SECOND MID TERM EXAM., SEMESTER I, 2024

DEPT. MATH., COLLEGE OF SCIENCE
KING SAUD UNIVERSITY

MATH: 107 FULL MARK: 25 TIME: 90 MINUTES

Q1. [3+3=6]

Let $A(2,2,0)$, $B(-1,0,2)$, and $C(0,4,3)$ be three points.

(a) Find the area of the triangle ABC.

(b) Find the distance from C to the line l which passes through A and B.

Q2. [4+4=8]

(a) Find parametric equations for the line of intersection of the planes

$$P_1\colon\ x-2y+3z=1 \text{ and } P_2\colon\ x+y+z=1.$$

(b) Determine whether the two lines

$$l_1:\quad x=1+2t,\quad y=1-4t,\quad z=5-t, \text{ where } t\in\Re,$$
$$l_2:\quad x=4-s,\quad y=-1+6s,\quad z=4+s, \text{ where } s\in\Re$$

intersect, and if so, find the point of intersection.

Q3. [3] Identify the surface given by the equation $z^2-4y^2-64x^2+64=0$. Sketch the surface by finding traces on the coordinate planes.

Q4. [4+4=8]

(a) Find the velocity, acceleration and speed of a moving point P at time $t=\frac{\pi}{2}$ along

$$\mathbf{r}(t)=t(\cos t\mathbf{i}+\sin t\mathbf{j}+t\mathbf{k}).$$

(b) Find $\mathbf{r}(t)$ subject to the given conditions:

$$\mathbf{r}''(t)=6t\mathbf{i}+3\mathbf{j}, \text{ where } \mathbf{r}'(0)=4\mathbf{i}-\mathbf{j}+\mathbf{k}, \text{ and } \mathbf{r}(0)=7\mathbf{j}.$$

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Model for second Midterm Exam S1 1446

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Q1 [3+3]

Let $A(2,2,0)$, $B(-1,0,2)$ and $C(0,4,3)$ be three points.

(a) Find the area of the triangle ABC

(b) Find the distance from C to the line ℓ which passes through A and B.

Ans:

(a) $\vec{AB} = \langle -3,-2,2\rangle$, $\vec{AC} = \langle -2,2,3\rangle$

$$\vec{AB}\times\vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3 & -2 & 2 \\ -2 & 2 & 3 \end{vmatrix}$$

$$= -10\vec{i} + 5\vec{j} - 10\vec{k} \quad 1.5$$

$$\text{Area of } \triangle ABC = \frac{1}{2}\|\vec{AB}\times\vec{AC}\|$$

$$= \frac{1}{2}\sqrt{100+25+100}$$

$$= \frac{1}{2}\sqrt{225} = \frac{15}{2} = 7.5 \quad 1.5$$

(b)

$$\text{The distance } d = \frac{\|\vec{AB}\times\vec{AC}\|}{\|\vec{AB}\|} \quad 2$$

$$= \frac{15}{\sqrt{9+4+4}} = \frac{15}{\sqrt{17}} \quad 1$$

$$\simeq 3.64$$

Q2 [3+4]

(a) Find the parametric Eqns for the line of intersection of the planes $P_1: x-2y+3z=1$ and $P_2: x+y+z=1$

(b) Determine whether the two lines $\ell_1: x=1+2t,\ y=1-4t,\ z=5-t;\ t\in\mathbb{R}$, $\ell_2: x=4-s,\ y=-1+6s,\ z=4+s;\ s\in\mathbb{R}$ intersect and if so, find the point of intersection.

Ans:

$P_1: x-2y+3z=1 \Rightarrow \vec{n_1} = \langle 1,-2,3\rangle$

$P_2: x+y+z=1 \Rightarrow \vec{n_2} = \langle 1,1,1\rangle$

So, $\vec{a} = \vec{n_1}\times\vec{n_2}$

$$\vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 3 \\ 1 & 1 & 1 \end{vmatrix} \quad 1$$

$$\vec{a} = -5\vec{i} + 2\vec{j} + 3\vec{k}$$

is the direction vector for the intersection line of the two planes. To get a point on that line, let $z=0$

$\Rightarrow x-2y=1$ (1), $x+y=1$ (2)

Solving (1) and (2), we get

$x=1,\ y=0$

$\therefore P(1,0,0)$ is a point lies on the intersection line. 2

Hence, the parametric Eqns of this line are given by

$x=1-5t,\ y=2t,\ z=3t$ where $t\in\mathbb{R}$.

