 x = 1 - \sin^2 x$ and the substitution $u = \sin x$ to solve the integral $\int \cos^n x \ dx$.

2. If n is an even number:

- Use the identity $\sin^2 x = \dfrac{1 - \cos 2x}{2}$ to solve the integral $\int \sin^n x \ dx$.
- Use the identity $\cos^2 x = \dfrac{1 + \cos 2x}{2}$ to solve the integral $\int \cos^n x \ dx$.

Example (1): Evaluate the following integrals:

1. $\int \cos^3 x \ dx$.

   Solution:

   $$\int \cos^3 x \ dx = \int \cos^2 x \ \cos x \ dx = \int \left(1 - \sin^2 x\right) \ \cos x \ dx$$

   Using the substitution $u = \sin x$.

   $du = \cos x \ dx$.

   $$\int \cos^3 x \ dx = \int \left(1 - \sin^2 x\right) \ \cos x \ dx = \int (1 - u^2) \ du$$

   $$= u - \frac{u^3}{3} + c = \sin x - \frac{\sin^3 x}{3} + c.$$

2. $\int \sin^5 x \ dx$.

   Solution:

   $$\int \sin^5 x \ dx = \int \sin^4 x \ \sin x \ dx = \int \left(\sin^2 x\right)^2 \ \sin x \ dx$$

   $$= \int \left(1 - \cos^2 x\right)^2 \ \sin x \ dx$$

   Using the substitution $u = \cos x$.

   $du = -\sin x \ dx \implies -du = \sin x \ dx$.

$$\int \sin^5 x \; dx = \int \left(1-\cos^2 x\right)^2 \; \sin x \; dx = -\int \left(1-u^2\right)^2 \; du$$

$$= -\int \left(1-2u^2+u^4\right) \; du = -\left(u - 2\;\frac{u^3}{3} + \frac{u^5}{5}\right) + c$$

$$= -\cos x + 2\;\frac{\cos^3 x}{3} - \frac{\cos^5 x}{5} + c.$$

3. $\int \cos^4 2x \; dx.$

Solution: Using the identity $\cos^2 x = \dfrac{1+\cos 2x}{2}$.

$$\int \cos^4 2x \; dx = \int \left(\cos^2 2x\right)^2 \; dx = \int \left(\frac{1+\cos 4x}{2}\right)^2 \; dx$$

$$= \int \frac{1}{4}\left[1 + 2\cos 4x + \cos^2 4x\right] \; dx$$

$$= \int \frac{1}{4}\left[1 + 2\cos 4x + \frac{1}{2}\left(1+\cos 8x\right)\right] \; dx$$

$$= \int \left[\frac{1}{4} + \frac{1}{2} \; \cos 4x + \frac{1}{8} + \frac{1}{8}\cos 8x\right] \; dx$$

$$= \int \left[\frac{3}{8} + \frac{1}{2} \; \cos 4x + \frac{1}{8}\cos 8x\right] \; dx$$

$$= \int \frac{3}{8} \; dx + \frac{1}{2}\int \cos 4x \; dx + \frac{1}{8}\int \cos 8x \; dx$$

$$= \int \frac{3}{8} \; dx + \frac{1}{2}\;\frac{1}{4}\int \cos 4x \; (4) \; dx + \frac{1}{8}\;\frac{1}{8}\int \cos 8x \; (8) \; dx$$

$$= \frac{3}{8} \; x + \frac{1}{8} \; \sin 4x + \frac{1}{64} \; \sin 8x + c.$$

Second- Integrals of the form $\int \sin^n x \; \cos^m x \; dx.$

1. If n is an odd number, use the identity $\sin^2 x = 1 - \cos^2 x$ and the substitution $u = \cos x$ to solve the integral.
2. If m is an odd number, use the identity $\cos^2 x = 1 - \sin^2 x$ and the substitution $u = \sin x$ to solve the integral.
3. If n and m are both even numbers:

   Use $\sin^2 x = \dfrac{1-\cos 2x}{2}$ and $\cos^2 x = \dfrac{1+\cos 2x}{2}$ to solve the integral.

Example (2): Evaluate the following integrals:

1. $\displaystyle\int \sin^5 x\ \cos^2 x\ dx.$

   Solution:

   $\displaystyle\int \sin^5 x\ \cos^2 x\ dx = \int \sin^4 x\ \cos^2 x\ \sin x\ dx$

   $\displaystyle= \int \left(\sin^2 x\right)^2\ \cos^2 x\ \sin x\ dx = \int \left(1-\cos^2 x\right)^2\ \cos^2 x\ \sin x\ dx$

   Using the substitution $u = \cos x$.

   $du = -\sin x\ dx \implies -du = \sin x\ dx.$

   $\displaystyle\int \left(1-\cos^2 x\right)^2\ \cos^2 x\ \sin x\ dx = -\int \left(1-u^2\right)^2\ u^2\ du$

   $\displaystyle= -\int \left(1-2u^2+u^4\right)\ u^2\ du = -\int \left(u^2-2u^4+u^6\right)\ du$

   $\displaystyle= -\left(\frac{u^3}{3} - 2\ \frac{u^5}{5} + \frac{u^7}{7}\right) + c = -\frac{\cos^3 x}{3} + 2\ \frac{\cos^5 x}{5} - \frac{\cos^7 x}{7} + c.$

2. $\displaystyle\int \sqrt{\sin x}\ \cos^3 x\ dx$

   Solution:

   $\displaystyle\int \sqrt{\sin x}\ \cos^3 x\ dx = \int \sqrt{\sin x}\ \cos^2 x\ \cos x\ dx$

   $\displaystyle= \int \sqrt{\sin x}\left(1-\sin^2 x\right)\ \cos x\ dx$

   Using the substitution $u = \sin x$.

   $du = \cos x\ dx.$

   $\displaystyle\int \sqrt{\sin x}\left(1-\sin^2 x\right)\ \cos x\ dx = \int \sqrt{u}\ \left(1-u^2\right)\ du$

   $\displaystyle= \int u^{\frac{1}{2}}\ \left(1-u^2\right)\ du = \int \left(u^{\frac{1}{2}} - u^{\frac{5}{2}}\right)\ du$

   $\displaystyle= \frac{u^{\frac{3}{2}}}{\frac{3}{2}} - \frac{u^{\frac{7}{2}}}{\frac{7}{2}} + c = \frac{2}{3}\ (\sin x)^{\frac{3}{2}} - \frac{2}{7}\ (\sin x)^{\frac{7}{2}} + c.$

3. $\displaystyle\int \sin^5 x\ \cos^7 x\ dx$

   Solution:

   $\displaystyle\int \sin^5 x\ \cos^7 x\ dx = \int \sin^4 x\ \cos^7 x\ \sin x\ dx$

$$= \int \left(\sin^2 x\right)^2 \ \cos^7 x \ \sin x \ dx = \int \left(1 - \cos^2 x\right)^2 \ \cos^7 x \ \sin x \ dx$$

Using the substitution $u = \cos x$.

$du = -\sin x \ dx \implies \ -du = \sin x \ dx.$

$$\int \left(1 - \cos^2 x\right)^2 \ \cos^7 x \ \sin x \ dx = -\int \left(1 - u^2\right)^2 \ u^7 \ du$$

$$= -\int \left(1 - 2u^2 + u^4\right) \ u^7 \ du = -\int \left(u^7 - 2u^9 + u^{11}\right) \ du$$

$$= -\left(\frac{u^8}{8} - 2 \ \frac{u^{10}}{10} + \frac{u^{12}}{12}\right) + c = -\frac{\cos^8 x}{8} + 2 \ \frac{\cos^{10} x}{10} - \frac{\cos^{12} x}{12} + c.$$

4. $\int \sin^2 x \ \cos^4 x \ dx$

Solution:

$$\int \sin^2 x \ \cos^4 x \ dx = \int \left(\frac{1 - \cos 2x}{2}\right) \left(\cos^2 x\right)^2 \ dx$$

$$= \int \left(\frac{1 - \cos 2x}{2}\right) \ \left(\frac{1 + \cos 2x}{2}\right)^2 \ dx$$

$$= \frac{1}{8} \int (1 - \cos^2 2x) \ (1 + \cos 2x) \ dx$$

$$= \frac{1}{8} \int \sin^2 2x \ (1 + \cos 2x) \ dx$$

$$= \frac{1}{8} \int \sin^2 2x \ dx + \frac{1}{8} \int \sin^2 2x \ \cos 2x \ dx$$

$$= \frac{1}{8} \int \left(\frac{1 - \cos 4x}{2}\right) \ dx + \frac{1}{8} \ \frac{1}{2} \int \sin^2 2x \ \cos 2x \ (2) \ dx$$

$$= \frac{1}{16} \int 1 \ dx - \frac{1}{16} \ \frac{1}{4} \int \cos 4x \ (4) \ dx + \frac{1}{16} \int \sin^2 2x \ \cos 2x \ (2) \ dx$$

$$= \frac{1}{16} \ x - \frac{1}{64} \sin 4x + \frac{1}{16} \ \frac{\sin^3 2x}{3} + c.$$

Third- Integrals of the form $\int \tan^m x \ \sec^n x \ dx.$

1. If $m = 0$ and n is an odd number, then use integration by parts to solve the integral $\int \sec^n x \ dx.$
2. If $n = 0$ and $m \geq 2$, then:

$$\int \tan^m x \ \sec^n x \ dx = \int \tan^m x \ dx = \int \tan^{m-2} x \ (\sec^2 x - 1) \ dx$$

$$= \int \tan^{m-2} x \ \sec^2 x \ dx - \int \tan^{m-2} x \ dx$$

$$= \frac{1}{m-1} \ \tan^{m-1} x - \int \tan^{m-2} x \ dx.$$

3. If $n \geq 2$ is an even number, use the identity $\sec^2 x = 1 + \tan^2 x$ and the substitution $u = \tan x$ to solve the integral.
4. If m is an odd number and $n \geq 1$, use the identity $\tan^2 x = \sec^2 x - 1$ and the substitution $u = \sec x$ to solve the integral.
5. If n is an odd number and m is an even number, use the identity $\tan^2 x = \sec^2 x - 1$, to transform the integral as a power of $\sec x$, then use (1).

Example (3): Evaluate the following integrals:

1. $\int \sec^3 x \ dx.$

   Solution: Using integration by parts.

$$\int \sec^3 x \ dx = \int \sec x \ \sec^2 x \ dx$$

$u = \sec x$ $\qquad dv = \sec^2 x \ dx$
$du = \sec x \ \tan x \ dx$ $\qquad v = \tan x$

$$\int \sec^3 x \ dx = \sec x \ \tan x - \int \tan^2 x \ \sec x \ dx$$

$$= \sec x \ \tan x - \int \left(\sec^2 x - 1\right) \ \sec x \ dx$$

$$= \sec x \ \tan x - \int \left(\sec^3 x - \sec x\right) \ dx$$

$$= \sec x \ \tan x - \int \sec^3 x \ dx + \int \sec x \ dx$$

$$2 \int \sec^3 x \ dx = \sec x \ \tan x + \int \sec x \ dx$$

$$\int \sec^3 x \ dx = \frac{1}{2} \left(\sec x \ \tan x + \ln|\sec x + \tan x|\right) + c \ .$$

2. $\int \tan^4 x \ dx.$

   Solution:

$$\int \tan^4 x \ dx = \int \tan^2 x \ \tan^2 x \ dx = \int \tan^2 x \ \left(\sec^2 x - 1\right) \ dx$$

$$= \int \tan^2 x \ \sec^2 x \ dx - \int \tan^2 x \ dx$$

$= \int (\tan x)^2 \sec^2 x \; dx - \int \left(\sec^2 x - 1\right) \; dx$

$= \int (\tan x)^2 \sec^2 x \; dx - \int \sec^2 x \; dx + \int 1 \; dx = \dfrac{\tan^3 x}{3} - \tan x + x + c.$

3. $\int \tan^2 x \; \sec^6 x \; dx.$

Solution:

$\int \tan^2 x \; \sec^6 x \; dx = \int \tan^2 x \; \sec^4 x \; \sec^2 x \; dx$

$= \int \tan^2 x \; \left(\sec^2 x\right)^2 \; \sec^2 x \; dx = \int \tan^2 x \; \left(1 + \tan^2 x\right)^2 \; \sec^2 x \; dx$

Using the substitution $u = \tan x$.

$du = \sec^2 x \; dx.$

$\int \tan^2 x \; \left(1 + \tan^2 x\right)^2 \; \sec^2 x \; dx = \int u^2 \left(1 + u^2\right)^2 \; du$

$= \int u^2 \left(1 + 2u^2 + u^4\right) \; du = \int \left(u^2 + 2u^4 + u^6\right) \; du$

$= \dfrac{u^3}{3} + 2 \; \dfrac{u^5}{5} + \dfrac{u^7}{7} + c = \dfrac{\tan^3 x}{3} + 2 \; \dfrac{\tan^5 x}{5} + \dfrac{\tan^7 x}{7} + c.$

4. $\int \tan^5 x \; \sec^3 x \; dx$

Solution:

$\int \tan^5 x \; \sec^3 x \; dx = \int \tan^4 x \; \sec^2 x \; \sec x \; \tan x \; dx$

$= \int \left(\tan^2 x\right)^2 \; \sec^2 x \; \sec x \; \tan x \; dx$

$= \int \left(\sec^2 x - 1\right)^2 \; \sec^2 x \; \sec x \; \tan x \; dx$

Using the substitution $u = \sec x$.

$du = \sec x \; \tan x \; dx.$

$\int \left(\sec^2 x - 1\right)^2 \; \sec^2 x \; \sec x \; \tan x \; dx = \int (u^2 - 1)^2 \; u^2 \; du$

$= \int \left(u^4 - 2u^2 + 1\right) \; u^2 \; du = \int \left(u^6 - 2u^4 + u^2\right) \; du$

$= \dfrac{u^7}{7} - 2 \; \dfrac{u^5}{5} + \dfrac{u^3}{3} + c = \dfrac{\sec^7 x}{7} - 2 \; \dfrac{\sec^5 x}{5} + \dfrac{\sec^3 x}{3} + c.$

Fourth- Integrals of the form $\int \cot^m x \ \csc^n x \ dx.$

1. If $m = 0$ and n is an odd number, then use integration by parts to solve the integral $\int \csc^n x \ dx.$
2. If $n = 0$ and $m \geq 2$, then:

   $\int \cot^m x \ \csc^n x \ dx = \int \cot^m x \ dx = \int \cot^{m-2} x \ (\csc^2 x - 1) \ dx$

   $= \int \cot^{m-2} x \ \csc^2 x \ dx - \int \cot^{m-2} x \ dx$

   $= -\frac{1}{m-1} \ \cot^{m-1} x - \int \cot^{m-2} x \ dx.$
3. If $n \geq 2$ is an even number, use the identity $\csc^2 x = 1 + \cot^2 x$ and the substitution $u = \cot x$ to solve the integral.
4. If m is an odd number and $n \geq 1$, use the identity $\cot^2 x = \csc^2 x - 1$ and the substitution $u = \csc x$ to solve the integral.
5. If n is an odd number and m is an even number, use the identity $\cot^2 x = \csc^2 x - 1$, to transform the integral as a power of $\csc x$, then use (1).

Example (4): Evaluate the following integrals:

1. $\int \cot^4 x \ \csc^4 x \ dx.$

   Solution:

   $\int \cot^4 x \ \csc^4 x \ dx = \int \cot^4 x \ \csc^2 x \ \csc^2 x \ dx$

   $= \int \cot^4 x \ \left(1 + \cot^2 x\right) \ \csc^2 x \ dx$

   Using the substitution $u = \cot x$.

   $du = -csc^2 x \ dx \implies -du = \csc^2 x \ dx.$

   $\int \cot^4 x \ \left(1 + \cot^2 x\right) \ \csc^2 x \ dx = -\int u^4 \ (1 + u^2) \ du$

   $= -\int (u^4 + u^6) \ du = -\left(\frac{u^5}{5} + \frac{u^7}{7}\right) + c = -\frac{\cot^5 x}{5} - \frac{\cot^7 x}{7} + c.$

2. $\int \cot^5 x \ \csc^5 x \ dx.$

   Solution:

   $\int \cot^5 x \ \csc^5 x \ dx = \int \cot^4 x \ \csc^4 x \ \csc x \ \cot x \ dx$

$$= \int \left(\cot^2 x\right)^2 \csc^4 x \ \csc x \ \cot x \ dx$$

$$= \int \left(\csc^2 x - 1\right)^2 \csc^4 x \ \csc x \ \cot x \ dx$$

Using the substitution $u = \csc x$.

$du = -\csc x \ \cot x \ dx \implies -du = \csc x \ \cot x \ dx$.

$$\int \left(\csc^2 x - 1\right)^2 \csc^4 x \ \csc x \ \cot x \ dx = -\int (u^2-1)^2 \ u^4 \ du$$

$$= -\int (u^4 - 2u^2 + 1) \ u^4 \ du = -\int (u^8 - 2u^6 + u^4) \ du$$

$$= -\left(\frac{u^9}{9} - 2\,\frac{u^7}{7} + \frac{u^5}{5}\right) + c = -\frac{\csc^9 x}{9} + 2\,\frac{\csc^7 x}{7} - \frac{\csc^5 x}{5} + c.$$

Fifth- Integrals of the form

$\int \sin mx \ \cos nx \ dx$, $\int \sin mx \ \sin nx \ dx$, $\int \cos mx \ \cos nx \ dx$.

Use the following identities to solve these type of integrals:

$$\sin mx \ \cos nx = \frac{1}{2}\left(\sin[(m-n)x] + \sin[(m+n)x]\right).$$

$$\sin mx \ \sin nx = \frac{1}{2}\left(\cos[(m-n)x] - \cos[(m+n)x]\right).$$

$$\cos mx \ \cos nx = \frac{1}{2}\left(\cos[(m-n)x] + \cos[(m+n)x]\right).$$

Example (5): Evaluate the following integrals:

1. $\int \sin 7x \ \cos 5x \ dx$.

   Solution:

$$\int \sin 7x \ \cos 5x \ dx = \int \frac{1}{2}\left(\sin[(7-5)x] + \sin[(7+5)x]\right) \ dx$$

$$= \frac{1}{2}\int \sin 2x \ dx + \frac{1}{2}\int \sin 12x \ dx$$

$$= \frac{1}{2}\,\frac{1}{2}\int \sin 2x \ (2) \ dx + \frac{1}{2}\,\frac{1}{12}\int \sin 12x \ (12) \ dx$$

$$= -\frac{1}{4}\cos 2x - \frac{1}{24}\cos 12x + c.$$

2. $\displaystyle\int \sin 4x \ \sin 3x \ dx.$

Solution:

$$\int \sin 4x \ \sin 3x \ dx = \int \frac{1}{2} \left(\cos[(4-3)x] - \cos[(4+3)]x\right) \ dx$$

$$= \frac{1}{2} \int \cos x \ dx - \frac{1}{2} \int \cos 7x \ dx$$

$$= \frac{1}{2} \int \cos x \ dx - \frac{1}{2} \ \frac{1}{7} \int \cos 7x \ (7) \ dx = \frac{1}{2} \sin x - \frac{1}{14} \sin 7x + c.$$

EXERCISES (5.2)

Evaluate the following integrals:

(1) $\int \sin^3 x \ \cos^4 x \ dx$ (2) $\int \sin^2 x \ \cos^5 x \ dx$

(3) $\int \sin^5 x \ \cos^3 x \ dx$ (4) $\int \sqrt[3]{\cos x} \ \sin^3 x \ dx$

(5) $\int \sqrt{\tan x} \ \sec^4 x \ dx$ (6) $\int \tan^5 x \ \sec^5 x \ dx$

(7) $\int \tan^3 x \ \sec^4 x \ dx$ (8) $\int \cot^2 x \ \csc^6 x \ dx$

(9) $\int \cot^5 x \ \csc^3 x \ dx$ (10) $\int \sin 3x \ \cos 5x \ dx$

(11) $\int \sin 2x \ \sin 6x \ dx$ (12) $\int \cos 4x \ \cos 7x \ dx$

(13) $\int \sin^2 x \ dx$ (14) $\int \cos^2 x \ dx$

(15) $\int \sec^5 x \ dx$ (16) $\int \tan^6 x \ dx$

5.3 Trigonometric Substitutions

Trigonometric substitutions are used to solve integrals involving $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$ and $\sqrt{x^2 - a^2}$, where $a > 0$.
These substitutions transform these type of integrals into integrals involving powers of trigonometric functions.
More specific:

1. Use $x = a\sin\theta$, where $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ to transform $\sqrt{a^2 - x^2}$ into:

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2\sin^2\theta} = \sqrt{a^2(1 - \sin^2\theta)} = \sqrt{a^2\cos^2\theta} = a\cos\theta.$$

2. Use $x = a\tan\theta$, where $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ to transform $\sqrt{a^2 + x^2}$ into:

$$\sqrt{a^2 + x^2} = \sqrt{a^2 + a^2\tan^2\theta} = \sqrt{a^2(1 + \tan^2\theta)} = \sqrt{a^2\sec^2\theta} = a\sec\theta.$$

3. Use $x = a\sec\theta$, where $\theta \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$ to transform $\sqrt{x^2 - a^2}$ into:

$$\sqrt{x^2 - a^2} = \sqrt{a^2\sec^2\theta - a^2} = \sqrt{a^2(\sec^2\theta - 1)} = \sqrt{a^2\tan^2\theta} = a\tan\theta\ .$$

Example: Evaluate the following integrals:

1. $\displaystyle\int \frac{1}{x^2\sqrt{16 - x^2}}\, dx.$

Solution:

$$\int \frac{1}{x^2\sqrt{16 - x^2}}\, dx = \int \frac{1}{x^2\sqrt{4^2 - x^2}}\, dx$$

Using the substitution $x = 4\sin\theta \implies \sin\theta = \frac{x}{4}$.