(b)

The two lines are intersected if they have a common point $P(x,y,z)$ such that:

$$\begin{cases} 1+2t = 4-s, \\ 1-4t = -1+6s, \\ 5-t = 4+s \end{cases} \Rightarrow \begin{cases} 2t+s=3 & (1) \\ 4t+6s=2 & (2) \\ t+s=1 & (3) \end{cases} \quad 2$$

Solving (1) and (2), we get

$t=2$ and $s=-1$

$\because t=2$ and $s=-1$ is a solution of each of the three Eqns, the two lines intersect. Substitute $t=2$ in line ℓ_1 or $s=-1$ in line ℓ_2, the intersection point is $P(5,-7,3)$. 2

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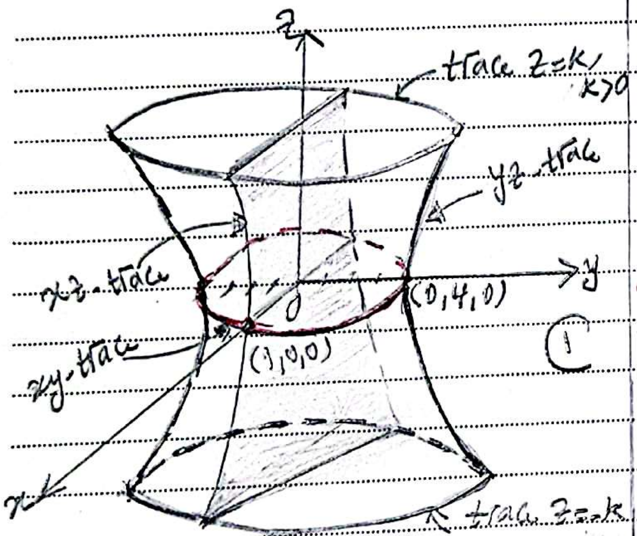
Q3 [3] Identify the surface given by the equation $z^2 - 4y^2 - 64x^2 + 64 = 0$. Sketch the surface by finding traces on the coordinate planes.

Ans: we have $\frac{x^2}{1} + \frac{y^2}{16} - \frac{z^2}{64} = 1$ which is a hyperboloid of one sheet (1)

Trace	Eqn of Trace	Description
* on xy-plane ($z=0$)	$\frac{x^2}{1} + \frac{y^2}{16} = 1$	ellipse
* on yz-plane ($x=0$)	$\frac{y^2}{16} - \frac{z^2}{64} = 1$	hyperbola
* on xz-plane ($y=0$)	$\frac{x^2}{1} - \frac{z^2}{64} = 1$	hyperbola

(1)



(1)

Hyperboloid of one sheet $\frac{x^2}{1} + \frac{y^2}{16} - \frac{z^2}{64} = 1$, whose its axis is z-axis.

Q4 [5+4]

(a) Find the velocity, acceleration and speed of a moving point P at time $t = \frac{\pi}{2}$ along

$\vec{r}(t) = t(\cos t\,\vec{i} + \sin t\,\vec{j} + t\,\vec{k})$

(b) Find $\vec{r}(t)$ subject to the given conditions:

$\vec{r}''(t) = 6t\,\vec{i} + 3\vec{j}$, where $\vec{r}'(0) = 4\vec{i} - \vec{j} + k$, and $\vec{r}(0) = 7\vec{j}$.

Ans:

$\vec{r}(t) = t\cos t\,\vec{i} + t\sin t\,\vec{j} + t^2\vec{k}$

The velocity is $\vec{v}(t) = \vec{r}'(t)$,

$\vec{v}(t) = (\cos t - t\sin t)\vec{i} + (\sin t + t\cos t)\vec{j} + 2t\vec{k}$ (1)

Acceleration is $\vec{a}(t) = \vec{r}''(t)$,

$\vec{a}(t) = (-2\sin t - t\cos t)\vec{i} + (2\cos t - t\sin t)\vec{j} + 2\vec{k}$ (1)

$\vec{v}(\frac{\pi}{2}) = -\frac{\pi}{2}\vec{i} + \vec{j} + \pi\vec{k}$ (1)

The speed at $t = \pi/2$ is $\|v(\frac{\pi}{2})\| = \sqrt{\frac{5\pi^2 + 4}{4}}$ (1)

$= \frac{1}{2}\sqrt{5\pi^2 + 4} \simeq 3.65$

and $\vec{a}(\pi/2) = -2\vec{i} - \frac{\pi}{2}\vec{j} + 2\vec{k}$. (1)

(b) $\because \vec{r}''(t) = 6t\,\vec{i} + 3\vec{j}$

$\Rightarrow \vec{r}'(t) = \int (6t\,\vec{i} + 3\vec{j})\,dt + c_1$

$\therefore \vec{r}'(t) = 3t^2\vec{i} + 3t\vec{j} + c_1$

at $t=0$, $\vec{r}'(0) = c_1 = \langle 4, -1, 1 \rangle$

$\therefore \vec{r}'(t) = (3t^2 + 4)\vec{i} + (3t - 1)\vec{j} + \vec{k}$ (2)

$\Rightarrow \vec{r}(t) = \int [(3t^2+4)\vec{i} + (3t-1)\vec{j} + \vec{k}]\,dt + c_2$

$\therefore \vec{r}(t) = (t^3 + 4t)\vec{i} + (\frac{3}{2}t^2 - t)\vec{j} + t\vec{k} + c_2$

at $t=0$, $\vec{r}(0) = c_2 = \langle 0, 7, 0 \rangle$

$\therefore \vec{r}(t) = (t^3 + 4t)\vec{i} + (\frac{3}{2}t^2 - t + 7)\vec{j} + t\vec{k}$. (2)