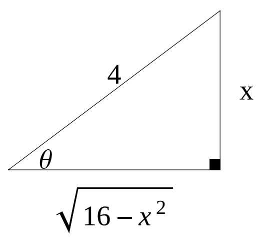
$dx = 4\cos\theta\ d\theta$.

$$\sqrt{16 - x^2} = \sqrt{16 - 16\sin^2\theta} = \sqrt{16\left(1 - \sin^2\theta\right)} = \sqrt{16\cos^2\theta} = 4\cos\theta.$$

$$\int \frac{1}{x^2\sqrt{16 - x^2}}\, dx = \int \frac{4\cos\theta}{16\sin^2\theta\ 4\cos\theta}\, d\theta$$

$$= \int \frac{1}{16\sin^2\theta}\, d\theta = \frac{1}{16}\int \csc^2\theta\ d\theta = -\frac{1}{16}\cot\theta + c$$

$\sin\theta = \frac{x}{4}$.
From the triangle:
$\cot\theta = \dfrac{\sqrt{16 - x^2}}{x}$.



$$\int \frac{1}{x^2\sqrt{16-x^2}}\,dx = -\frac{1}{16}\,\frac{\sqrt{16-x^2}}{x} + c.$$

2. $\displaystyle\int \frac{1}{\sqrt{x^2+9}}\,dx.$

Solution:

$$\int \frac{1}{\sqrt{x^2+9}}\,dx = \int \frac{1}{\sqrt{x^2+3^2}}\,dx$$

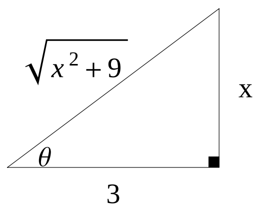
Using the substitution $x = 3\tan\theta \implies \tan\theta = \dfrac{x}{3}$

$dx = 3\sec^2\theta\ d\theta.$

$\sqrt{x^2+9} = \sqrt{9\tan^2\theta+9} = \sqrt{9\left(\tan^2\theta+1\right)} = \sqrt{9\sec^2\theta} = 3\sec\theta.$

$$\int \frac{1}{\sqrt{x^2+9}}\,dx = \int \frac{3\sec^2\theta}{3\sec\theta}\,d\theta = \int \sec\theta\ d\theta = \ln|\sec\theta+\tan\theta| + c$$

$\tan\theta = \dfrac{x}{3}$.
From the triangle:
$\sec\theta = \dfrac{\sqrt{x^2+9}}{3}$.



$$\int \frac{1}{\sqrt{x^2+9}}\,dx = \ln\left|\frac{\sqrt{x^2+9}}{3} + \frac{x}{3}\right| + c.$$

3. $\displaystyle\int \frac{\sqrt{x^2-25}}{x^4}\,dx.$

Solution:

$$\int \frac{\sqrt{x^2-25}}{x^4}\,dx = \int \frac{\sqrt{x^2-5^2}}{\left(x^2\right)^2}\,dx$$

Using the substitution $x = 5\sec\theta \implies \sec\theta = \dfrac{x}{5}.$

$dx = 5\sec\theta\ \tan\theta\ d\theta.$

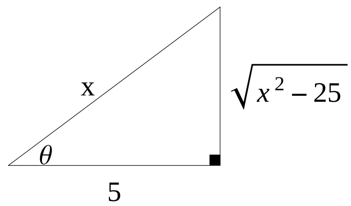
$\sqrt{x^2-25} = \sqrt{25\sec^2\theta-25} = \sqrt{25\left(\sec^2\theta-1\right)} = \sqrt{25\tan^2\theta} = 5\tan\theta.$

$$\int \frac{\sqrt{x^2-5^2}}{\left(x^2\right)^2}\,dx = \int \frac{5\ \tan\theta}{5^4\sec^4\theta}\,5\sec\theta\ \tan\theta\ d\theta = \int \frac{5^2\sec\theta\ \tan^2\theta}{5^4\sec^4\theta}\,d\theta$$

$$= \int \frac{\tan^2\theta}{5^2\sec^3\theta}\,d\theta = \frac{1}{25}\int \tan^2\theta\ \frac{1}{\sec^3\theta}\,d\theta = \frac{1}{25}\int \frac{\sin^2\theta\ \cos^3\theta}{\cos^2\theta}\,d\theta$$

$$= \frac{1}{25}\int (\sin\theta)^2 \cos\theta \; d\theta = \frac{1}{25}\;\frac{(\sin\theta)^3}{3} + c = \frac{1}{75}\;(\sin\theta)^3 + c$$

$\sec\theta = \dfrac{x}{5}$.
From the triangle:
$\sin\theta = \dfrac{\sqrt{x^2-25}}{x}$.



$$\int \frac{\sqrt{x^2-25}}{x^4}\; dx = \frac{1}{75}\;\left(\frac{\sqrt{x^2-25}}{x}\right)^3 + c.$$

4. $\displaystyle\int \frac{1}{(x^2+8x+25)^{\frac{3}{2}}}\; dx.$

Solution:

$$\int \frac{1}{(x^2+8x+25)^{\frac{3}{2}}}\; dx = \int \frac{1}{((x^2+8x+16)+9)^{\frac{3}{2}}}\; dx$$

$$= \int \frac{1}{((x+4)^2+3^2)^{\frac{3}{2}}}\; dx$$

Using the substitution $x+4 = 3\tan\theta \implies \tan\theta = \dfrac{x+4}{3}$.

$dx = 3\sec^2\theta \; d\theta$.

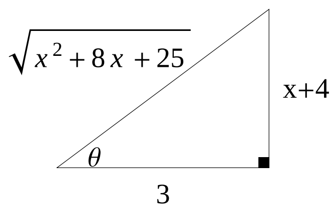
$$\left((x+4)^2+3^2\right)^{\frac{3}{2}} = \left(9\tan^2\theta + 9\right)^{\frac{3}{2}} = \left[9\left(\tan^2\theta+1\right)\right]^{\frac{3}{2}}$$

$$= \left[3^2\sec^2\right]^{\frac{3}{2}} = \left(3^2\right)^{\frac{3}{2}}\left(\sec^2\theta\right)^{\frac{3}{2}} = 3^3\sec^3\theta.$$

$$\int \frac{1}{((x+4)^2+3^2)^{\frac{3}{2}}}\; dx = \int \frac{3\sec^2\theta}{3^3\;\;\sec^3\theta}\; d\theta$$

$$= \frac{1}{3^2}\int \frac{1}{\sec\theta}\; d\theta = \frac{1}{9}\int \cos\theta\; d\theta = \frac{1}{9}\sin\theta + c$$

$\tan\theta = \dfrac{x+4}{3}$.
From the triangle:
$\sin\theta = \dfrac{x+4}{\sqrt{x^2+8x+25}}$.



$$\int \frac{1}{(x^2+8x+25)^{\frac{3}{2}}}\; dx = \frac{1}{9}\;\frac{x+4}{\sqrt{x^2+8x+25}} + c.$$

5. $\displaystyle\int \frac{1}{\sqrt{4-x^2}}\,dx.$

Solution:

$$\int \frac{1}{\sqrt{4-x^2}}\,dx = \int \frac{1}{\sqrt{2^2-x^2}}\,dx$$

Using the substitution $x = 2\sin\theta \implies \sin\theta = \frac{x}{2}$.

$dx = 2\cos\theta\ d\theta.$

$$\sqrt{4-x^2} = \sqrt{4-4\sin^2\theta} = \sqrt{4\left(1-\sin^2\theta\right)} = \sqrt{4\cos^2\theta} = 2\cos\theta.$$

$$\int \frac{1}{\sqrt{4-x^2}}\,dx = \int \frac{2\cos\theta}{2\cos\theta}\,d\theta = \int 1\ d\theta = \theta + c = \sin^{-1}\left(\frac{x}{2}\right) + c.$$

Note that $\sin\theta = \frac{x}{2} \implies \theta = \sin^{-1}\left(\frac{x}{2}\right).$

EXERCISES (5.3)

Evaluate the following integrals:

(1) $\displaystyle\int \frac{\sqrt{4-x^2}}{x^2}\,dx$ (2) $\displaystyle\int \frac{1}{x^2\sqrt{9+x^2}}\,dx$

(3) $\displaystyle\int \frac{1}{x^3\sqrt{x^2-25}}\,dx$ (4) $\displaystyle\int \frac{1}{(16+x^2)^{\frac{3}{2}}}\,dx$

(5) $\displaystyle\int \frac{\sqrt{9x^2-1}}{x}\,dx$ (6) $\displaystyle\int \sqrt{36-x^2}\,dx$

(7) $\displaystyle\int \sqrt{32-x^2-4x}\,dx$ (8) $\displaystyle\int \frac{1}{(x^2+4x+13)^{\frac{5}{2}}}\,dx$

(9) $\displaystyle\int \frac{x^2}{(1-x^2)^{\frac{3}{2}}}\,dx$ (10) $\displaystyle\int \frac{1}{x\sqrt{x^2-9}}\,dx$

(11) $\displaystyle\int \frac{1}{\sqrt{x^2-16}}\,dx$ (12) $\displaystyle\int \frac{1}{\sqrt{x^2+4}}\,dx$

5.4 Integrals of Rational Functions

(The method of partial fractions)

Linear factor: It is a polynomial of degree one, and it has the form $px + q$, where $p, q \in \mathbb{R}$ and $p \neq 0$.

Example (1): $2x$, $x - 3$ and $4x + 1$ are all linear factors.

Irreducible quadratic factor: It is a polynomial of degree two, and it has the form $ax^2 + bx + c$, where $a, b, c \in \mathbb{R}$, $a \neq 0$ and $b^2 - 4ac < 0$.

Example (2): x^2+1, x^2+9 and x^2+x+1 are all irreducible quadratics factors. While the quadratic factor $x^2 - 4$ is reducible, because $x^2 - 4 = (x-2)(x+2)$.

The method of partial fractions is used to solve integrals of rational functions, a rational function is a quotient of two different polynomials.

Guidelines for finding partial fraction decomposition of $\dfrac{f(x)}{g(x)}$:

1. Degree of $f(x)$ must be strictly less than degree of $g(x)$, if degree of $f(x)$ is greater than or equal to degree of $g(x)$, use long division of polynomials to obtain the proper form.
2. Write $g(x)$ as a product of linear factors $px + q$ or irreducible quadratic factors $ax^2 + bx + c$, with repeated factors as:

   $g(x) = (px + q)^m \; (ax^2 + bx + c)^n$, where $m, n \in \mathbb{N}$.
3. The partial fraction decomposition of $(px+q)^m$ contains a sum of partial fraction of the form $\dfrac{A_1}{px+q} + \dfrac{A_2}{(px+q)^2} + \cdots + \dfrac{A_m}{(px+q)^m}$.
4. The partial fraction decomposition of $(ax^2 + bx + c)^n$ contains a sum of partial fraction of the form $\dfrac{B_1x + C_1}{ax^2+bx+c} + \cdots + \dfrac{B_nx + C_n}{(ax^2+bx+c)^n}$.

Example (3): Write the partial fraction decomposition of the following:

1. $\dfrac{2x-1}{x^2-2x-3}$.

   Solution:

   $$\frac{2x-1}{x^2-2x-3} = \frac{2x-1}{(x+1)(x-3)} = \frac{A_1}{x+1} + \frac{A_2}{x-3}.$$
2. $\dfrac{x+4}{x^3+4x^2+4x}$.

Solution:

$$\frac{x+4}{x^3+4x^2+4x} = \frac{x+4}{x(x+2)^2} = \frac{A_1}{x} + \frac{A_2}{x+2} + \frac{A_3}{(x+2)^2}.$$

3. $\dfrac{x^2+1}{(x-1)(x^2-1)}$.

Solution:

$$\frac{x^2+1}{(x-1)(x^2-1)} = \frac{x^2+1}{(x-1)(x-1)(x+1)} = \frac{x^2+1}{(x+1)(x-1)^2}$$

$$= \frac{A_1}{x+1} + \frac{A_2}{x-1} + \frac{A_3}{(x-1)^2}.$$

4. $\dfrac{x-2}{x^4+4x^2}$.

Solution:

$$\frac{x-2}{x^4+4x^2} = \frac{x-2}{x^2(x^2+4)} = \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{Bx+C}{x^2+4}.$$

5. $\dfrac{x+1}{(x^2+1)^2\ (x^2+9)}$.

Solution:

$$\frac{x+1}{(x^2+1)^2\ (x^2+9)} = \frac{B_1x+C_1}{x^2+9} + \frac{B_2x+C_2}{x^2+1} + \frac{B_3x+C_3}{(x^2+1)^2}.$$

6. $\dfrac{x^4+2}{x^3+x}$.

Solution: Using long division of polynomials,

$$\frac{x^4+2}{x^3+x} = x + \frac{-x^2+2}{x(x^2+1)} = x + \frac{A}{x} + \frac{Bx+c}{x^2+1}.$$

Example (4): Evaluate the following integrals:

1. $\displaystyle\int \frac{x-7}{x^2+x-6}\ dx.$

Solution: Using the method of partial fractions.

$$\frac{x-7}{x^2+x-6} = \frac{x-7}{(x-2)(x+3)} = \frac{A_1}{x-2} + \frac{A_2}{x+3}$$

$$\frac{x-7}{x^2+x-6} = \frac{A_1(x+3)}{(x-2)(x+3)} + \frac{A_2(x-2)}{(x+3)(x-2)}$$

$x - 7 = A_1(x+3) + A_2(x-2)$

Put $x = 2$, then:

$2 - 7 = A_1(2+3) \implies -5 = 5A_1 \implies A_1 = -1.$

Put $x = -3$, then:

$-3-7 = A_2(-3-2) \implies -10 = -5A_2 \implies A_2 = 2.$

$$\frac{x-7}{(x-2)(x+3)} = \frac{-1}{x-2} + \frac{2}{x+3}$$

$$\int \frac{x-7}{x^2+x-6}\, dx = \int \left(\frac{-1}{x-2} + \frac{2}{x+3}\right) dx$$

$$= \int \frac{-1}{x-2}\, dx + \int \frac{2}{x+3}\, dx = -\int \frac{1}{x-2}\, dx + 2\int \frac{1}{x+3}\, dx$$

$$= -\ln|x-2| + 2\ln|x+3| + c.$$

2. $\displaystyle\int \frac{7x^2+11x+5}{x^3+2x^2+x}\, dx.$

Solution: Using the method of partial fractions.

$$\frac{7x^2+11x+5}{x^3+2x^2+x} = \frac{7x^2+11x+5}{x(x^2+2x+1)} = \frac{7x^2+11x+5}{x(x+1)^2}$$

$$= \frac{A_1}{x} + \frac{A_2}{x+1} + \frac{A_3}{(x+1)^2}$$

$$\frac{7x^2+11x+5}{x^3+2x^2+x} = \frac{A_1(x+1)^2}{x(x+1)^2} + \frac{A_2\ x(x+1)}{(x+1)x(x+1)} + \frac{A_3\ x}{x(x+1)^2}$$

$$7x^2+11x+5 = A_1(x+1)^2 + A_2\ x(x+1) + A_3\ x$$

$$7x^2+11x+5 = A_1(x^2+2x+1) + A_2x^2 + A_2x + A_3x$$

$$7x^2+11x+5 = A_1x^2 + 2A_1x + A_1 + A_2x^2 + A_2x + A_3x$$

$$7x^2+11x+5 = (A_1+A_2)\,x^2 + (2A_1+A_2+A_3)\,x + A_1$$

By comparing the coefficients of the two polynomials in each side:

$$\begin{array}{lll} A_1 + A_2 = 7 & \longrightarrow & (1) \\ 2A_1 + A_2 + A_3 = 11 & \longrightarrow & (2) \\ A_1 = 5 & \longrightarrow & (3) \end{array}$$

From equation (1): $A_2 = 7 - A_1 = 7 - 5 = 2.$

From equation (2): $10 + 2 + A_3 = 11 \implies A_3 = 11 - 12 = -1.$

$$\frac{7x^2+11x+5}{x^3+2x^2+x} = \frac{5}{x} + \frac{2}{x+1} + \frac{-1}{(x+1)^2}$$

$$\int \frac{7x^2+11x+5}{x^3+2x^2+x}\, dx = \int \left(\frac{5}{x} + \frac{2}{x+1} + \frac{-1}{(x+1)^2}\right) dx$$

$$= 5\int \frac{1}{x}\, dx + 2\int \frac{1}{x+1}\, dx - \int (x+1)^{-2}\, dx$$

$$= 5\ln|x| + 2\ln|x+1| - \frac{(x+1)^{-1}}{-1} + c.$$

3. $\displaystyle\int \frac{x-2}{x^3+x}\,dx.$

Solution: Using the method of partial fractions.

$$\frac{x-2}{x^3+x} = \frac{x-2}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$\frac{x-2}{x^3+x} = \frac{A(x^2+1)}{x(x^2+1)} + \frac{(Bx+C)x}{x(x^2+1)}$$

$$x-2 = A(x^2+1) + (Bx+C)x = Ax^2 + A + Bx^2 + Cx$$

$$x-2 = (A+B)x^2 + Cx + A$$

By comparing the coefficients of the two polynomials in each side:

$$\begin{array}{lcl} A+B=0 & \longrightarrow & (1) \\ C=1 & \longrightarrow & (2) \\ A=-2 & \longrightarrow & (3) \end{array}$$

From equation (1): $-2+B=0 \implies B=2.$

$$\frac{x-2}{x^3+x} = \frac{-2}{x} + \frac{2x+1}{x^2+1}$$

$$\int \frac{x-2}{x^3+x}\,dx = \int \left(\frac{-2}{x} + \frac{2x+1}{x^2+1}\right)\,dx$$

$$= \int \frac{-2}{x}\,dx + \int \frac{2x+1}{x^2+1}\,dx = -2\int \frac{1}{x}\,dx + \int \frac{2x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx$$

$$= -2\ln|x| + \ln(x^2+1) + \tan^{-1} x + c.$$

4. $\displaystyle\int \frac{3}{(x^2+1)(x^2+4)}\,dx.$

Solution: Using the method of partial fractions.

$$\frac{3}{(x^2+1)(x^2+4)} = \frac{B_1x+C_1}{x^2+1} + \frac{B_2x+C_2}{x^2+4}$$

$$\frac{3}{(x^2+1)(x^2+4)} = \frac{(B_1x+C_1)(x^2+4)}{(x^2+1)(x^2+4)} + \frac{(B_2x+C_2)(x^2+1)}{(x^2+4)(x^2+1)}$$

$$3 = (B_1x+C_1)(x^2+4) + (B_2x+C_2)(x^2+1)$$

$$3 = B_1x^3 + 4B_1x + C_1x^2 + 4C_1 + B_2x^3 + B_2x + C_2x^2 + C_2$$

$$3 = (B_1+B_2)x^3 + (C_1+C_2)x^2 + (4B_1+B_2)x + (4C_1+C_2)$$

By comparing the coefficients of the two polynomials in each side:

$$\begin{array}{lcl} B_1 + B_2 = 0 & \longrightarrow & (1) \\ C_1 + C_2 = 0 & \longrightarrow & (2) \\ 4B_1 + B_2 = 0 & \longrightarrow & (3) \\ 4C_1 + C_2 = 3 & \longrightarrow & (4) \end{array}$$

Subtracting equation (1) from equation (3): $3B_1 = 0 \implies B_1 = 0$.

From equation (1): $B_2 = 0$.

Subtracting equation (2) from equation (4): $3C_1 = 3 \implies C_1 = 1$.

From equation (1): $C_2 = -1$.

$$\frac{3}{(x^2+1)(x^2+4)} = \frac{1}{x^2+1} + \frac{-1}{x^2+4}$$

$$\int \frac{3}{(x^2+1)(x^2+4)}\, dx = \int \left(\frac{1}{x^2+1} - \frac{1}{x^2+4} \right)\, dx$$

$$= \int \frac{1}{x^2+1}\, dx - \int \frac{1}{x^2+4}\, dx = \tan^{-1} x - \frac{1}{2}\, \tan^{-1}\left(\frac{x}{2}\right) + c.$$

EXERCISES (5.4)

Evaluate the following integrals:

(1) $\displaystyle\int \frac{8x+2}{x^2+2x-8}\,dx$ (2) $\displaystyle\int \frac{5x^2+3x-2}{x^3+2x^2}\,dx$

(3) $\displaystyle\int \frac{4x^2+2x+12}{x^3+4x}\,dx$ (4) $\displaystyle\int \frac{5}{x^4+13x^2+36}\,dx$

(5) $\displaystyle\int \frac{1}{(x-1)(x^2+1)}\,dx$ (6) $\displaystyle\int \frac{-2x^3+5x^2-2x+3}{x^4+x^2}\,dx$

(7) $\displaystyle\int \frac{x^3-1}{x^2+x}\,dx$ (8) $\displaystyle\int \frac{x^2+1}{x^3-x^2}\,dx$

(9) $\displaystyle\int \frac{1}{(x+1)\ (x-1)^2}\,dx$ (10) $\displaystyle\int \frac{4x^2-13x+6}{(x+2)\ (x-2)^2}\,dx$

(11) $\displaystyle\int \frac{\cos x}{\sin^2 x+5\sin x+6}\,dx$ (12) $\displaystyle\int \frac{4\ e^x}{e^{2x}+2e^x-3}\,dx$

5.5 Miscellaneous Substitutions

First- Integrals involving fractional powers:
If the integral contains a fractional powers for the variable x, use the substitution $u = x^{\frac{1}{n}}$, where n is the least common multiple (lcm) of the denominators of these fraction powers.

Example (1): Evaluate the following integrals:

1. $\displaystyle\int \frac{dx}{\sqrt{x} + \sqrt[3]{x}}$.

   Solution:

   $$\int \frac{dx}{\sqrt{x} + \sqrt[3]{x}} = \int \frac{1}{x^{\frac{1}{2}} + x^{\frac{1}{3}}}\ dx$$

   Using the substitution $u = x^{\frac{1}{6}} \implies x = u^6$.

   $dx = 6u^5\ du$.

   $$\int \frac{1}{x^{\frac{1}{2}} + x^{\frac{1}{3}}}\ dx = \int \frac{6u^5}{u^3 + u^2}\ du$$

   $$= \int \frac{6u^5}{u^2(u+1)}\ du = \int \frac{6u^3}{u+1}\ du$$

   Using long division of polynomials.

   $$\int \frac{6u^3}{u+1}\ du = \int \left(6u^2 - 6u + 6 - \frac{6}{u+1}\right)\ du$$

   $$= 6\ \frac{u^3}{3} - 6\ \frac{u^2}{2} + 6u - 6\ln|u+1| + c = 2u^3 - 3u^2 + 6u - 6\ln|u+1| + c$$

   $$= 2\left(x^{\frac{1}{6}}\right)^3 - 3\left(x^{\frac{1}{6}}\right)^2 + 6x^{\frac{1}{6}} - 6\ln\left|x^{\frac{1}{6}} + 1\right| + c$$

   $$= 2x^{\frac{1}{2}} - 3x^{\frac{1}{3}} + 6x^{\frac{1}{6}} - 6\ln\left|x^{\frac{1}{6}} + 1\right| + c.$$

2. $\displaystyle\int \frac{dx}{\sqrt{x} + \sqrt[6]{x}}$.

   Solution:

   $$\int \frac{dx}{\sqrt{x} + \sqrt[6]{x}} = \int \frac{1}{x^{\frac{1}{2}} + x^{\frac{1}{6}}}\ dx$$

   Using the substitution $u = x^{\frac{1}{6}} \implies x = u^6$.

   $dx = 6u^5\ du$.

   $$\int \frac{1}{x^{\frac{1}{2}} + x^{\frac{1}{6}}}\ dx = \int \frac{6u^5}{u^3 + u}\ du$$

$$= \int \frac{6u^5}{u(u^2+1)}\ du = \int \frac{6u^4}{u^2+1}\ du$$

Using long division of polynomials.

$$\int \frac{6u^4}{u^2+1}\ du = \int \left(6u^2 - 6 + \frac{6}{u^2+1}\right)\ du$$

$$= 6\ \frac{u^3}{3} - 6u + 6\tan^{-1} u + c = 2u^3 - 6u + 6\tan^{-1} u + c$$

$$= 2\left(x^{\frac{1}{6}}\right)^3 - 6x^{\frac{1}{6}} + 6\tan^{-1}\left(x^{\frac{1}{6}}\right) + c = 2x^{\frac{1}{2}} - 6x^{\frac{1}{6}} + 6\tan^{-1}\left(x^{\frac{1}{6}}\right) + c.$$

Second - Integrals involving $\sqrt[n]{g(x)}$:
If the integral contains $\sqrt[n]{g(x)}$, use the substitution $u = \sqrt[n]{g(x)}$.

Example (2): Evaluate the following integrals:

1. $\int \sqrt{1+\sqrt{x}}\ dx.$

   Solution: Using the substitution

   $$u = \sqrt{1+\sqrt{x}} \implies 1+\sqrt{x} = u^2 \implies \sqrt{x} = u^2 - 1 \implies x = (u^2-1)^2$$

   $$dx = 2(u^2-1)(2u)\ du = (4u^3 - 4u)\ du$$

   $$\int \sqrt{1+\sqrt{x}}\ dx = \int u(4u^3 - 4u)\ du = \int (4u^4 - 4u^2)\ du$$

   $$= 4\ \frac{u^5}{5} - 4\ \frac{u^3}{3} + c = \frac{4}{5}\left(\sqrt{1+\sqrt{x}}\right)^5 - \frac{4}{3}\left(\sqrt{1+\sqrt{x}}\right)^3 + c.$$

2. $\int \frac{1}{\sqrt{e^x+1}}\ dx.$

   Solution: Using the substitution

   $$u = \sqrt{e^x+1} \implies e^x + 1 = u^2 \implies e^x = u^2 - 1 \implies x = \ln|u^2-1|$$

   $$dx = \frac{2u}{u^2-1}$$

   $$\int \frac{dx}{\sqrt{e^x+1}}\ dx = \int \frac{1}{u}\ \frac{2u}{u^2-1}\ du = \int \frac{2}{u^2-1}\ du$$

   Using the method of partial fractions.

   $$\frac{2}{u^2-1} = \frac{2}{(u-1)(u+1)} = \frac{A_1}{u-1} + \frac{A_2}{u+1}$$

   $$\frac{2}{u^2-1} = \frac{A_1(u+1)}{(u-1)(u+1)} + \frac{A_2(u-1)}{(u+1)(u-1)}$$

$2 = A_1(u+1) + A_2(u-1)$

Put $u = 1 : 2 = A_1(1+1) \implies 2A_1 = 2 \implies A_1 = 1.$

Put $u = -1 : 2 = A_2(-1-1) \implies -2A_2 = 2 \implies A_2 = -1.$

$$\int \frac{2}{u^2-1}\, du = \int \left(\frac{1}{u-1} + \frac{-1}{u+1} \right)\, du$$

$$= \ln|u-1| - \ln|u+1| + c$$

$$\int \frac{dx}{\sqrt{e^x+1}}\, dx = \ln\left|\sqrt{e^x+1} - 1\right| - \ln\left|\sqrt{e^x+1} + 1\right| + c.$$

EXERCISES (5.5)

Evaluate the following integrals:

(1) $\displaystyle\int \frac{1}{x^{\frac{1}{3}} + x^{\frac{1}{6}}}\, dx$ (2) $\displaystyle\int \frac{1}{x^{\frac{1}{4}} + x^{\frac{1}{3}}}\, dx$

(3) $\displaystyle\int \frac{1}{x^{\frac{1}{2}} + x^{\frac{1}{4}}}\, dx$ (4) $\displaystyle\int \sqrt{3 + \sqrt{x}}\, dx$

(5) $\displaystyle\int \frac{1}{\sqrt{2 + \sqrt{x}}}\, dx$ (6) $\displaystyle\int \frac{\sqrt{1 + \sqrt{x}}}{\sqrt{x}}\, dx$

(7) $\displaystyle\int \frac{1 - \sqrt{x}}{1 + \sqrt{x}}\, dx$ (8) $\displaystyle\int \sqrt{e^x - 1}\, dx$

Chapter 6

INDETERMINATE FORMS AND IMPROPER INTEGRALS

6.1 Indeterminate forms

First - Indeterminate form $\left(\frac{0}{0}\right)$ or $\left(\frac{\infty}{\infty}\right)$:

Theorem (L'Hôpital's rule):
Let f, g be both differentiable on an interval I that contains c (except possibly at c itself), if $g'(x) \neq 0$ for every $x \in I - \{c\}$ and if $\frac{f(x)}{g(x)}$ has the indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, and if $\lim_{x \to c} \frac{f'(x)}{g'(x)}$ exists or equals ∞ or $-\infty$ then:
$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$.

Example (1): Evaluate the following limits:

1. $\lim_{x \to 1} \frac{\ln x}{x^2 - 1}$.

   Solution:

   $\lim_{x \to 1} \frac{\ln x}{x^2 - 1} \qquad \left(\frac{0}{0}\right)$

   Using L'Hôpital's rule:

   $\lim_{x \to 1} \frac{\left(\frac{1}{x}\right)}{2x} = \lim_{x \to 1} \frac{1}{2x^2} = \frac{1}{2(1)^2} = \frac{1}{2}$

   Hence, $\lim_{x \to 1} \frac{\ln x}{x^2 - 1} = \lim_{x \to 1} \frac{\left(\frac{1}{x}\right)}{2x} = \frac{1}{2}$.

2. $\lim_{x\to 0} \frac{x-\sin x}{3x^2}$.

   Solution:

   $\lim_{x\to 0} \frac{x-\sin x}{3x^2} \qquad \left(\frac{0}{0}\right)$

   Using L'Hôpital's rule:

   $\lim_{x\to 0} \frac{1-\cos x}{6x} \qquad \left(\frac{0}{0}\right)$

   Using L'Hôpital's rule again:

   $\lim_{x\to 0} \frac{\sin x}{6} = \frac{0}{6} = 0.$

   Hence, $\lim_{x\to 0} \frac{x-\sin x}{3x^2} = \lim_{x\to 0} \frac{1-\cos x}{6x} = \lim_{x\to 0} \frac{\sin x}{6} = 0.$

3. $\lim_{x\to\infty} \frac{x}{\ln x}$.

   Solution:

   $\lim_{x\to\infty} \frac{x}{\ln x} \qquad \left(\frac{\infty}{\infty}\right)$

   Using L'Hôpital's rule:

   $\lim_{x\to\infty} \frac{1}{\left(\frac{1}{x}\right)} = \lim_{x\to\infty} x = \infty.$

   Hence, $\lim_{x\to\infty} \frac{x}{\ln x} = \lim_{x\to\infty} \frac{1}{\left(\frac{1}{x}\right)} = \infty.$

4. $\lim_{x\to\infty} \frac{x^2}{e^x}$.

   Solution:

   $\lim_{x\to\infty} \frac{x^2}{e^x} \qquad \left(\frac{\infty}{\infty}\right)$

   Using L'Hôpital's rule:

   $\lim_{x\to\infty} \frac{2x}{e^x} \qquad \left(\frac{\infty}{\infty}\right)$

   Using L'Hôpital's rule again:

   $\lim_{x\to\infty} \frac{2}{e^x} = 0.$

   Hence, $\lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{2x}{e^x} = \lim_{x\to\infty} \frac{2}{e^x} = 0.$

   Note that $e^x \to \infty$, when $x \to \infty$.

Second - Indeterminate form $(0.\infty)$:
This form can be transformed into $\left(\frac{0}{0}\right)$ or $\left(\frac{\infty}{\infty}\right)$, then use L'Hôpital's rule.

Example (2): Evaluate the following limits:

1. $\lim\limits_{x\to\infty} (x^2-1)e^{-x^2}$.

   Solution:

   $\lim\limits_{x\to\infty} (x^2-1)e^{-x^2} \qquad (\infty.0)$

   $\lim\limits_{x\to\infty} (x^2-1)e^{-x^2} = \lim\limits_{x\to\infty} \dfrac{x^2-1}{e^{x^2}} \qquad \left(\dfrac{\infty}{\infty}\right)$

   Using L'Hôpital's rule:

   $\lim\limits_{x\to\infty} \dfrac{2x}{2x\ e^{x^2}} = \lim\limits_{x\to\infty} \dfrac{1}{e^{x^2}} = 0.$

   Hence, $\lim\limits_{x\to\infty} \dfrac{x^2-1}{e^{x^2}} = \lim\limits_{x\to\infty} \dfrac{1}{e^{x^2}} = 0.$

   Note that $e^{x^2} \to \infty$, when $x \to \infty$.

2. $\lim\limits_{x\to 0^+} x^2\ \ln x$.

   Solution:

   $\lim\limits_{x\to 0^+} x^2\ \ln x \qquad (0.-\infty)$

   $\lim\limits_{x\to 0^+} x^2\ \ln x = \lim\limits_{x\to 0^+} \dfrac{\ln x}{x^{-2}} \qquad \left(\dfrac{-\infty}{\infty}\right)$

   Using L'Hôpital's rule:

   $\lim\limits_{x\to 0^+} \dfrac{\left(\frac{1}{x}\right)}{-2x^{-3}} = \lim\limits_{x\to 0^+} \dfrac{x^3}{-2x} = \lim\limits_{x\to 0^+} \dfrac{x^2}{-2} = 0.$

   Hence, $\lim\limits_{x\to 0^+} \dfrac{\ln x}{x^{-2}} = \lim\limits_{x\to 0^+} \dfrac{\left(\frac{1}{x}\right)}{-2x^{-3}} = 0.$

Third - Indeterminate form $(\infty-\infty)$:
This form can be transformed into $\left(\frac{0}{0}\right)$, then use L'Hôpital's rule.

Example (3): Evaluate the following limits:

1. $\lim\limits_{x\to 1^+} \left(\dfrac{1}{x-1} - \dfrac{1}{\ln x}\right)$.

Solution:

$$\lim_{x\to 1^+}\left(\frac{1}{x-1}-\frac{1}{\ln x}\right) \qquad (\infty-\infty)$$

$$\lim_{x\to 1^+}\left(\frac{1}{x-1}-\frac{1}{\ln x}\right) = \lim_{x\to 1^+}\frac{\ln x-(x-1)}{(x-1)\ln x}$$

$$= \lim_{x\to 1^+}\frac{\ln x - x + 1}{(x-1)\ln x} \qquad \left(\frac{0}{0}\right)$$

Using L'Hôpital's rule:

$$\lim_{x\to 1^+}\frac{\left(\frac{1}{x}-1\right)}{(x-1)\ \frac{1}{x}+\ln x}$$

$$= \lim_{x\to 1^+}\frac{\left(\frac{1-x}{x}\right)}{\left(\frac{x-1+x\ln x}{x}\right)} = \lim_{x\to 1^+}\frac{1-x}{x-1+x\ln x} \qquad \left(\frac{0}{0}\right)$$

Using L'Hôpital's rule again:

$$\lim_{x\to 1^+}\frac{-1}{1+1+\ln x} = \frac{-1}{1+1+0} = -\frac{1}{2}.$$

Hence, $\displaystyle\lim_{x\to 1^+}\frac{\ln x - x + 1}{(x-1)\ln x} = \lim_{x\to 1^+}\frac{\left(\frac{1}{x}-1\right)}{(x-1)\ \frac{1}{x}+\ln x}$

$$= \lim_{x\to 1^+}\frac{-1}{1+1+\ln x} = -\frac{1}{2}.$$

2. $\displaystyle\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{e^x-1}\right).$

Solution:

$$\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{e^x-1}\right) \qquad (\infty-\infty)$$

$$\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{e^x-1}\right) = \lim_{x\to 0^+}\left(\frac{e^x-1-x}{x(e^x-1)}\right) \qquad \left(\frac{0}{0}\right)$$

Using L'Hôpital's rule:

$$\lim_{x\to 0^+}\left(\frac{e^x-1}{(e^x-1)+xe^x}\right) = \lim_{x\to 0^+}\left(\frac{e^x-1}{(x+1)e^x-1}\right) \qquad \left(\frac{0}{0}\right)$$

Using L'Hôpital's rule again:

$$\lim_{x\to 0^+}\left(\frac{e^x}{e^x+(x+1)e^x}\right) = \lim_{x\to 0^+}\left(\frac{e^x}{e^x\ \ (1+(x+1))}\right)$$

$$= \lim_{x\to 0^+}\frac{1}{2+x} = \frac{1}{2}.$$

Hence, $\displaystyle\lim_{x\to 0^+}\left(\frac{e^x-1-x}{x(e^x-1)}\right) = \lim_{x\to 0^+}\left(\frac{e^x-1}{(e^x-1)+xe^x}\right)$

$$= \lim_{x\to 0^+} \left(\frac{e^x}{e^x + (x+1)e^x} \right) = \frac{1}{2}.$$

Fourth - Indeterminate form (1^∞) , (∞^0) , (0^0) :
Using logarithmic function, all these forms can be transformed into $(0.\infty)$.

Example (4): Evaluate the following limits:

1. $\lim\limits_{x\to 0^+} x^x$.

   Solution:

   $\lim\limits_{x\to 0^+} x^x \qquad (0^0)$

   Put $y = x^x$.

   $\ln|y| = \ln|x^x| = x\ln|x|$

   $\lim\limits_{x\to 0^+} \ln|y| = \lim\limits_{x\to 0^+} x\ln|x| \qquad (0.-\infty)$

   $$\lim_{x\to 0^+} x\ln|x| = \lim_{x\to 0^+} \frac{\ln|x|}{x^{-1}} \qquad \left(\frac{-\infty}{\infty}\right)$$

   Using L'Hôpital's rule:

   $$\lim_{x\to 0^+} \frac{\left(\frac{1}{x}\right)}{-x^{-2}} = \lim_{x\to 0^+} \frac{x^2}{-x} = \lim_{x\to 0^+} -x = 0$$

   Hence, $\lim\limits_{x\to 0^+} \dfrac{\ln|x|}{x^{-1}} = \lim\limits_{x\to 0^+} \dfrac{\left(\frac{1}{x}\right)}{-x^{-2}} = 0.$

   Therefore, $\lim\limits_{x\to 0^+} x^x = e^0 = 1.$

2. $\lim\limits_{x\to\infty} \left(1+e^{2x}\right)^{\frac{1}{x}}$.

   Solution:

   $\lim\limits_{x\to\infty} \left(1+e^{2x}\right)^{\frac{1}{x}} \qquad (\infty^0)$

   Put $y = \left(1+e^{2x}\right)^{\frac{1}{x}}$.

   $$\ln|y| = \ln\left|\left(1+e^{2x}\right)^{\frac{1}{x}}\right| = \frac{1}{x}\ \ln\left|1+e^{2x}\right| = \frac{\ln\left|1+e^{2x}\right|}{x}$$

   $$\lim_{x\to\infty} \ln|y| = \lim_{x\to\infty} \frac{\ln\left|1+e^{2x}\right|}{x} \qquad \left(\frac{\infty}{\infty}\right)$$

   Using L'Hôpital's rule:

   $$\lim_{x\to\infty} \frac{\left(\frac{2e^{2x}}{1+e^{2x}}\right)}{1} = \lim_{x\to\infty} \frac{2e^{2x}}{1+e^{2x}} \qquad \left(\frac{\infty}{\infty}\right)$$

Using L'Hôpital's rule:

$\lim\limits_{x\to\infty} \dfrac{4e^{2x}}{2e^{2x}} = \lim\limits_{x\to\infty} 2 = 2.$

Hence, $\lim\limits_{x\to\infty} \dfrac{\ln\left|1+e^{2x}\right|}{x} = \lim\limits_{x\to\infty} \dfrac{\left(\frac{2e^{2x}}{1+e^{2x}}\right)}{1} = \lim\limits_{x\to\infty} \dfrac{4e^{2x}}{2e^{2x}} = 2.$

Therefore, $\lim\limits_{x\to\infty} \left(1+e^{2x}\right)^{\frac{1}{x}} = e^2.$

3. $\lim\limits_{x\to\infty} \left(1+\dfrac{3}{x}\right)^x.$

Solution:

$\lim\limits_{x\to\infty} \left(1+\dfrac{3}{x}\right)^x \qquad (1^\infty)$

Put $y = \left(1+\dfrac{3}{x}\right)^x.$

$\ln|y| = \ln\left|\left(1+\dfrac{3}{x}\right)^x\right| = x\ln\left|1+\dfrac{3}{x}\right|$

$\lim\limits_{x\to\infty} \ln|y| = \lim\limits_{x\to\infty} x\ln\left|1+\dfrac{3}{x}\right| \qquad (\infty.0)$

$\lim\limits_{x\to\infty} x\ln\left|1+\dfrac{3}{x}\right| = \lim\limits_{x\to\infty} \dfrac{\ln\left|1+\frac{3}{x}\right|}{\left(\frac{1}{x}\right)} \qquad \left(\dfrac{0}{0}\right)$

Using L'Hôpital's rule:

$\lim\limits_{x\to\infty} \dfrac{\left(\frac{\frac{-3}{x^2}}{1+\frac{3}{x}}\right)}{\left(\frac{-1}{x^2}\right)} = \lim\limits_{x\to\infty} \dfrac{3}{1+\frac{1}{x}} = \dfrac{3}{1+0} = \dfrac{3}{1} = 3.$

Hence, $\lim\limits_{x\to\infty} \dfrac{\ln\left|1+\frac{3}{x}\right|}{\left(\frac{1}{x}\right)} = \lim\limits_{x\to\infty} \dfrac{\left(\frac{\frac{-3}{x^2}}{1+\frac{3}{x}}\right)}{\left(\frac{-1}{x^2}\right)} = 3.$

Therefore, $\lim\limits_{x\to\infty} \left(1+\dfrac{3}{x}\right)^x = e^3.$

EXERCISES (6.1)

Evaluate the following limits:

(1) $\lim_{x\to 0} \dfrac{x+1-e^x}{x^2}$ (2) $\lim_{x\to 0} \dfrac{\int_0^x \sin(t^2)\ dt}{x^3}$

(3) $\lim_{x\to 0} \dfrac{e^x + e^{-x} - 2\cos x}{x^2 + \sin x}$ (4) $\lim_{x\to \infty} \dfrac{x^2 + \ln x}{e^{2x}}$

(5) $\lim_{x\to \infty} \dfrac{1 + x^2 + e^{2x}}{2 + x + e^{3x}}$ (6) $\lim_{x\to 0^+} \left(\dfrac{1}{x} - \dfrac{1}{\sin x}\right)$

(7) $\lim_{x\to \infty} \left(\dfrac{x^2}{x-1} - \dfrac{x^2}{x+1}\right)$ (8) $\lim_{x\to \infty} \left(x\ e^{\frac{1}{x}} - x\right)$

(9) $\lim_{x\to \infty} x^3\ 5^{-x}$ (10) $\lim_{x\to \infty} x^2 \left[1 - \cos\left(\dfrac{1}{x}\right)\right]$

(11) $\lim_{x\to 0^+} (2x)^{\frac{1}{\ln x}}$ (12) $\lim_{x\to 0^+} (e^x - 1)^x$

(13) $\lim_{x\to \infty} \left(1 + \sqrt{x}\right)^{\frac{1}{x}}$ (14) $\lim_{x\to \infty} (e^x + x)^{\frac{3}{x}}$

(15) $\lim_{x\to 0^+} (1+x)^{\cot x}$ (16) $\lim_{x\to 1^-} x^{\frac{1}{x-1}}$

6.2 Improper Integrals

First - The case of unbounded interval:

1. If f is a continuous function on $[a, \infty)$, then the improper integral $\int_a^\infty f(x)\ dx$ is defined as $\int_a^\infty f(x)\ dx = \lim_{t\to\infty} \int_a^t f(x)\ dx$.

   If the limit exists then the improper integral is convergent, but if the limit does not exist or equals $\pm\infty$ then the improper integral is divergent.

2. If f is a continuous function on $(-\infty, b]$, then the improper integral $\int_{-\infty}^b f(x)\ dx$ is defined as $\int_{-\infty}^b f(x)\ dx = \lim_{t\to-\infty} \int_t^b f(x)\ dx$.

   If the limit exists then the improper integral is convergent, but if the limit does not exist or equals $\pm\infty$ then the improper integral is divergent.

3. The improper integral $\int_{-\infty}^\infty f(x)\ dx$ is defined as:

   $$\int_{-\infty}^\infty f(x)\ dx = \int_{-\infty}^a f(x)\ dx + \int_a^\infty f(x)\ dx$$

   The improper integral is convergent when both integrals in the right hand side are convergent, but if one of the integrals in the right hand side is divergent then the improper integral is also divergent.

Example (1): Discuss the convergence of the following improper integrals:

1. $\int_2^\infty \frac{1}{(x-1)^2}\ dx$

   Solution:

   $$\int_2^\infty \frac{1}{(x-1)^2}\ dx = \lim_{t\to\infty} \int_2^t \frac{1}{(x-1)^2}\ dx$$

   $$= \lim_{t\to\infty} \int_2^t (x-1)^{-2}\ dx = \lim_{t\to\infty} \left[\frac{(x-1)^{-1}}{-1}\right]_2^t = \lim_{t\to\infty} \left[\frac{-1}{(x-1)}\right]_2^t$$

   $$= \lim_{t\to\infty} \left[\frac{-1}{t-1} - \frac{-1}{2-1}\right] = (0+1) = 1.$$

   The improper integral is convergent.

2. $\displaystyle\int_2^\infty \frac{1}{x-1}\,dx.$

Solution:

$$\int_2^\infty \frac{1}{x-1}\,dx = \lim_{t\to\infty}\int_2^t \frac{1}{x-1}\,dx = \lim_{t\to\infty}\left[\ln|x-1|\right]_2^t$$

$$= \lim_{t\to\infty}\left[\ln|t-1| - \ln(1)\right] = \lim_{t\to\infty}\ln|t-1| = \infty.$$

The improper integral is divergent.

3. $\displaystyle\int_{-\infty}^0 e^{2x}\,dx.$

Solution:

$$\int_{-\infty}^0 e^{2x}\,dx = \lim_{t\to-\infty}\int_t^0 e^{2x}\,dx = \lim_{t\to-\infty}\frac{1}{2}\int_t^0 e^{2x}\,(2)\,dx$$

$$= \lim_{t\to-\infty}\left[\frac{1}{2}e^{2x}\right]_t^0 = \lim_{t\to-\infty}\left[\frac{e^0}{2} - \frac{e^{2t}}{2}\right] = \frac{1}{2} - 0 = \frac{1}{2}.$$

The improper integral is convergent.

4. $\displaystyle\int_{-\infty}^\infty \frac{1}{x^2+9}\,dx.$

Solution:

$$\int_{-\infty}^\infty \frac{1}{x^2+9}\,dx = \int_{-\infty}^0 \frac{1}{x^2+9}\,dx + \int_0^\infty \frac{1}{x^2+9}\,dx$$

$$= \lim_{s\to-\infty}\int_s^0 \frac{1}{x^2+9}\,dx + \lim_{t\to\infty}\int_0^t \frac{1}{x^2+9}\,dx$$

$$= \lim_{s\to-\infty}\int_s^0 \frac{1}{x^2+3^2}\,dx + \lim_{t\to\infty}\int_0^t \frac{1}{x^2+3^2}\,dx$$

$$= \lim_{s\to-\infty}\left[\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)\right]_s^0 + \lim_{t\to\infty}\left[\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)\right]_0^t$$

$$= \lim_{s\to-\infty}\left[\frac{1}{3}\tan^{-1}\left(\frac{0}{3}\right) - \frac{1}{3}\tan^{-1}\left(\frac{s}{3}\right)\right]$$

$$+ \lim_{t\to\infty}\left[\frac{1}{3}\tan^{-1}\left(\frac{t}{3}\right) - \frac{1}{3}\tan^{-1}\left(\frac{0}{3}\right)\right]$$

$$= \left[0 - \frac{1}{3}\,\left(-\frac{\pi}{2}\right)\right] + \left[\frac{1}{3}\,\left(\frac{\pi}{2}\right) - 0\right] = \frac{\pi}{6} + \frac{\pi}{6} = \frac{\pi}{3}$$

The improper integral is convergent.

Second - The case of unbounded function:

1. If f is continuous function on the interval $[a, b)$ and $\lim_{x\to b^-} f(x) = \pm\infty$, then the improper integral $\int_a^b f(x)\, dx$ is defined as

   $$\int_a^b f(x)\, dx = \lim_{t\to b^-} \int_a^t f(x)\, dx$$

   If the limit exists then the improper integral is convergent, but if the limit does not exist or equals $\pm\infty$ then the improper integral is divergent.

2. If f is continuous function on the interval $(a, b]$ and $\lim_{x\to a^+} f(x) = \pm\infty$, then the improper integral $\int_a^b f(x)\, dx$ is defined as

   $$\int_a^b f(x)\, dx = \lim_{t\to a^+} \int_t^b f(x)\, dx$$

   If the limit exists then the improper integral is convergent, but if the limit does not exist or equals $\pm\infty$ then the improper integral is divergent.

3. If f is continuous function on the interval $[a, b]$ except at $c \in (a, b)$ and $\lim_{x\to c^\pm} f(x) = \pm\infty$, then the improper integral $\int_a^b f(x)\, dx$ is defined as

   $$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$$

   The improper integral is convergent when both integrals in the right hand side are convergent, but if one of the integrals in the right hand side is divergent then the improper integral is also divergent.

Example (2): Discuss the convergence of the following improper integrals:

1. $\displaystyle\int_0^2 \frac{1}{\sqrt{2-x}}\, dx.$

   Solution: Note that $\displaystyle\lim_{x\to 2^-} \frac{1}{\sqrt{2-x}} = \infty.$

   $$\int_0^2 \frac{1}{\sqrt{2-x}}\, dx = \lim_{t\to 2^-} \int_0^t \frac{1}{\sqrt{2-x}}\, dx$$

   $$= \lim_{t\to 2^-} \left(-\int_0^t (2-x)^{-\frac{1}{2}}\, (-1)\, dx \right) = \lim_{t\to 2^-} \left(-\left[\frac{(2-x)^{\frac{1}{2}}}{\frac{1}{2}} \right]_0^t \right)$$

   $$= \lim_{t\to 2^-} \left(-2\left[(2-t)^{\frac{1}{2}} - (2-0)^{\frac{1}{2}} \right] \right) = -2[0 - \sqrt{2}] = 2\sqrt{2}.$$

   The improper integral is convergent.

2. $\displaystyle\int_3^4 \frac{1}{x-3}\, dx.$

Solution: Note that $\displaystyle\lim_{t\to 3^+} \frac{1}{x-3} = \infty.$

$$\int_3^4 \frac{1}{x-3}\, dx = \lim_{t\to 3^+} \int_t^4 \frac{1}{x-3}\, dx = \lim_{t\to 3^+} \left[\ln|x-3|\right]_t^4$$

$$= \lim_{t\to 3^+} \left[\ln(1) - \ln|t-3|\right] = \lim_{t\to 3^+} \left[0 - \ln|t-3|\right] = -(-\infty) = \infty.$$

The improper integral is divergent.

3. $\displaystyle\int_0^3 \frac{1}{\sqrt[3]{x-1}}\, dx.$

Solution: Note that $\displaystyle\lim_{x\to 1^+} \frac{1}{\sqrt[3]{x-1}} = \infty$ and $\displaystyle\lim_{x\to 1^-} \frac{1}{\sqrt[3]{x-1}} = -\infty.$

$$\int_0^3 \frac{1}{\sqrt[3]{x-1}}\, dx = \int_0^1 \frac{1}{\sqrt[3]{x-1}}\, dx + \int_1^3 \frac{1}{\sqrt[3]{x-1}}\, dx$$

$$= \lim_{t\to 1^-} \int_0^t \frac{1}{\sqrt[3]{x-1}}\, dx + \lim_{s\to 1^+} \int_s^3 \frac{1}{\sqrt[3]{x-1}}\, dx$$

$$= \lim_{t\to 1^-} \int_0^t (x-1)^{-\frac{1}{3}}\, dx + \lim_{s\to 1^+} \int_s^3 (x-1)^{-\frac{1}{3}}\, dx$$

$$= \lim_{t\to 1^-} \left[\frac{3}{2}(x-1)^{\frac{2}{3}}\right]_0^t + \lim_{s\to 1^+} \left[\frac{3}{2}(x-1)^{\frac{2}{3}}\right]_s^3$$

$$= \lim_{t\to 1^-} \left[\frac{3}{2}(t-1)^{\frac{2}{3}} - \frac{3}{2}(0-1)^{\frac{2}{3}}\right] + \lim_{s\to 1^+} \left[\frac{3}{2}(3-1)^{\frac{2}{3}} - \frac{3}{2}(t-1)^{\frac{2}{3}}\right]$$

$$= \left[\frac{3}{2}\ (0) - \frac{3}{2}\ (1)\right] + \left[\frac{3}{2}\ \sqrt[3]{4} - \frac{3}{2}\ (0)\right] = \frac{3}{2}\sqrt[3]{4} - \frac{3}{2}.$$

The improper integral is convergent.

4. $\displaystyle\int_0^\infty \frac{1}{\sqrt{x}\ (x+1)}\, dx.$

Solution: Note that $\displaystyle\lim_{x\to 0^+} \frac{1}{\sqrt{x}(x+1)} = \infty.$

$$\int_0^\infty \frac{1}{\sqrt{x}\ (x+1)}\, dx = \int_0^1 \frac{1}{\sqrt{x}\ (x+1)}\, dx + \int_1^\infty \frac{1}{\sqrt{x}\ (x+1)}\, dx$$

$$= \lim_{t\to 0^+} \int_t^1 \frac{1}{\sqrt{x}\ (x+1)}\, dx + \lim_{s\to \infty} \int_1^s \frac{1}{\sqrt{x}\ (x+1)}\, dx$$

$$= \lim_{t\to 0^+}\left(2\int_t^1 \frac{\left(\frac{1}{2\sqrt{x}}\right)}{\left(\sqrt{x}\right)^2+1}\,dx\right) + \lim_{s\to\infty}\left(2\int_1^s \frac{\left(\frac{1}{2\sqrt{x}}\right)}{\left(\sqrt{x}\right)^2+1}\,dx\right)$$

$$= \lim_{t\to 0^+}\left(2\left[\tan^{-1}(\sqrt{x})\right]_t^1\right) + \lim_{s\to\infty}\left(2\left[\tan^{-1}(\sqrt{x})\right]_1^s\right)$$

$$= \lim_{t\to 0^+}\left(2\left[\tan^{-1}(\sqrt{1}) - \tan^{-1}(\sqrt{t})\right]\right)$$

$$+ \lim_{s\to\infty}\left(2\left[\tan^{-1}(\sqrt{s}) - \tan^{-1}(\sqrt{1})\right]\right)$$

$$= 2\left[\frac{\pi}{4} - 0\right] + 2\left[\frac{\pi}{2} - \frac{\pi}{4}\right] = 2\left(\frac{\pi}{4}\right) + 2\left(\frac{\pi}{4}\right) = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

The improper integral is convergent.

EXERCISES (6.2)

Discuss whether the following improper integrals converge or diverge? Evaluate the convergent improper integral:

(1) $\displaystyle\int_3^{\infty} \frac{1}{(x-2)^4}\,dx$ (2) $\displaystyle\int_5^{\infty} \frac{1}{x-3}\,dx$

(3) $\displaystyle\int_{-\infty}^{-2} \frac{1}{(x+1)^2}\,dx$ (4) $\displaystyle\int_1^{\infty} e^{-x}\,dx$

(5) $\displaystyle\int_{-\infty}^{0} e^{x}\,dx$ (6) $\displaystyle\int_0^{\infty} x\,e^{-x}\,dx$

(7) $\displaystyle\int_2^{\infty} \frac{1}{x\,\ln^2 x}\,dx$ (8) $\displaystyle\int_{-\infty}^{\infty} \frac{x}{\sqrt{x^2+1}}\,dx$

(9) $\displaystyle\int_{-\infty}^{\infty} \frac{x}{1+x^4}\,dx$ (10) $\displaystyle\int_0^{2} \frac{x}{\sqrt{4-x^2}}\,dx$

(11) $\displaystyle\int_0^{2} \frac{x}{(4-x^2)^{\frac{5}{2}}}\,dx$ (12) $\displaystyle\int_0^{1} \ln x\,dx$

(13) $\displaystyle\int_1^{2} \frac{1}{x-1}\,dx$ (14) $\displaystyle\int_2^{3} \frac{1}{\sqrt{x-2}}\,dx$

(15) $\displaystyle\int_0^{4} \frac{1}{\sqrt[3]{x-2}}\,dx$ (16) $\displaystyle\int_0^{\infty} \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx$

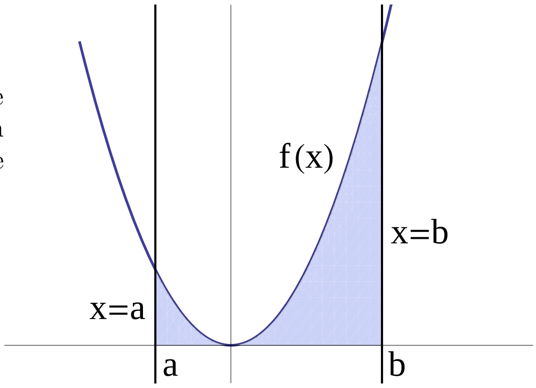
Chapter 7

APPLICATIONS OF DEFINITE INTEGRALS

7.1 Area of a plane region

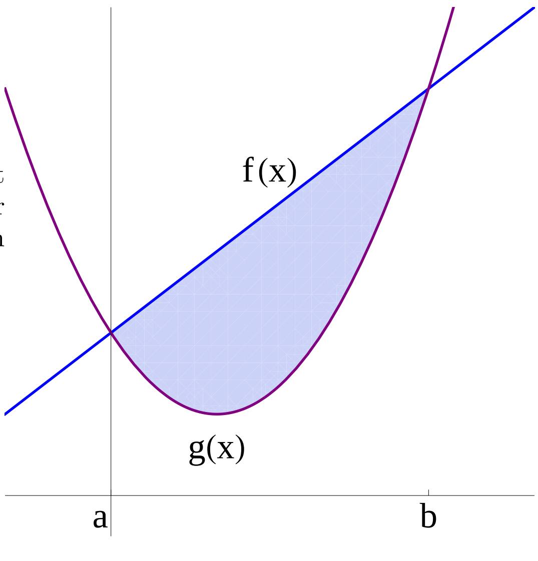
If f is a positive continuous function on the interval $[a, b]$, then the area of the region bounded by the graphs of f, the x-axis, the line $x = a$ and the line $x = b$ is

$$\mathbf{A} = \int_a^b f(x)\ dx.$$



If the two functions f and g intersect at $x = a$ and $x = b$, and if $f(x) \geq g(x)$ for every $x \in (a, b)$, then the area of the region bounded by the graphs of f and g is

$$\mathbf{A} = \int_a^b [f(x) - g(x)]\ \ dx.$$

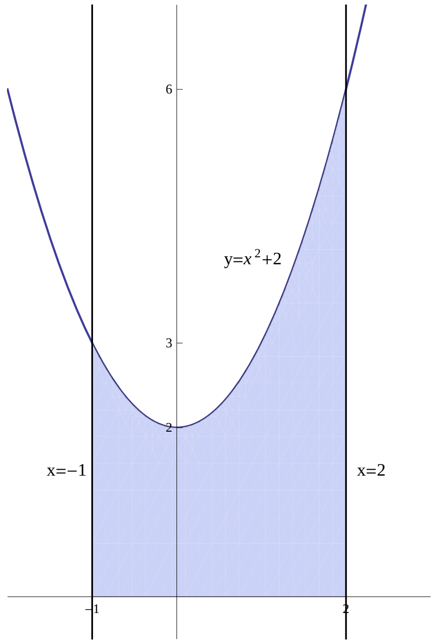


Example :

1. Find the area of the region bounded by the graphs of $y = x^2 + 2$, $x = -1$, $x = 2$ and $y = 0$.

   Solution:

   $y = x^2 + 2$ represents a parabola opens upwards, with vertex $(0, 2)$.
   $x = -1$ represents a straight line parallel to the y-axis, and passes through the point $(-1, 0)$.
   $x = 2$ represents a straight line parallel to the y-axis, and passes through the point $(2, 0)$.
   $y = 0$ represents the x-axis.



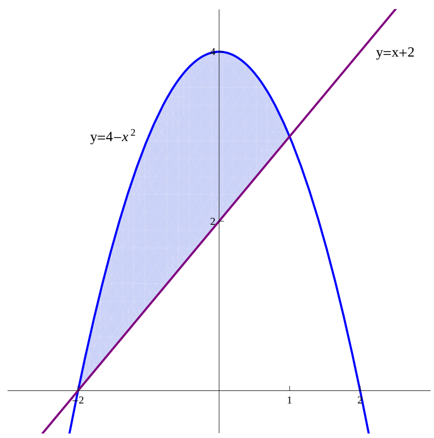
$$\mathbf{A} = \int_{-1}^{2} (x^2 + 2)\ dx = \left[\frac{x^3}{3} + 2x\right]_{-1}^{2}$$
$$= \left[\left(\frac{2^3}{3} + 2(2)\right) - \left(\frac{(-1)^3}{3} + 2(-1)\right)\right]$$
$$= \left[\left(\frac{8}{3} + 4\right) - \left(\frac{-1}{3} - 2\right)\right]$$
$$= \frac{8}{3} + 4 + \frac{1}{3} + 2 = 9.$$

2. Find the area of the region bounded by the graphs of $y = 4 - x^2$ and $y = x + 2$.

   Solution:

   $y = 4 - x^2$ represents a parabola opens downwards, with vertex $(0, 4)$.
   $y = x + 2$ represents a straight line passes through $(0, 2)$, with slope equals 1.

   Points of intersection of $y = 4 - x^2$ and $y = x + 2$:
   $x + 2 = 4 - x^2 \implies x^2 + x - 2 = 0$
   $\implies (x + 2)(x - 1) = 0$
   $\implies x = -2\ ,\ x = 1.$



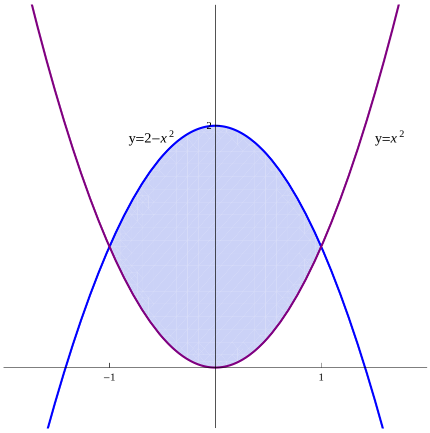
$$A = \int_{-2}^{1} \left[(4 - x^2) - (x + 2)\right]\ dx = \int_{-2}^{1} (-x^2 - x + 2)\ dx$$

$$= \left[-\frac{x^3}{3} - \frac{x^2}{2} + 2x\right]_{-2}^{1} = \left[\left(-\frac{1}{3} - \frac{1}{2} + 2\right) - \left(\frac{8}{3} - \frac{4}{2} - 4\right)\right]$$

$$= -\frac{1}{3} - \frac{8}{3} - \frac{1}{2} + 2 + 6 = -3 + 8 - \frac{1}{2} = 5 - \frac{1}{2} = \frac{9}{2}.$$

3. Find the area of the region bounded by the graphs of $y = 2 - x^2$ and $y = x^2$.

Solution:



$y = 2 - x^2$ represents a parabola opens downwards, with vertex $(0, 2)$.
$y = x^2$ represents a parabola opens upwards, with vertex $(0, 0)$.

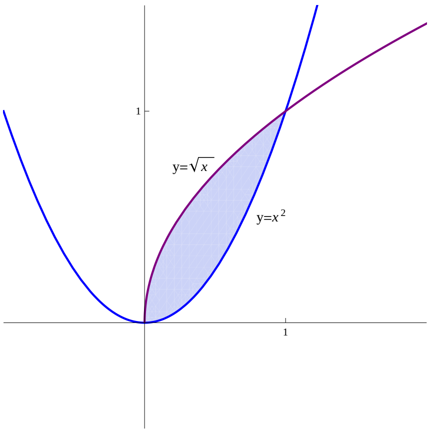
Points of intersection of $y = 2 - x^2$ and $y = x^2$:
$x^2 = 2 - x^2 \implies 2x^2 - 2 = 0 \implies x^2 - 1 = 0$
$\implies (x-1)(x+1) = 0 \implies x = -1\ ,\ x = 1.$

$$\mathbf{A} = \int_{-1}^{1} \left[(2 - x^2) - x^2\right]\, dx = \int_{-1}^{1} (2 - 2x^2)\, dx = \left[2x - 2\frac{x^3}{3}\right]_{-1}^{1}$$

$$= \left[\left(2 - \frac{2}{3}\right) - \left(-2 + \frac{2}{3}\right)\right] = 2 - \frac{2}{3} + 2 - \frac{2}{3} = 4 - \frac{4}{3} = \frac{12-4}{3} = \frac{8}{3}.$$

4. Find the area of the region bounded by the graphs of $y = x^2$ and $y = \sqrt{x}$.

Solution:



$y = x^2$ represents a parabola opens upwards, with vertex $(0, 0)$.
$y = \sqrt{x}$ represents the upper-half of the parabola $x = y^2$ which opens to the right, with vertex $(0, 0)$.
Points of intersection of $y = x^2$ and $y = \sqrt{x}$:
$x^2 = \sqrt{x} \implies x^4 = x \implies x^4 - x = 0$
$\implies x(x^3 - 1) = 0 \implies x = 0\ ,\ x = 1.$

$$\mathbf{A} = \int_{0}^{1} \left[\sqrt{x} - x^2\right]\, dx = \int_{0}^{1} \left[x^{\frac{1}{2}} - x^2\right]\, dx = \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{x^3}{3}\right]_{0}^{1}$$

$$= \left[\left(\frac{2}{3} - \frac{1}{3}\right) - (0 - 0)\right] = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}.$$

EXERCISES (7.1)

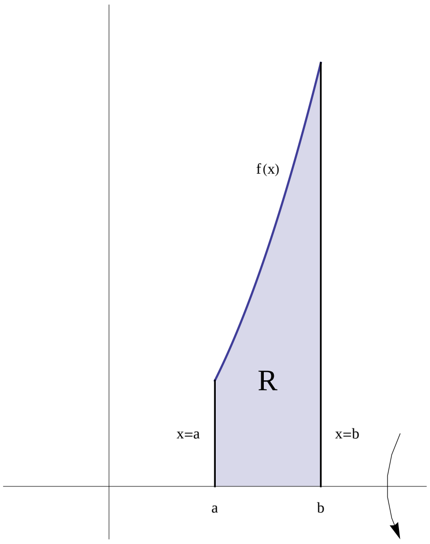
Find the area of the region bounded by the graphs of the follwing functions:

1. $y = x^2 + 1$, $y = 0$, $x = -2$, $x = 3$
2. $y = e^x$, $y = 0$, $x = 0$, $x = 5$
3. $y = \ln x$, $y = 0$, $x = 4$
4. $y = 4 - x^2$, $y = 0$
5. $y = x^2 + 1$, $y = 2$
6. $y = 6 - x^2$, $y = 2$
7. $y = x^2 + 3$, $y = 1$, $x = 0$, $x = 4$
8. $y = 1 - x^2$, $y = 4$, $x = -1$, $x = 1$
9. $y = x^2 + 2$, $y = x + 2$
10. $y = x^2 + 1$, $y = 1 - x$
11. $y = x^2 + 4$, $y = 4 - x$, $x = 4$
12. $y = x^2 + 1$, $y = 1 - x^2$, $x = -1$, $x = 3$
13. $y = x^2 - 3$, $y = 5 - x^2$
14. $y = 2x^2 + 1$, $y = x^2 + 5$
15. $y = x^2$, $y = x^2 - 4x + 4$, $y = 0$
16. $y = \sqrt{x - 1} + 2$, $y = 2$, $y = 3$, $x = 0$

7.2 Solid of Revolution

First - Disk Method:

If f is a positive continuous function defined on the interval $[a, b]$, and if R is the region bounded by the graph of f, the x-axis, the two lines $x = a$ and $x = b$, then the volume V of the solid of revolution generated by revolving R about the x-axis is $V = \pi \int_a^b [f(x)]^2\, dx$.



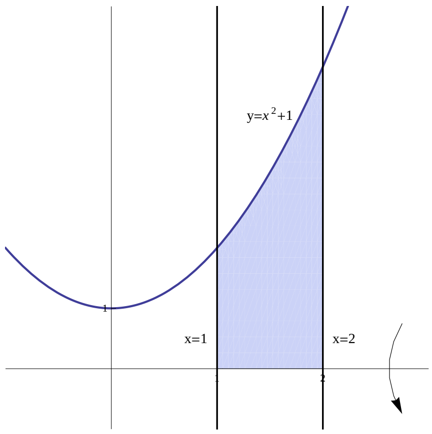
Remark (1): The disk method is used when the region R touches entirely the axis of revolution.

Example (1):

1. Find the volume of the solid of revolution generated by revolving the region bounded by the graphs of $y = x^2 + 1$, $x = 1$, $x = 2$ and $y = 0$, about the x-axis.

   solution:

   $y = x^2 + 1$ represents a parabola opens upwards, with vertex $(0, 1)$.
   $x = 1$ represents a straight line parallel to the y-axis and passes through $(1, 0)$.
   $x = 2$ represents a straight line parallel to the y-axis and passes through $(2, 0)$.
   $y = 0$ represents the y-axis.

   Using Disk method:

   $$\mathbf{V} = \pi \int_1^2 \left(x^2 + 1\right)^2\ dx = \pi \int_1^2 \left(x^4 + 2x^2 + 1\right)\, dx$$

   $$= \pi \left[\frac{x^5}{5} + 2\frac{x^3}{3} + x\right]_1^2 = \pi \left[\left(\frac{32}{5} + \frac{16}{3} + 1\right) - \left(\frac{1}{5} + \frac{2}{3} + 1\right)\right]$$

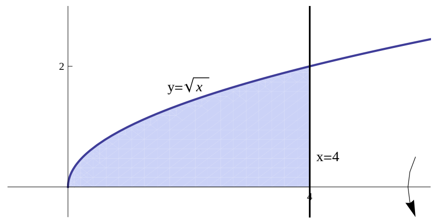
$$= \pi\left[\frac{32}{5} - \frac{1}{5} + \frac{16}{3} - \frac{2}{3} + 1\right] = \pi\left(\frac{31}{5} + \frac{14}{3} + 1\right)$$

$$= \left(\frac{93 + 70 + 15}{15}\right)\ \pi = \frac{178}{15}\ \pi.$$

2. Find the volume of the solid of revolution generated by revolving the region bounded by the graphs of $y = \sqrt{x}$, $x = 4$ and $y = 0$, about the x-axis.

solution:

$y = \sqrt{x}$ represents the upper-half of the parabola $x = y^2$, which opens to the right, with vertex $(0, 0)$.
$x = 4$ represents a straight line parallel to the y-axis and passes through $(4, 0)$.
$y = 0$ represents the y-axis.



Using Disk method:

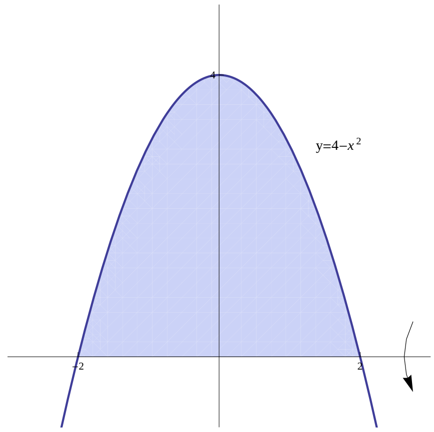
$$\mathbf{V} = \pi\int_0^4 \left(\sqrt{x}\right)^2\ dx = \pi\int_0^4 x\ dx = \pi\left[\frac{x^2}{2}\right]_0^4 = \pi\left(\frac{16}{2} - \frac{0}{2}\right) = 8\pi$$

3. Find the volume of the solid of revolution generated by revolving the region bounded by the graphs of $y = 4 - x^2$ and $y = 0$, about the x-axis.

solution:

$y = y = 4 - x^2$ represents a parabola opens downwards, with vertex $(4, 0)$.
$y = 0$ represents the y-axis.

Points of intersection of $y = 4 - x^2$ and $y = 0$:
$4 - x^2 = 0 \implies x^2 = 4 \implies x = -2\ ,\ x = 2.$



Using Disk method:

$$\mathbf{V} = \pi\int_{-2}^{2} (4 - x^2)^2\ dx = \pi\int_{-2}^{2} (16 - 8x^2 + x^4)\ dx$$

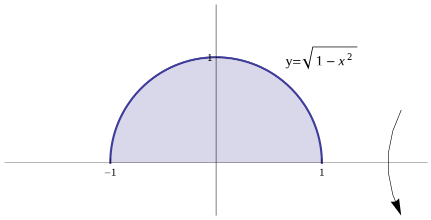
$$= \pi\left[16x - \frac{8}{3}x^3 + \frac{x^5}{5}\right]_{-2}^{2} = \pi\left[\left(32 - \frac{64}{3} + \frac{32}{5}\right) - \left(-32 + \frac{64}{3} - \frac{32}{3}\right)\right]$$

$$= \pi\left(32 - \frac{64}{3} + \frac{32}{5} + 32 - \frac{64}{3} + \frac{32}{5}\right) = \pi\left(64 - \frac{128}{3} + \frac{64}{5}\right) = \frac{512}{15}\pi$$

4. Find the volume of the solid of revolution generated by revolving the region bounded by the graphs of $y = \sqrt{1-x^2}$ and $y = 0$, about the x-axis.

solution:

$y = \sqrt{1-x^2}$ represents the upper half of the circle $x^2 + y^2 = 1$ of center $(0,0)$ and radius 1.
$y = 0$ represents the x-axis.



Points of intersection of $y = \sqrt{1-x^2}$ and $y = 0$:
$\sqrt{1-x^2} = 0 \implies 1 - x^2 = 0 \implies x^2 = 1$
$\implies x = -1 \ , \ x = 1$

Using Disk method:

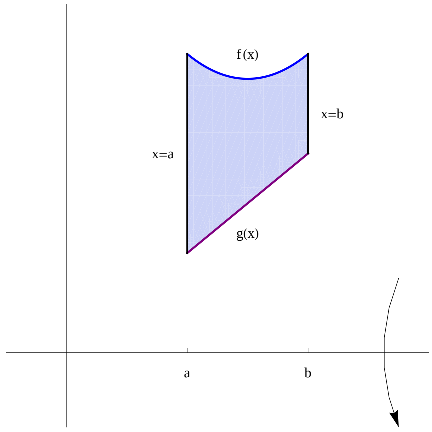
$$\mathbf{V} = \pi \int_{-1}^{1} \left(\sqrt{1-x^2}\right)^2 \, dx = \pi \int_{-1}^{1} (1 - x^2) \, dx = \pi \left[x - \frac{x^3}{3}\right]_{-1}^{1}$$

$$= \pi\left[\left(1 - \frac{1}{3}\right) - \left(-1 + \frac{1}{3}\right)\right] = \pi\left(1 - \frac{1}{3} + 1 - \frac{1}{3}\right)$$

$$= \pi\left(2 - \frac{2}{3}\right) = \frac{4}{3}\pi.$$

Second - Washer Method:

If f, g are two positive continuous functions on $[a, b]$, and if $f(x) \geq g(x)$ for every $x \in [a, b]$, and R is the region bounded by the graphs of f, g, then the volume V of the solid of revolution generated by revolving R about the x-axis is $\mathbf{V} = \pi \int_a^b \left([f(x)]^2 - [g(x)]^2\right)\, dx$.



Remark (2) : The washer method is used when the region R does not touch entirely the axis of revolution.

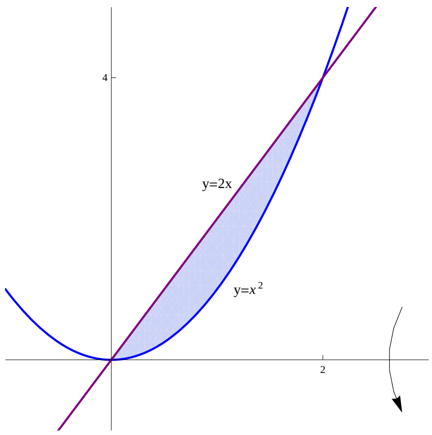
Example (2):

1. Find the volume of the solid of revolution generated by revolving the region bounded by the graphs of $y = x^2$ and $y = 2x$, about the x-axis.

   solution:

   $y = x^2$ represents a parabola opens upwards, with vertex $(0, 0)$.
   $y = 2x$ represents a straight line passes through $(0, 0)$, with slope 2.

   Points of intersection of $y = x^2$ and $y = 2x$:
   $x^2 = 2x \implies x^2 - 2x = 0 \implies x(x - 2) = 0$
   $\implies x = 0 \ , \ x = 2.$

   Using Washer method :

   $$\mathbf{V} = \pi \int_0^2 \left[(2x)^2 - (x^2)^2\right]\, dx = \pi \int_0^2 \left(4x^2 - x^4\right)\, dx$$

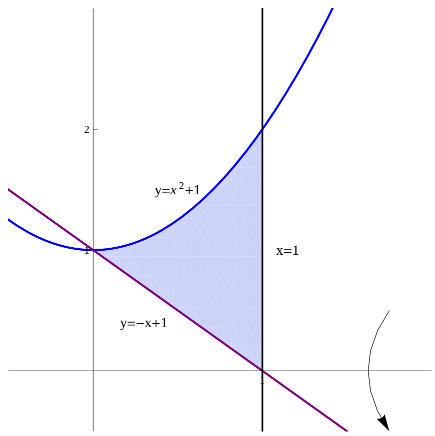
   $$= \pi \left[\frac{4}{3}x^3 - \frac{x^5}{5}\right]_0^2 = \pi \left[\left(\frac{4}{3}(8) - \frac{32}{5}\right) - \left(\frac{4}{3}(0) - \frac{0}{5}\right)\right]$$

   $$= \pi \left(\frac{32}{3} - \frac{32}{5}\right) = 32\pi \left(\frac{1}{3} - \frac{1}{5}\right) = 32\pi \left(\frac{2}{15}\right) = \frac{64}{15}\pi.$$

2. Find the volume of the solid of revolution generated by revolving the region bounded by the graphs of $y = x^2 + 1$, $y = -x + 1$ and $x = 1$, about the x-axis.

solution:

$y = x^2 + 1$ represents a parabola opens upwards, with vertex $(0, 1)$.
$y = -x + 1$ represents a straight line passes through $(0, 1)$, with slope -1.
$x = 1$ represents a straight line parallel to the y-axis and passes through $(1, 0)$.



Points of intersection of $y = x^2 + 1$ and $y = -x + 1$:
$x^2 + 1 = -x + 1 \implies x^2 + x = 0$
$\implies x(x + 1) = 0 \implies x = -1\ ,\ x = 0.$

Point of intersection of $y = -x + 1$ and $x = 1$ is $(1, 0)$.

Using Washer method :

$$\mathbf{V} = \pi \int_0^1 \left[(x^2 + 1)^2 - (-x + 1)^2\right]\ dx = \pi \int_0^1 \left[(x^4 + 2x^2 + 1) - (x^2 - 2x + 1)\right]\ dx$$

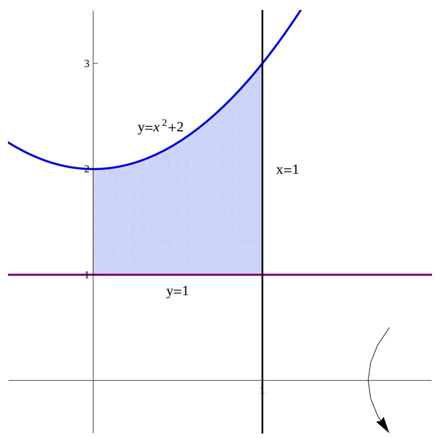
$$= \pi \int_0^1 \left(x^4 + x^2 + 2x\right)\ dx = \pi \left[\frac{x^5}{5} + \frac{x^3}{3} + x^2\right]_0^1$$

$$= \pi \left[\left(\frac{1}{5} + \frac{1}{3} + 1\right) - (0 + 0 + 0)\right] = \frac{23}{15}\pi.$$

3. Find the volume of the solid of revolution generated by revolving the region bounded by the graphs of $y = x^2 + 2$, $y = 1$, $x = 1$ and $x = 0$, about the x-axis.

solution:

$y = x^2 + 2$ represents a parabola opens upwards, with vertex $(0, 2)$.
$y = 1$ represents a straight line parallel to the x-axis and passes through $(0, 1)$.
$x = 1$ represents a straight line parallel to the y-axis and passes through $(1, 0)$.
$x = 0$ represents the y-axis.



Using Washer method :

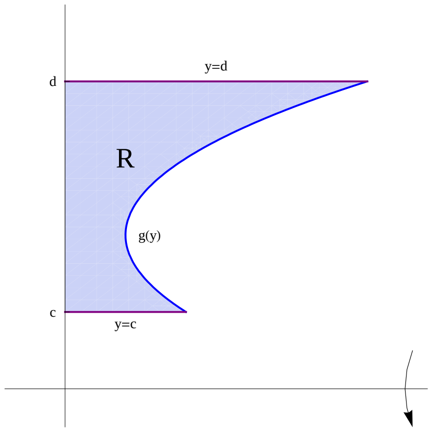
$$V = \pi \int_0^1 \left[(x^2+2)^2 - (1)^2\right] \, dx = \pi \int_0^1 (x^4 + 4x^2 + 4 - 1) \, dx$$

$$= \pi \int_0^1 (x^4 + 4x^2 + 3) \, dx = \pi \left[\frac{x^5}{5} + \frac{4}{3}x^3 + 3x\right]_0^1$$

$$= \pi \left[\left(\frac{1}{5} + \frac{4}{3} + 3\right) - (0+0+0)\right] = \frac{68}{15}\pi.$$

Third - Cylindrical Shells Method:

If g is a positive continuous function on the interval $[c, d]$, and R is the region bounded by the graphs of g, $y = c$, $y = d$ and the y-axis, then the volume V of the solid of revolution generated by revolving R about the x-axis is $\mathbf{V} = 2\pi \int_c^d y \; g(y) \; dy$.



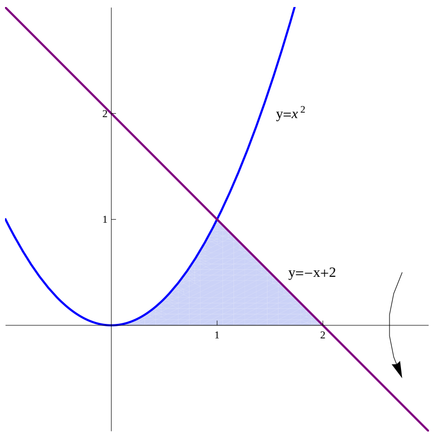
Example (3):

1. Find the volume of the solid of revolution generated by revolving the region bounded by the graphs of $y = x^2$, $y = -x+2$ and $y = 0$, about the x-axis.

   solution:

   $y = x^2$ represents a parabola opens upwards, with vertex $(0, 0)$.
   $y = -x + 2$ represents a straight line passes through $(0, 2)$, with slope -1.
   $y = 0$ represents the x-axis.

   Points of intersection of $y = x^2$ and $y = -x + 2$:
   $x^2 = -x + 2 \implies x^2 + x - 2 = 0$
   $\implies (x + 2)(x - 1) = 0 \implies x = -2 \; , \; x = 1$
   $\implies y = 4 \; , \; y = 1.$



   $y = x^2 \implies x = \sqrt{y}.$

   $y = -x + 2 \implies x = -y + 2.$

   Using cylindrical shells method :

   $$\mathbf{V} = 2\pi \int_0^1 y \left[(-y + 2) - \sqrt{y}\right] \; dy = 2\pi \int_0^1 y \left(-y - y^{\frac{1}{2}} + 2\right) \; dy$$

   $$= 2\pi \int_0^1 \left(-y^2 - y^{\frac{3}{2}} + 2y\right) \; dy = 2\pi \left[-\frac{y^3}{3} - \frac{2}{5} y^{\frac{5}{2}} + y^2\right]_0^1$$

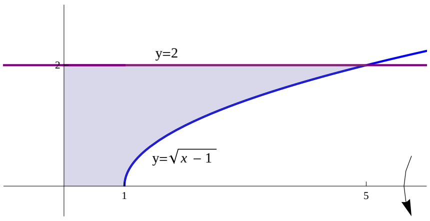
   $$= 2\pi \left[\left(-\frac{1}{3} - \frac{2}{5} + 1\right) - (0 - 0 + 0)\right]$$

   $$= 2\pi \left(\frac{-5 - 6 + 15}{15}\right) = 2\pi \left(\frac{4}{15}\right) = \frac{8\pi}{15}.$$

2. Find the volume of the solid of revolution generated by revolving the region bounded by the graphs of $y = \sqrt{x-1}$, $y = 2$, $y = 0$ and $x = 0$, about the x-axis.

solution:

$\sqrt{x-1}$ represents the upper half of the parabola $x = y^2 + 1$, which opens to the right, with vertex $(1, 0)$.
$y = 2$ represents a straight line parallel to the x-axis and passes through $(0, 2)$.
$y = 0$ represents the x-axis.
$x = 0$ represents the y-axis.



$y = \sqrt{x-1} \implies y^2 = x - 1 \implies x = y^2 + 1.$

Using cylindrical shells method :

$$\mathbf{V} = 2\pi \int_0^2 y\ (y^2+1)\ dy = 2\pi \int_0^2 (y^3 + y)\ dy = 2\pi \left[\frac{y^4}{4} + \frac{y^2}{2}\right]_0^2$$

$$= 2\pi \left[\left(\frac{16}{4} + \frac{4}{2}\right) - (0+0)\right] = 2\pi\,(4+2) = 12\pi.$$

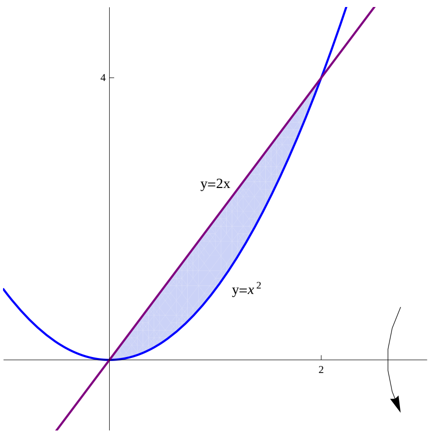
3. Find the volume of the solid of revolution generated by revolving the region bounded by the graphs of $y = x^2$, and $y = 2x$, about the x-axis.

solution:

$y = x^2$ represents a parabola opens upwards, with vertex $(0, 0)$.
$y = 2x$ represents a straight line passes through $(0, 0)$, with slope 1.

Points of intersection of $y = x^2$ and $y = 2x$:
$x^2 = 2x \implies x^2 - 2x = 0 \implies x(x-2) = 0.$
$\implies x = 0\ ,\ x = 2 \implies y = 0\ ,\ y = 4.$

$y = x^2 \implies x = \sqrt{y}.$
$y = 2x \implies x = \frac{1}{2}y.$



Using cylindrical shells method :

$$\mathbf{V} = 2\pi \int_0^4 y\ \left(\sqrt{y} - \frac{1}{2}y\right)\ dy = 2\pi \int_0^4 \left(y^{\frac{3}{2}} - \frac{1}{2}y^2\right)\ dy$$

$$= 2\pi \left[\frac{2}{5}y^{\frac{5}{2}} - \frac{1}{2}\frac{y^3}{3}\right]_0^4 = 2\pi \left[\left(\frac{2}{5}(4)^{\frac{5}{2}} - \frac{(4)^3}{6}\right) - (0-0)\right]$$

$$= 2\pi \left(\frac{64}{5} - \frac{64}{6}\right) = 128\pi \left(\frac{1}{5} - \frac{1}{6}\right) = \frac{128}{30}\pi = \frac{64}{15}\pi.$$

EXERCISES (7.2)

Find the volume of the solid of revolution generated by revolving the region bounded by the graphs of the following functions, about the x-axis:

1. $y = x^2$, $y = 0$, $x = 2$
2. $y = e^x$, $y = 0$, $x = 0$, $x = 2$
3. $y = \sqrt{x-1}$, $y = 0$, $x = 5$
4. $y = \frac{r}{h}x$, $y = 0$, $x = h$ where $r, h \in \mathbb{R}^+$ (The volume of the cone).
5. $y = -x^2 + 2x$, $y = 0$
6. $y = x^2 + 1$, $y = 3x + 1$
7. $y = 4 - x^2$, $y = 4 + x^2$, $x = 2$
8. $y = 1 - x^2$, $y = 3$, $x = 0$, $x = 1$
9. $y = x^2$, $y = 2 - x^2$
10. $y = 4 - x^2$, $y = x + 4$, $x = 2$
11. $y = x^2 - 4x + 4$, $y = x$, $y = 0$
12. $y = \sqrt{x}$, $y = 1$, $y = 2$, $x = 0$
13. $x = y^2 - 4y + 5$, $y = 1$, $y = 4$, $x = 0$

7.3 Arc Length

If f is a differentiable function on $[a, b]$, then the arc length (curve length) of f from $x = a$ to $x = b$ is $\mathbf{L} = \int_a^b \sqrt{1 + [f'(x)]^2}\ dx$.

Example :

1. Find the arc length of the function $y = 1 + \frac{2}{3}x^{\frac{3}{2}}$, from $x = 0$ to $x = 3$.

   solution:

   $f(x) = 1 + \frac{2}{3}x^{\frac{3}{2}} \implies f'(x) = x^{\frac{1}{2}}$

   $\mathbf{L} = \int_0^3 \sqrt{1 + \left(x^{\frac{1}{2}}\right)^2}\ dx = \int_0^3 \sqrt{1+x}\ dx = \int_0^3 (1+x)^{\frac{1}{2}}\ dx$

   $= \left[\frac{2}{3}(1+x)^{\frac{3}{2}}\right]_0^3 = \left[\frac{2}{3}(1+3)^{\frac{3}{2}} - \frac{2}{3}(1+0)^{\frac{3}{2}}\right]$

   $= \frac{2}{3}(8) - \frac{2}{3}(1) = \frac{16-2}{3} = \frac{14}{3}.$

2. Find the arc length of the function $y = \cosh x$, from $x = 0$ to $x = \ln 2$.

   solution:

   $f(x) = \cosh x \implies f'(x) = \sinh x$

   $\mathbf{L} = \int_0^{\ln 2} \sqrt{1 + (\sinh x)^2}\ dx = \int_0^{\ln 2} \sqrt{1 + \sinh^2 x}\ dx$

   $= \int_0^{\ln 2} \sqrt{\cosh^2 x}\ dx = \int_0^{\ln 2} \cosh x\ dx = [\sinh x]_0^{\ln 2}$

   $= \sinh(\ln 2) - \sinh(0) = \sinh(\ln 2) = \frac{e^{\ln 2} - e^{-\ln 2}}{2} = \frac{2 - \frac{1}{2}}{2} = \frac{3}{4}.$

   Note that: $\cosh^2 x - \sinh^2 x = 1 \implies \cosh^2 x = 1 + \sinh^2 x.$

3. Find the arc length of the function $y = \frac{1}{4}x^2 - \frac{1}{2}\ln x$, from $x = 1$ to $x = 2$.

   solution:

   $f(x) = \frac{1}{4}x^2 - \frac{1}{2}\ln x \implies f'(x) = \frac{1}{2}x - \frac{1}{2}\frac{1}{x} = \frac{x}{2} - \frac{1}{2x}$

   $\mathbf{L} = \int_1^2 \sqrt{1 + \left(\frac{x}{2} - \frac{1}{2x}\right)^2}\ dx = \int_1^2 \sqrt{1 + \left(\frac{x^2}{4} - \frac{1}{2} + \frac{1}{4x^2}\right)}\ dx$

$$= \int_1^2 \sqrt{\frac{x^2}{4} + \frac{1}{2} + \frac{1}{4x^2}}\ dx = \int_1^2 \sqrt{\left(\frac{x}{2} + \frac{1}{2x}\right)^2}\ dx = \int_1^2 \left|\frac{x}{2} + \frac{1}{2x}\right|\ dx$$

$$= \int_1^2 \left(\frac{x}{2} + \frac{1}{2x}\right)\ dx = \left[\frac{x^2}{4} + \frac{1}{2}\ln x\right]_1^2 = \left[\left(1 + \frac{1}{2}\ln 2\right) - \left(\frac{1}{4} + \frac{1}{2}\ln 1\right)\right]$$

$$= 1 + \frac{1}{2}\ln 2 - \frac{1}{4} = \frac{3}{4} + \frac{\ln 2}{2}.$$

4. Find the arc length of the function $y = \sqrt{4 - x^2}$, from $x = -2$ to $x = 2$.

solution:

$$f(x) = \sqrt{4 - x^2} \implies f'(x) = \frac{-2x}{2\sqrt{4 - x^2}} = \frac{-x}{\sqrt{4 - x^2}}$$

$$\mathbf{L} = \int_{-2}^2 \sqrt{1 + \left(\frac{-x}{\sqrt{4 - x^2}}\right)^2}\ dx = \int_{-2}^2 \sqrt{1 + \frac{x^2}{4 - x^2}}\ dx$$

$$= \int_{-2}^2 \sqrt{\frac{(4 - x^2) + x^2}{4 - x^2}}\ dx = \int_{-2}^2 \sqrt{\frac{4}{4 - x^2}}\ dx = 2\int_{-2}^2 \frac{1}{\sqrt{4 - x^2}}\ dx$$

$$= 2\left[\sin^{-1}\left(\frac{x}{2}\right)\right]_{-2}^2 = 2\left[\sin^{-1}(1) - \sin^{-1}(-1)\right] = 2\left(\frac{\pi}{2} - \frac{-\pi}{2}\right) = 2\pi.$$

EXERCISES (7.3)

Find the arc length of the following functions on the given intervals:

1. $y = \pi + \frac{2}{3}x\sqrt{x}$, on the interval $[0, 8]$
2. $y = \ln|\sec x|$ on the interval $\left[0, \frac{\pi}{4}\right]$
3. $y = \frac{1}{3}\left(x^2 + 2\right)^{\frac{3}{2}}$ on the interval $[0, 1]$
4. $y = \frac{e^{2x} + e^{-2x}}{4}$ on the interval $[0, 1]$
5. $y = \frac{x^3}{6} + \frac{1}{2x}$ on the interval $[1, 3]$
6. $y = \frac{1}{3}x^{\frac{3}{2}} - \sqrt{x}$ on the interval $[1, 4]$

7.4 Surface of revolution

If f is a positive differentiable function on $[a,b]$, then the area $\mathbf{S}$ of surface of revolution generated by revolving f from $x=a$ to $x=b$, about the x-axis is $\mathbf{S} = 2\pi \int_a^b f(x)\sqrt{1+[f'(x)]^2}\ dx$.

Example :

1. Find the area of surface of revolution generated by revolving $y = \frac{1}{3}x^3$, where $0 \le x \le 1$, about the x-axis.

   solution:

   $f(x) = \frac{1}{3}x^3 \implies f'(x) = x^2$

   $$\mathbf{S} = 2\pi \int_0^1 \frac{1}{3}x^3\sqrt{1+(x^2)^2}\ dx = \frac{2\pi}{3}\int_0^1 x^3\sqrt{1+x^4}\ dx$$

   $$= \frac{2\pi}{3}\frac{1}{4}\int_0^1 \left(1+x^4\right)^{\frac{1}{2}}(4x^3)\ dx = \frac{\pi}{6}\left[\frac{2}{3}(1+x^4)^{\frac{3}{2}}\right]_0^1$$

   $$= \frac{\pi}{6}\left[\frac{2}{3}(1+1)^{\frac{3}{2}} - \frac{2}{3}(1+0)^{\frac{3}{2}}\right] = \frac{\pi}{6}\left(\frac{2}{3}\sqrt{8} - \frac{2}{3}\right) = \frac{\pi}{9}\left(\sqrt{8}-1\right).$$

2. Find the area of surface of revolution generated by revolving $y = \sqrt{x}$, where $1 \le x \le 4$, about the x-axis.

   solution:

   $f(x) = \sqrt{x} \implies f'(x) = \frac{1}{2\sqrt{x}}$

   $$\mathbf{S} = 2\pi \int_1^4 \sqrt{x}\ \sqrt{1+\left(\frac{1}{2\sqrt{x}}\right)^2}\ dx = 2\pi\int_1^4 \sqrt{x}\ \sqrt{1+\frac{1}{4x}}\ dx$$

   $$= 2\pi\int_1^4 \sqrt{x}\ \sqrt{\frac{4x+1}{4x}}\ dx = 2\pi\int_1^4 \sqrt{x}\ \frac{\sqrt{4x+1}}{2\sqrt{x}}\ dx$$

   $$= 2\pi\ \frac{1}{2}\int_1^4 \sqrt{4x+1}\ dx = \pi\int_1^4 (4x+1)^{\frac{1}{2}}\ dx = \pi\ \frac{1}{4}\int_1^4 (4x+1)^{\frac{1}{2}}(4)\ dx$$

   $$= \frac{\pi}{4}\left[\frac{2}{3}(4x+1)^{\frac{3}{2}}\right]_1^4 = \frac{\pi}{4}\left[\frac{2}{3}(17)^{\frac{3}{2}} - \frac{2}{3}(5)^{\frac{3}{2}}\right] = \frac{\pi}{6}\left[(17)^{\frac{3}{2}} - (5)^{\frac{3}{2}}\right].$$

3. Find the area of surface of revolution generated by revolving $y = \sqrt{9 - x^2}$, where $-3 \leq x \leq 3$, about the x-axis.

solution:

$$f(x) = \sqrt{9 - x^2} \implies f'(x) = \frac{-2x}{2\sqrt{9 - x^2}} = \frac{-x}{\sqrt{9 - x^2}}$$

$$\mathbf{S} = 2\pi \int_{-3}^{3} \sqrt{9 - x^2} \sqrt{1 + \left(\frac{-x}{\sqrt{9 - x^2}} \right)^2} \ dx$$

$$= 2\pi \int_{-3}^{3} \sqrt{9 - x^2} \sqrt{1 + \frac{x^2}{9 - x^2}} \ dx$$

$$= 2\pi \int_{-3}^{3} \sqrt{9 - x^2} \sqrt{\frac{(9 - x^2) + x^2}{9 - x^2}} \ dx = 2\pi \int_{-3}^{3} \sqrt{9 - x^2} \sqrt{\frac{9}{9 - x^2}} \ dx$$

$$= 2\pi \int_{-3}^{3} \sqrt{9 - x^2} \ \frac{3}{\sqrt{9 - x^2}} \ dx = 6\pi \int_{-3}^{3} 1 \ dx$$

$$= 6\pi \left[x \right]_{-3}^{3} = 6\pi[3 - (-3)] = 6\pi \ (6) = 36\pi.$$

EXERCISES (7.4)

Find the area of surface of revolution generated by revolving the following functions on the given intervals, about the x-axis :

1. $y = \frac{1}{3}\left(3\sqrt{x} - x^{\frac{3}{2}}\right)$ on the interval $[1, 3]$
2. $y = \frac{x^3}{6} + \frac{1}{2x}$ on the interval $[1, 2]$
3. $y = \frac{x^4}{4} + \frac{1}{8x^2}$ on the interval $[1, 3]$
4. $y = r$ on the interval $[0, h]$, where $r, h \in \mathbb{R}^+$ (Surface area of a cylinder).
5. $y = \frac{r}{h}x$ on the interval $[0, h]$, where $r, h \in \mathbb{R}^+$ (surface area of a cone).

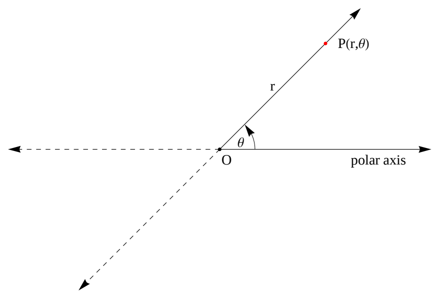
Chapter 8

POLAR COORDINATES

8.1 Polar Coordinates

Every point in the Cartesian plane is represented by the ordered pair (a, b), where a is the x coordinate and b is the y coordinate.
Every point in the plane can be represented by another method called the polar coordinates.
The polar plane consists of the pole and polar axis, the pole is the origin in the Cartesian plane and the polar axis is the x-axis in the Cartesian plane.

If P is any point in the plane, the polar axis is moved in the positive direction (counter clockwise) until the point P is reached. The distance between P and the pole is denoted by r and the angle made by the polar axis is denoted by θ. The ordered pair (r, θ) is the polar representation of the point P.

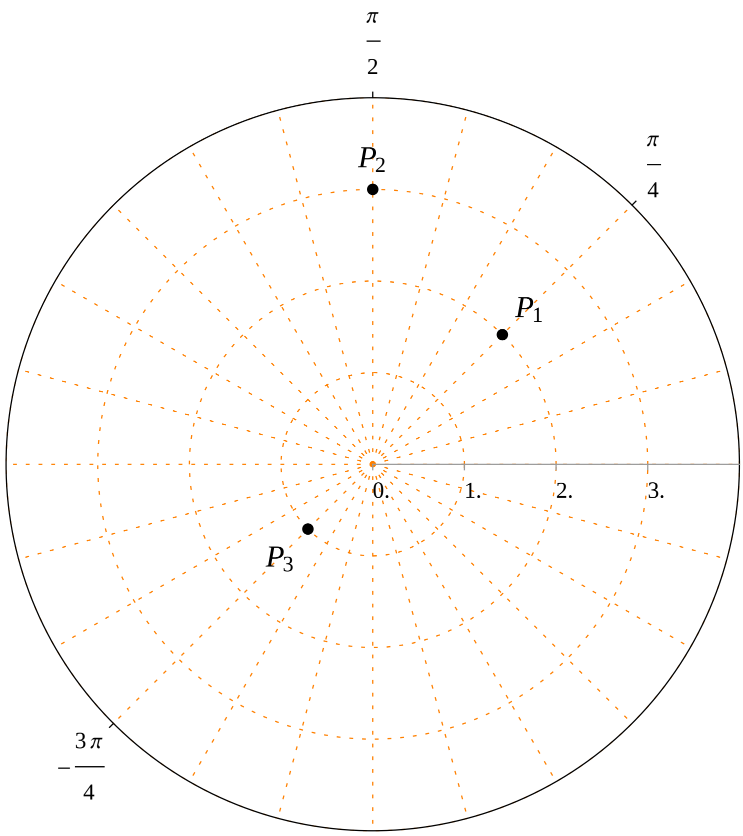


Remarks :

- The angle θ in the polar coordinates is measured by radians.
- The polar representation of the pole is $(0, \theta)$, where θ is any angle.
- The polar representation is not unique. For example :

  The ordered pairs $\left(2, \frac{\pi}{4}\right)$, $\left(-2, \frac{5\pi}{4}\right)$, $\left(-2, -\frac{3\pi}{4}\right)$ represent the same point in the plane.

Example : Locate the following points in the polar plane:
$P_1\left(2, \frac{\pi}{4}\right)$, $P_2\left(3, \frac{\pi}{2}\right)$, $P_3\left(1, -\frac{3\pi}{4}\right)$.
Solution :



EXERCISES (8.1)

Locate the following points in the polar plane:

(1) $(3, 0)$ (2) $(-2, 0)$ (3) $\left(1, \frac{\pi}{2}\right)$ (4) $\left(-1, \frac{\pi}{2}\right)$

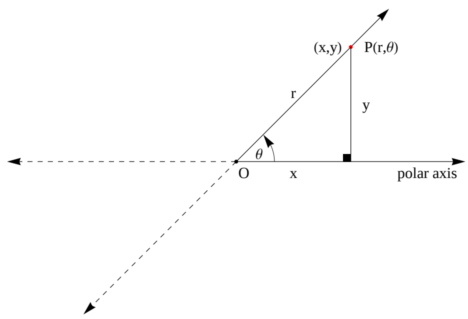
(5) $(2, \pi)$ (6) $(-1, \pi)$ (7) $\left(1, \frac{3\pi}{2}\right)$ (8) $\left(-2, \frac{3\pi}{2}\right)$

(9) $\left(2, \frac{\pi}{6}\right)$ (10) $\left(-1, \frac{\pi}{6}\right)$ (11) $\left(-2, \frac{\pi}{4}\right)$ (12) $\left(3, \frac{\pi}{4}\right)$

(13) $\left(1, \frac{\pi}{3}\right)$ (14) $\left(-4, \frac{\pi}{3}\right)$ (15) $\left(1, -\frac{\pi}{6}\right)$ (16) $\left(-2, -\frac{\pi}{6}\right)$

(17) $\left(-2, -\frac{\pi}{4}\right)$ (18) $\left(2, -\frac{\pi}{4}\right)$ (19) $\left(-2, -\frac{\pi}{2}\right)$ (20) $(-3, -\pi)$

8.2 Relation between polar and Cartesian coordinates



1. If (x, y) is the Cartesian representation of the point P, then the polar representation of P can be calculated from the following two identities:

- $r^2 = x^2 + y^2 \implies r = \sqrt{x^2 + y^2}$.
- $\tan\theta = \dfrac{y}{x} \implies \theta = \tan^{-1}\left(\dfrac{y}{x}\right)$, where $x \neq 0$.

2. If (r, θ) is the polar representation of the point P, then the Cartesian representation of P can be calculated from the following two identities:

- $\cos\theta = \dfrac{x}{r} \implies x = r\cos\theta$
- $\sin\theta = \dfrac{y}{r} \implies y = r\sin\theta$

Example (1) :

1. Find the polar coordinates of the point which its Cartesian coordinates are $(1, \sqrt{3})$.

   Solution :

   $x = 1\ ,\ y = \sqrt{3}$.

   $r = \sqrt{(1)^2 + (\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2$.

   $\tan\theta = \dfrac{\sqrt{3}}{1} = \sqrt{3} \implies \theta = \tan^{-1}(\sqrt{3}) = \dfrac{\pi}{3}$.

   The polar coordinates are $\left(2, \dfrac{\pi}{3}\right)$.

2. Find the Cartesian coordinates of the point which its polar coordinates are $\left(2, \frac{3\pi}{4}\right)$.

   Solution :

   $r = 2\ ,\ \theta = \frac{3\pi}{4}\ .$

   $x = 2\cos\left(\frac{3\pi}{4}\right) = 2\ \left(-\frac{1}{\sqrt{2}}\right) = -\frac{2}{\sqrt{2}} = -\sqrt{2}\ .$

   $y = 2\sin\left(\frac{3\pi}{4}\right) = 2\ \left(\frac{1}{\sqrt{2}}\right) = \frac{2}{\sqrt{2}} = \sqrt{2}\ .$

   The Cartesian coordinates are $\left(-\sqrt{2}, \sqrt{2}\right)$.

Example (2) :

1. Convert the polar equation $r = 3\sec\theta$ into a Cartesian equation.

   Solution :

   $r = 3\sec\theta \implies r = \frac{3}{\cos\theta} \implies r\cos\theta = 3 \implies x = 3\ .$

   The Cartesian equation is $x = 3$, and it represents a straight line perpendicular to the x-axis.

2. Convert the polar equation $r = 2$ into a Cartesian equation.

   Solution :

   $r = 2 \implies r^2 = 4 \implies x^2 + y^2 = 4\ .$

   The Cartesian equation is $x^2 + y^2 = 4$, and it represents a circle of center $(0, 0)$ and its radius is 2.

3. Convert the polar equation $r = 2\sin\theta$ into a Cartesian equation.

   Solution :

   $r = 2\sin\theta \implies r^2 = 2\,(r\ \sin\theta) \implies x^2 + y^2 = 2y$

   $\implies x^2 + y^2 - 2y = 0 \implies x^2 + (y^2 - 2y + 1) = 1 \implies x^2 + (y - 1)^2 = 1\ .$

   The Cartesian equation is $x^2 + (y - 1)^2 = 1$, and it represents a circle of center $(0, 1)$ and its radius is 1.

EXERCISES (8.2)

1. Convert the following Cartesian coordinates into polar coordinates:

   (i) $(x, y) = (2, 2)$ *(ii)* $(x, y) = \left(\sqrt{3}, 1\right)$ *(iii)* $(x, y) = \left(1, \sqrt{3}\right)$ *(iv)* $(x, y) = (-1, 1)$
   (v) $(x, y) = (5, 0)$ *(vi)* $(x, y) = (0, 3)$ *(vii)* $(x, y) = (0, -2)$ *(viii)* $(x, y) = (-1, 0)$

2. Convert the following polar coordinates into Cartesian coordinates:

   (i) $(r, \theta) = \left(2, \frac{\pi}{6}\right)$ *(ii)* $(r, \theta) = \left(3, \frac{\pi}{4}\right)$ *(iii)* $(r, \theta) = \left(1, -\frac{\pi}{4}\right)$

   (iv) $(r, \theta) = \left(2, \frac{\pi}{3}\right)$ *(v)* $(r, \theta) = \left(-2, -\frac{\pi}{2}\right)$ *(vi)* $(r, \theta) = \left(3, \frac{\pi}{2}\right)$

   (vii) $(r, \theta) = \left(1, \frac{3\pi}{4}\right)$ *(viii)* $(r, \theta) = \left(2, -\frac{3\pi}{3}\right)$

3. Convert the following Cartesian equations into polar equations:

   (i) $y = \sqrt{3}x$ *(ii)* $y = 2$ *(iii)* $x = -1$ *(iv)* $x^2 + y^2 = 16$
   (v) $x^2 + y^2 = 6x$ *(vi)* $x^2 + y^2 = 2y$ *(vii)* $x^2 + y^2 = -4x$ *(viii)* $x^2 + y^2 = -8y$

4. Convert the following polar equations into Cartesian equations:

   (i) $\theta = \frac{\pi}{6}$ *(ii)* $r = -2\sec\theta$ *(iii)* $r = 3\csc\theta$ *(iv)* $r = 4$
   (v) $r = -2$ *(vi)* $r = 2\sin\theta$ *(vii)* $r = -4\cos\theta$ *(viii)* $r = 6\sin\theta + 8\cos\theta$

8.3 Polar Curves

Test of symmetry :

1. The graph of the polar equation $r = r(\theta)$ is symmetric with respect to the polar axis if $r(\theta) = r(-\theta)$.
2. The graph of the polar equation $r = r(\theta)$ is symmetric with respect to the line $\theta = \frac{\pi}{2}$ if $r(\theta) = -r(-\theta)$.
3. The graph of the polar equation $r = r(\theta)$ is symmetric with respect to the pole if $r(\theta) = -r(\theta)$.

First - Straight Lines :

1. The straight line passing through the pole :

   Its polar equation is $\theta = \theta_0$.

   $\theta = \theta_0 \implies \tan(\theta) = \tan(\theta_0) \implies \frac{y}{x} = \tan(\theta_0) \implies y = \tan(\theta_0)\ x$.

   $\theta = \theta_0$ represents a straight line passing through the pole with slope equals $\tan(\theta_0)$.

2. The straight line perpendicular to the polar axis :

   Its polar equation is $r = a\ \sec\theta$, where $a \neq 0$ and $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

   $r = a\ \sec\theta = \frac{a}{\cos\theta} \implies r\cos\theta = a \implies x = a$.

   $r = a\sec\theta$ represents a straight line perpendicular to the polar axis at the point $(r, \theta) = (a, 0)$.

3. The straight line parallel to the polar axis :

   Its polar equation is $r = a\ \csc\theta$, where $a \neq 0$ and $\theta \in (0, \pi)$.

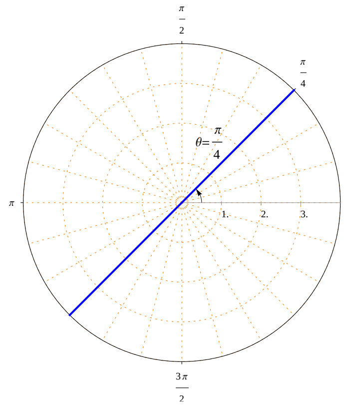
   $r = a\ \csc\theta = \frac{a}{\sin\theta} \implies r\sin\theta = a \implies y = a$.

   $r = a\csc\theta$ represents a straight line parallel to the polar axis and passing through the point $(r, \theta) = \left(a, \frac{\pi}{2}\right)$.

Example (1) : Convert the following polar equations into Cartesian equations and sketch them:

1. $\theta = \dfrac{\pi}{4}$.

   Solution :

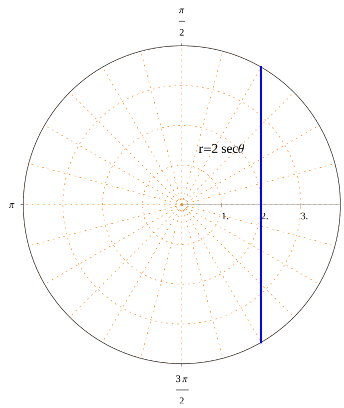
   

   $\theta = \dfrac{\pi}{4} \implies \tan(\theta) = \tan\left(\dfrac{\pi}{4}\right)$
   $\implies \dfrac{y}{x} = 1 \implies y = x$.

   The polar equation $\theta = \dfrac{\pi}{4}$ represents a straight line passing through the pole with slope equals 1 .

2. $r = 2\sec\theta$.

   Solution :

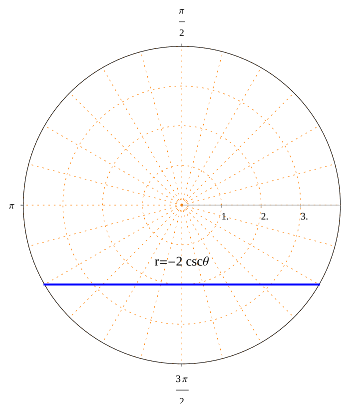
   

   $r = 2\sec\theta = \dfrac{2}{\cos\theta} \implies r\cos\theta = 2$
   $\implies x = 2$.

   The polar equation $r = 2\sec\theta$ represents a straight line perpendicular to the polar axis at the point $(r, \theta) = (2, 0)$.

3. $r = -2\csc\theta$.

   Solution :

   $r = -2\csc\theta = \dfrac{-2}{\sin\theta} \implies r\sin\theta = 2 \implies y = -2$.
   The polar equation $r = -2\csc\theta$ represents a straight line parallel to the polar axis and passing through the point $(r, \theta) = \left(-2, \dfrac{\pi}{2}\right)$.

Second - Circles :

1. Circles centered at the pole :

   The polar equation $r = a$, where $a \neq 0$, represents a circle centered at the pole with radius $|a|$.

   $r = a \implies r^2 = a^2 \implies x^2 + y^2 = a^2$

2. Circles of the form $r = a\cos\theta$, where $a \neq 0$ and $-\dfrac{\pi}{2} \leq \theta \leq \dfrac{\pi}{2}$:

   $r = a\cos\theta \implies r^2 = a(r\cos\theta) \implies x^2 + y^2 = ax$

   $\implies (x^2 - ax) + y^2 = 0 \implies \left(x^2 - ax + \dfrac{a^2}{4}\right) + y^2 = \dfrac{a^2}{4}$

   $\implies \left(x - \dfrac{a}{2}\right)^2 + y^2 = \dfrac{a^2}{4}$.

   The polar equation $r = a\cos\theta$ represents a circle centered at $(r, \theta) = \left(\dfrac{a}{2}, 0\right)$ with radius $\dfrac{|a|}{2}$.

   Note that the circle $r = a\cos\theta$ touches the pole.

   If $a > 0$, then $r = a\cos\theta$ is located to the right of the line $\theta = \dfrac{\pi}{2}$.

   If $a < 0$, then $r = a\cos\theta$ is located to the left of the line $\theta = \dfrac{\pi}{2}$.

3. Circles of the form $r = a\sin\theta$, where $a \neq 0$ and $0 \leq \theta \leq \pi$:

   $r = a\sin\theta \implies r^2 = a(r\sin\theta) \implies x^2 + y^2 = ay$

   $\implies x^2 + y^2 - ay = 0 \implies x^2 + \left(y^2 - ay + \dfrac{a^2}{4}\right) = \dfrac{a^2}{4}$

   $\implies x^2 + \left(y - \dfrac{a}{2}\right)^2 = \dfrac{a^2}{4}$.

   The polar equation $r = a\sin\theta$ represents a circle centered at $(r, \theta) = \left(\dfrac{a}{2}, \dfrac{\pi}{2}\right)$ with radius $\dfrac{|a|}{2}$.

   Note that the circle $r = a\sin\theta$ touches the pole. $r = a\sin\theta$

   If $a > 0$, then $r = a\sin\theta$ is located above the polar axis.

   If $a < 0$, then $r = a\sin\theta$ is located under the polar axis.

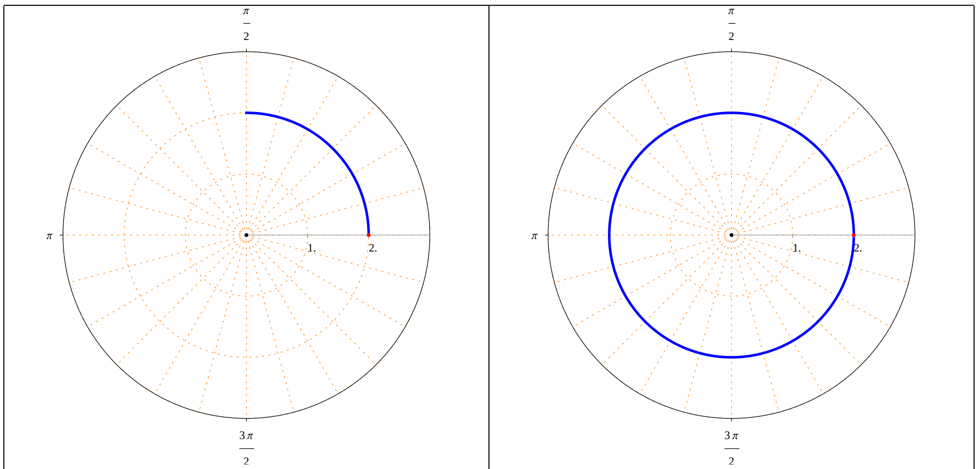
Example (2) : Sketch the following polar equations :

1. $r = 2$.

   Solution :

   The polar equation $r = 2$ represents a circle centered at the pole, with radius equals 2 .

   Note that the circle starts at the point $(r, \theta) = (2, 0)$.



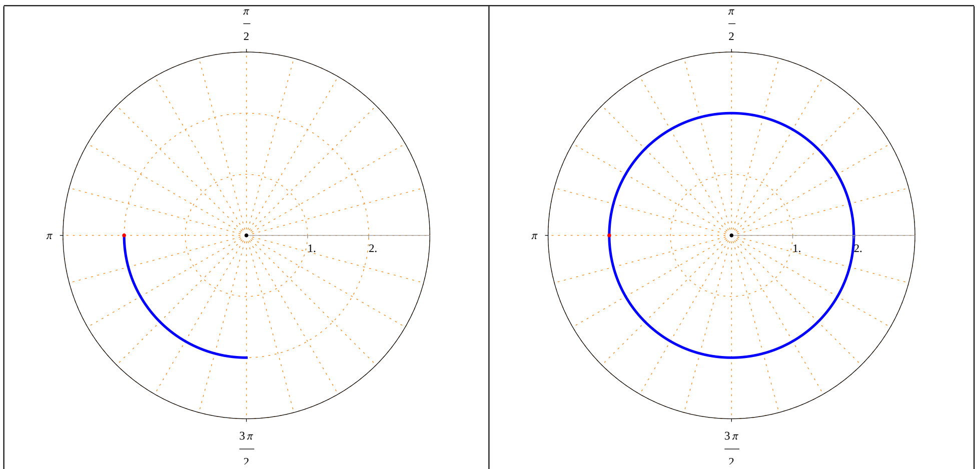
$r = 2 \ , \quad 0 \leq \theta \leq \dfrac{\pi}{2}$ $\qquad$ $r = 2 \ , \quad 0 \leq \theta \leq 2\pi$

2. $r = -2$.

   Solution :

   The polar equation $r = 2$ represents a circle centered at the pole, with radius equals 2 .

   Note that the circle starts at the point $(r, \theta) = (-2, 0)$.

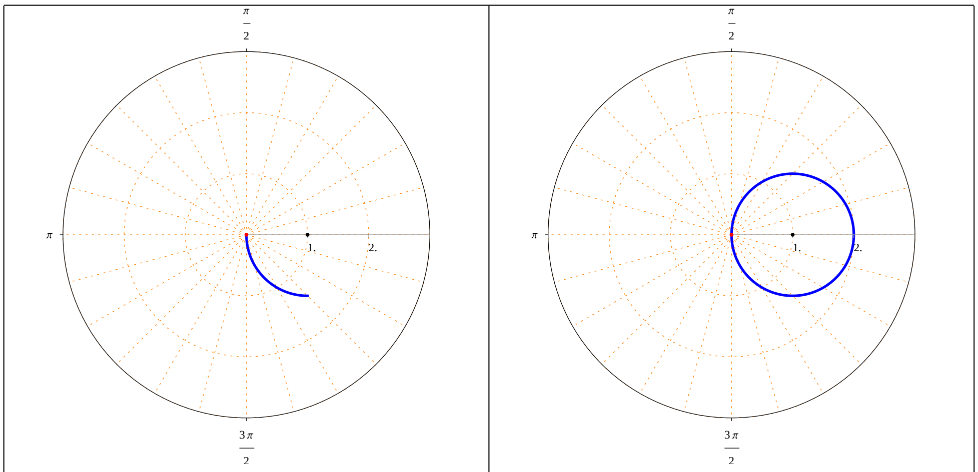


$r = -2 \ , \quad 0 \leq \theta \leq \dfrac{\pi}{2}$ $\qquad$ $r = -2 \ , \quad 0 \leq \theta \leq 2\pi$

3. $r = 2\cos\theta$, $-\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$

   Solution :

   The polar equation $r = 2\cos\theta$ represents a circle centered at $(r, \theta) = (1, 0)$, with radius equals 1 .



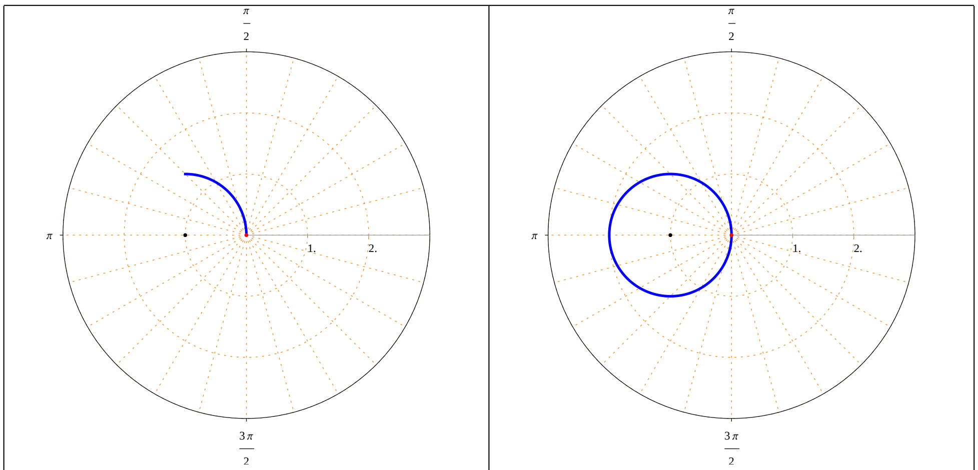
$r = 2\cos\theta$, $-\dfrac{\pi}{2} \le \theta \le -\dfrac{\pi}{4}$ $\qquad$ $r = 2\cos\theta$, $-\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$

4. $r = -2\cos\theta$, $-\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$

   Solution :

   The polar equation $r = -2\cos\theta$ represents a circle centered at

   $(r, \theta) = (-1, 0)$, with radius equals 1 .



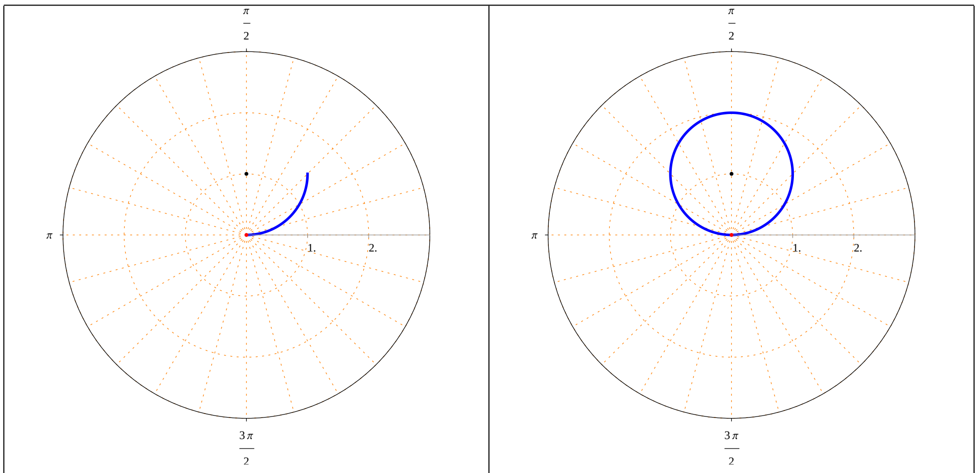
$r = -2\cos\theta$, $-\dfrac{\pi}{2} \le \theta \le -\dfrac{\pi}{4}$ $\qquad$ $r = -2\cos\theta$, $-\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$

5. $r = 2\sin\theta$, $0 \leq \theta \leq \pi$

Solution :

The polar equation $r = 2\sin\theta$ represents a circle centered at

$(r, \theta) = \left(1, \dfrac{\pi}{2}\right)$, with radius equals 1 .



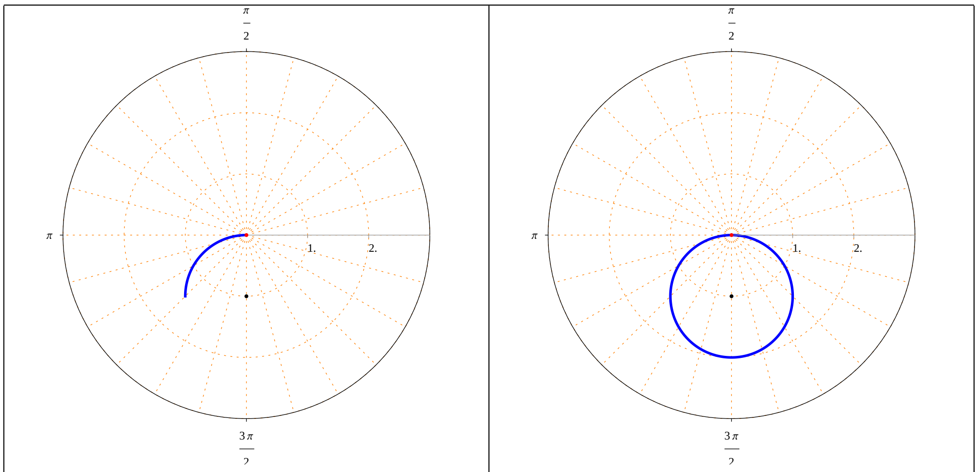
$r = 2\sin\theta$, $0 \leq \theta \leq \dfrac{\pi}{4}$ $\qquad$ $r = 2\sin\theta$, $0 \leq \theta \leq \pi$

6. $r = -2\sin\theta$, $0 \leq \theta \leq \pi$

Solution :

The polar equation $r = -2\sin\theta$ represents a circle centered at

$(r, \theta) = \left(-1, \dfrac{\pi}{2}\right)$, with radius equals 1 .



$r = -2\sin\theta$, $0 \leq \theta \leq \dfrac{\pi}{4}$ $\qquad$ $r = -2\sin\theta$, $0 \leq \theta \leq \pi$

Third - Cardioids :

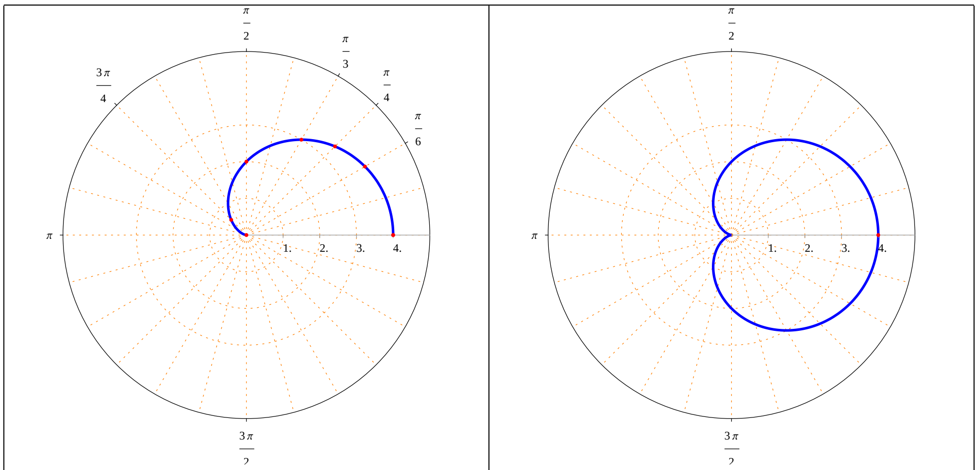
1. The polar equation $r = a(1 \pm \cos\theta)$, where $a \neq 0$ and $0 \leq \theta \leq 2\pi$ represents a cardioid symmetric with respect to the polar axis.
2. The polar equation $r = a(1 \pm \sin\theta)$, where $a \neq 0$ and $0 \leq \theta \leq 2\pi$ represents a cardioid symmetric with respect to the line $\theta = \dfrac{\pi}{2}$.

Example (3) : Sketch the following polar equations :

1. $r = 2 + 2\cos\theta$.

   Note that the cardioid is symmetric with respect to the polar axis.

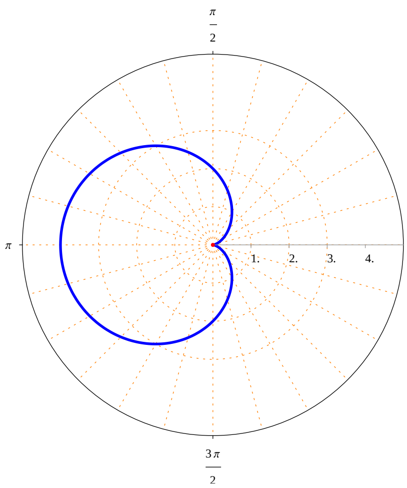
θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
$r(\theta)$	4	$2+\sqrt{3}$	$2+\sqrt{2}$	3	2	$2-\sqrt{2}$	0



$r = 2 + 2\cos\theta \;\; , \;\; 0 \leq \theta \leq \pi$ $\qquad$ $r = 2 + 2\cos\theta \;\; , \;\; 0 \leq \theta \leq 2\pi$

2. $r = 2 - 2\cos\theta$.

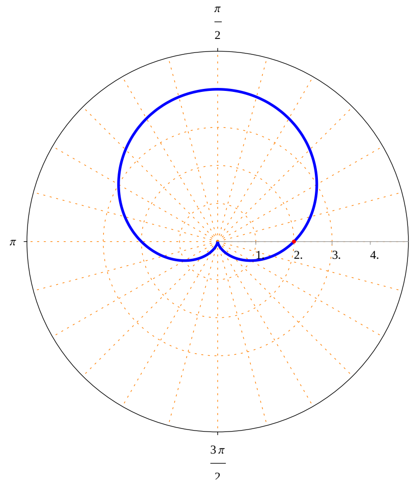
   Note that the cardioid is symmetric with respect to the polar axis.



$r = 2 - 2\cos\theta \;\; , \;\; 0 \leq \theta \leq 2\pi$

3. $r = 2 + 2\sin\theta$.

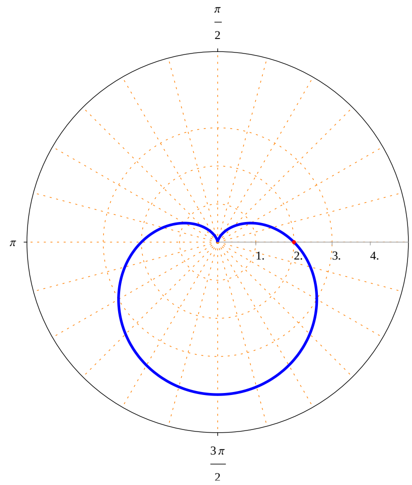
Note that the cardioid is symmetric with respect to the line $\theta = \dfrac{\pi}{2}$.



$r = 2 + 2\sin\theta \quad , \quad 0 \leq \theta \leq 2\pi$

4. $r = 2 - 2\sin\theta$.

Note that the cardioid is symmetric with respect to the line $\theta = \dfrac{\pi}{2}$.



$r = 2 - 2\sin\theta \quad , \quad 0 \leq \theta \leq 2\pi$

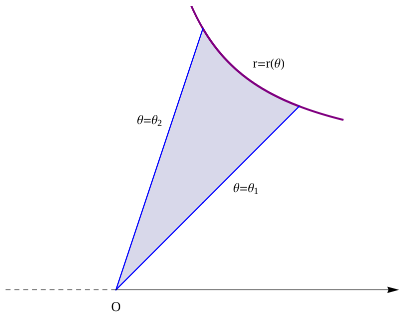
EXERCISES (8.3)

Identify the type of the following polar curves, and sketch them :

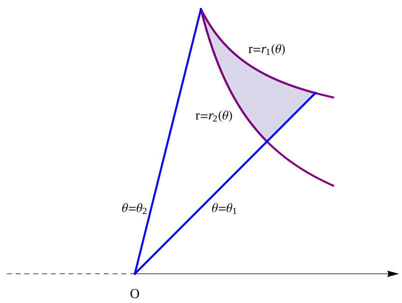
1. $\theta = -\dfrac{\pi}{4}$
2. $\theta = \dfrac{\pi}{3}$
3. $r = 3\sec\theta$
4. $r = -\csc\theta$
5. $r = 4$
6. $r = -3$
7. $r = 2\cos\theta$
8. $r = -4\cos\theta$
9. $r = 6\sin\theta$
10. $r = -\sin\theta$
11. $r = 3 + 3\cos\theta$
12. $r = 1 - \cos\theta$
13. $r = 1 + \sin\theta$
14. $r = 3 - 3\sin\theta$

8.4 Area of a region in the polar plane

If the function $r = r(\theta)$ is continuous, then the area of the region bounded by the graphs of $r(\theta)$ and the two lines $\theta = \theta_1$ and $\theta = \theta_2$ is $\mathbf{A} = \frac{1}{2}\int_{\theta_1}^{\theta_2} [r(\theta)]^2 \, d\theta$.



The area of the region bounded by the graphs of $r_1(\theta)$, $r_2(\theta)$, $\theta = \theta_1$ and $\theta = \theta_2$ is $\mathbf{A} = \frac{1}{2}\int_{\theta_1}^{\theta_2} \left([r_1(\theta)]^2 - [r_2(\theta)]^2\right) d\theta$.

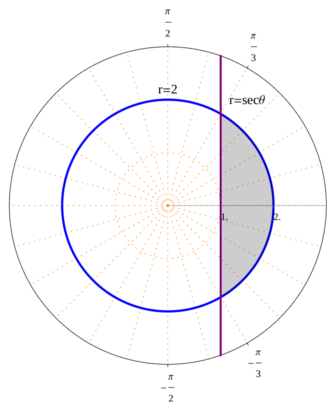


Example :

1. Find the area of the region inside the curve $r = 2$ and to the right of the straight line $r = \sec\theta$.

   Solution

   $r = 2$ represents a circle centered at the pole, with radius equals 2 .
   $r = \sec\theta$ represents a straight line perpendicular to the polar axis and passes through $(r, \theta) = (1, 0)$.

   Points of intersection of $r = 2$ and $r = \sec\theta$:
   $\sec\theta = 2 \implies \frac{1}{\cos\theta} = 2$
   $\implies \cos\theta = \frac{1}{2} \implies \theta = -\frac{\pi}{3} \quad , \quad \theta = \frac{\pi}{3}$.

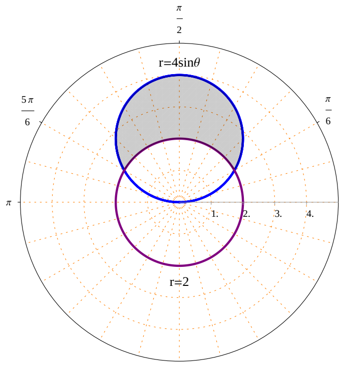
   Note that the shaded region is symmetric with respect to the polar axis .

   $$\mathbf{A} = 2\left(\frac{1}{2}\int_0^{\frac{\pi}{3}} \left[(2)^2 - (\sec\theta)^2\right] d\theta\right) = \int_0^{\frac{\pi}{3}} \left(4 - \sec^2\theta\right) d\theta$$

   $$= [4\theta - \tan\theta]_0^{\frac{\pi}{3}} = \left(4\left(\frac{\pi}{3}\right) - \tan\left(\frac{\pi}{3}\right)\right) - (4(0) - \tan(0)) = \frac{4\pi}{3} - \sqrt{3} \ .$$

2. Find the area of the region inside the curve $r = 4\sin\theta$ and outside the curve $r = 2$.

Solution



$r = 2$ represents a circle centered at the pole, with radius equals 2 .
$r = 4\sin\theta$ represents a circle centered at $(r,\theta) = \left(2, \frac{\pi}{2}\right)$, with radius equals 2 .

Points of intersection of $r = 2$ and $r = 4\sin\theta$:
$4\sin\theta = 2 \implies \sin\theta = \frac{1}{2}$
$\implies \theta = \frac{\pi}{6}$, $\theta = \frac{5\pi}{6}$.

Note that the shaded region is symmetric with respect to the line $\theta = \frac{\pi}{2}$.

$$\mathbf{A} = 2\left(\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left[(4\sin\theta)^2 - (2)^2\right] \, d\theta\right) = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(16\sin^2\theta - 4\right) \, d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left[16 \, \frac{1}{2} \, (1 - \cos 2\theta) - 4\right] \, d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (8 - 8\cos 2\theta - 4) \, d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (4 - 8\cos 2\theta) \, d\theta = [4\theta - 4\sin 2\theta]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= \left[4\left(\frac{\pi}{2}\right) - 4\sin\left(4 \, \frac{\pi}{2}\right)\right] - \left[4\left(\frac{\pi}{6}\right) - 4\sin\left(2 \, \frac{\pi}{6}\right)\right]$$

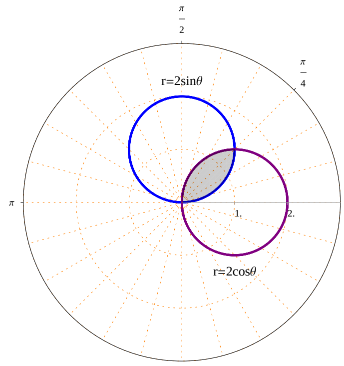
$$= (2\pi - 4\sin(2\pi)) - \left(\frac{2\pi}{3} - 4\sin\left(\frac{\pi}{3}\right)\right)$$

$$= 2\pi - 0 - \frac{2\pi}{3} + 4\left(\frac{\sqrt{3}}{2}\right) = \frac{4\pi}{3} + 2\sqrt{3} \, .$$

3. Find the area of the common region between the curves $r = 2\cos\theta$ and $r = 2\sin\theta$.

Solution

$r = 2\sin\theta$ represents a circle centered at $(r,\theta) = \left(1, \frac{\pi}{2}\right)$, with radius equals 1 .
$r = 2\cos\theta$ represents a circle centered at $(r,\theta) = (1,0)$, with radius equals 1 .



Points of intersection of $r = 2\sin\theta$ and $r = 2\cos\theta$:
$2\sin\theta = 2\cos\theta \implies \frac{\sin\theta}{\cos\theta} = 1$
$\implies \tan\theta = 1 \implies \theta = \frac{\pi}{4}$.

$$\mathbf{A} = \frac{1}{2}\int_0^{\frac{\pi}{4}} (2\sin\theta)^2\, d\theta + \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (2\cos\theta)^2\, d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} (4\sin^2\theta)\, d\theta + \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (4\cos^2\theta)\, d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} \left[4\ \frac{1}{2}(1-\cos 2\theta)\right] d\theta + \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left[4\ \frac{1}{2}(1+\cos 2\theta)\right] d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} (2 - 2\cos 2\theta)\, d\theta + \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (2 + 2\cos 2\theta)\, d\theta$$

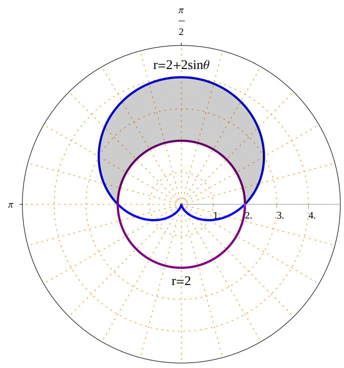
$$= \frac{1}{2}\left[2\theta - \sin 2\theta\right]_0^{\frac{\pi}{4}} + \frac{1}{2}\left[2\theta + \sin 2\theta\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{2} - 1\right) - (0-0)\right] + \frac{1}{2}\left[(\pi + 0) - \left(\frac{\pi}{2} + 1\right)\right]$$

$$= \frac{\pi}{4} - \frac{1}{2} + \frac{\pi}{2} - \frac{\pi}{4} - \frac{1}{2} = \frac{\pi}{2} - 1\ .$$

4. Find the area of the region inside the curve $r = 2 + 2\sin\theta$ and outside the curve $r = 2$.

Solution



$r = 2 + 2\sin\theta$ represents a Cardioid symmetric with respect to the line $\theta = \dfrac{\pi}{2}$.
$r = 2$ represents a circle centered at the pole, with radius equals 2 .

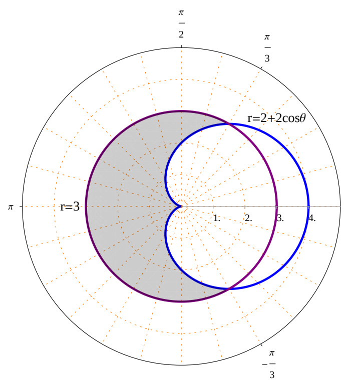
Points of intersection of $r = 2 + 2\sin\theta$ and $r = 2$:
$2 + 2\sin\theta = 2 \implies \sin\theta = 0$
$\implies \theta = 0 \quad , \quad \theta = \pi$.

Note that the shaded region is symmetric with respect to the line $\theta = \dfrac{\pi}{2}$.

$$\mathbf{A} = 2\left(\frac{1}{2}\int_0^{\frac{\pi}{2}} \left[(2 + 2\sin\theta)^2 - (2)^2\right]\, d\theta\right)$$
$$= \int_0^{\frac{\pi}{2}} \left[(4 + 8\sin\theta + 4\sin^2\theta) - 4\right]\, d\theta = \int_0^{\frac{\pi}{2}} (8\sin\theta + 4\sin^2\theta)\, d\theta$$
$$= \int_0^{\frac{\pi}{2}} \left[8\sin\theta + 4\,\frac{1}{2}(1 - \cos 2\theta)\right]\, d\theta = \int_0^{\frac{\pi}{2}} (2 + 8\sin\theta - 2\cos 2\theta)\, d\theta$$
$$= [2\theta - 8\cos\theta - \sin 2\theta]_0^{\frac{\pi}{2}}$$
$$= \left[2\left(\frac{\pi}{2}\right) - 8\cos\left(\frac{\pi}{2}\right) - \sin\left(2\,\frac{\pi}{2}\right)\right] - [2\,(0) - 8\cos(0) - \sin(2\,(0))]$$
$$= (\pi - 0 - 0) - (0 - 8 - 0) = 8 + \pi \ .$$

5. Find the area of the region inside the curve $r = 3$ and outside the curve $r = 2 + 2\cos\theta$.

Solution



$r = 2 + 2\cos\theta$ represents a Cardioid symmetric with respect to the polar axis .
$r = 3$ represents a circle centered at the pole, with radius equals 3 .

Points of intersection of $r = 2 + 2\cos\theta$ and $r = 3$:
$2 + 2\cos\theta = 3 \implies \cos\theta = \dfrac{1}{2}$
$\implies \theta = -\dfrac{\pi}{3} \quad , \quad \theta = \dfrac{\pi}{3}$.

Note that the shaded region is symmetric with respect to the polar axis .

$$\mathbf{A} = 2\left(\frac{1}{2}\int_{\frac{\pi}{3}}^{\pi}\left[(3)^2-(2+2\cos\theta)^2\right]\ d\theta\right)$$

$$= \int_{\frac{\pi}{3}}^{\pi}\left[9-(4+8\cos\theta+4\cos^2\theta)\right]\ d\theta$$

$$= \int_{\frac{\pi}{3}}^{\pi}\left[5-8\cos\theta-4\ \frac{1}{2}(1+\cos 2\theta)\right]\ d\theta$$

$$= \int_{\frac{\pi}{3}}^{\pi}(3-8\cos\theta-2\cos 2\theta)\ d\theta = \left[3\theta-8\sin\theta-\sin 2\theta\right]_{\frac{\pi}{3}}^{\pi}$$

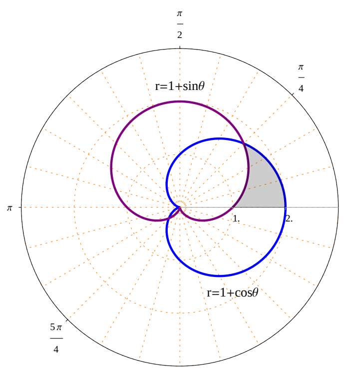
$$= \left[3\left(\pi\right)-8\sin\left(\pi\right)-\sin\left(2\pi\right)\right]-\left[3\left(\frac{\pi}{3}\right)-8\sin\left(\frac{\pi}{3}\right)-\sin\left(2\frac{\pi}{3}\right)\right]$$

$$= \left[3\pi-8(0)-(0)\right]-\left[\pi-8\ \frac{\sqrt{3}}{2}-2\ \frac{\sqrt{3}}{2}\ \frac{1}{2}\right]$$

$$= 3\pi-\pi+4\sqrt{3}+\frac{\sqrt{3}}{2} = 2\pi+\frac{9\sqrt{3}}{2}$$

6. Find the area of the region in the first quadrant, inside the curve $r = 1+\cos\theta$ and outside the curve $r = 1+\sin\theta$.

Solution



$r = 1+\cos\theta$ represents a Cardioid symmetric with respect to the polar axis .
$r = 1+\sin\theta$ represents a Cardioid symmetric with respect to the line $\theta = \frac{\pi}{2}$.

Points of intersection of $r = 1+\cos\theta$ and $r = 1+\sin\theta$:
$1+\cos\theta = 1+\sin\theta \implies \cos\theta = \sin\theta$
$\implies \theta = \frac{\pi}{4}\ \ ,\ \ \theta = \frac{5\pi}{4}$.

$$\mathbf{A} = \frac{1}{2}\int_0^{\frac{\pi}{4}}\left[(1+\cos\theta)^2-(1+\sin\theta)^2\right]\ d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}}\left[(1+2\cos\theta+\cos^2\theta)-(1+2\sin\theta+\sin^2\theta)\right]\ d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}}\left[2\cos\theta-2\sin\theta+\cos^2\theta-\sin^2\theta)\right]\ d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}}\left[2\cos\theta-2\sin\theta+\cos 2\theta)\right]\ d\theta$$

$$= \frac{1}{2}\left[2\sin\theta + 2\cos\theta + \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{2}\left[\left(2\sin\left(\frac{\pi}{4}\right) + 2\cos\left(\frac{\pi}{4}\right) + \frac{1}{2}\sin\left(\frac{\pi}{2}\right)\right) - \left(2(0) + 2(1) + \frac{1}{2}(0)\right)\right]$$

$$= \frac{1}{2}\left(2\ \frac{\sqrt{2}}{2} + 2\ \frac{\sqrt{2}}{2} + \frac{1}{2} - 0 - 2 - 0\right)$$

$$= \frac{1}{2}\left(2\sqrt{2} - \frac{3}{2}\right) = \sqrt{2} - \frac{3}{4}\ .$$

EXERCISES (8.4)

1. Sketch the region inside the curve $r = 3$ and outside the curve $r = 2$, then find its area.

2. Sketch the region inside the curve $r = 2$ and above the line $r = -\csc\theta$, then find its area.

3. Sketch the region inside the curve $r = 4$ and outside the curve $r = 4\sin\theta$, then find its area.

4. Sketch the region inside the curve $r = 4\cos\theta$ and outside the curve

   $r = 2\cos\theta$, then find its area.

5. Sketch the common region between the curves $r = \sqrt{3}\cos\theta$ and $r = \sin\theta$, then find its area.

6. Sketch the region inside the curve $r = 1$ and outside the curve $r = 1-\cos\theta$, then find its area.

7. Sketch the region inside the curve $r = 2 + 2\cos\theta$ and outside the curve $r = 3$, then find its area.

8. Sketch the region inside the curve $r = 3\sin\theta$ and outside the curve

   $r = 1 + \sin\theta$, then find its area.

9. Sketch the region inside the curve $r = 1 + \cos\theta$ and outside the curve $r = 1 - \cos\theta$, then find its area.

10. Sketch the region inside the curve $r = 1 + \cos\theta$ and outside the curve $r = 3\cos\theta$, then find its area.