Model Answer of MATH 107 for First MDT Exam S1 1446

Q. 1: [3+1+1+1]

Consider the following linear system of equations

$$x+y-z=2$$
$$x+y+z=3$$
$$x+y+(\lambda^2-5)z=\lambda$$

Determine for which values of λ this system has

(i) no solution (ii) infinitely many solutions (iii) unique solution.

Ans:

The augmented matrix is

$$\begin{bmatrix} 1 & 1 & -1 & 2 \\ 1 & 1 & 1 & 3 \\ 1 & 1 & \lambda^2-5 & \lambda \end{bmatrix} \quad -R_1+R_2,\ -R_1+R_3$$

$$\begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & \lambda^2-4 & \lambda-2 \end{bmatrix} \quad \frac{1}{2}R_2$$

$$\begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & \lambda^2-4 & \lambda-2 \end{bmatrix} \quad -(\lambda^2-4)R_2+R_3$$

$$\begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{\lambda(2-\lambda)}{2} \end{bmatrix}$$

If $\frac{\lambda(2-\lambda)}{2}=0$ then $\lambda=0$ or $\lambda=2$.

So, we have the following three cases.

(i) If $\lambda \neq 0$ or $\lambda \neq 2$, *i.e.* $\lambda \in \mathbb{R}-\{0,2\}$ the system has no solution.

(ii) If $\lambda=0$ or $\lambda=2$ the system has infinitely many solutions (the third row is $0 \quad 0 \quad 0 \quad 0$).

(iii) The system has not a unique solution.

Q. 2: [3+3]

For the following system of linear equations

$$\begin{aligned} x+y+z&=4\\ x-y+2z&=1\\ y+z&=3 \end{aligned}$$

Find the inverse of the coefficient matrix by using elementary row operations,

then find the solution of the given system.

Ans:

$$[A|I]=\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0\\ 1 & -1 & 2 & 0 & 1 & 0\\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \quad -R_1+R_2$$

$$\approx\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0\\ 0 & -2 & 1 & -1 & 1 & 0\\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \quad R_3 \leftrightarrow R_2$$

$$\approx\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0\\ 0 & 1 & 1 & 0 & 0 & 1\\ 0 & -2 & 1 & -1 & 1 & 0 \end{array}\right] \quad -R_2+R_1,\ 2R_2+R_3$$

$$\approx\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1\\ 0 & 1 & 1 & 0 & 0 & 1\\ 0 & 0 & 3 & -1 & 1 & 2 \end{array}\right] \quad \tfrac{1}{3}R_3$$

$$\approx\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1\\ 0 & 1 & 1 & 0 & 0 & 1\\ 0 & 0 & 1 & -\frac{1}{3} & \frac{1}{3} & \frac{2}{3} \end{array}\right] -R_3+R_2$$

$$\approx\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1\\ 0 & 1 & 0 & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3}\\ 0 & 0 & 1 & -\frac{1}{3} & \frac{1}{3} & \frac{2}{3} \end{array}\right] = [I|A^{-1}]$$

$$\therefore A^{-1}=\begin{bmatrix} 1 & 0 & -1\\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3}\\ -\frac{1}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

$\therefore X = A^{-1}B$

$$= \begin{bmatrix} 1 & 0 & -1 \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$\therefore$ The solution is $x = 1, y = 2, z = 1$.

Q. 3: [3]

Prove the identity

$$\begin{vmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{vmatrix} = a(b-a)(c-b)(d-c)$$

Ans:

$$\begin{vmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{vmatrix} = a \begin{vmatrix} 1 & a & a & a \\ 1 & b & b & b \\ 1 & b & c & c \\ 1 & b & c & d \end{vmatrix} \quad -R_1 + R_2, -R_2 + R_3, -R_3 + R_4$$

$$= a \begin{vmatrix} 1 & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{vmatrix} = a(b-a)(c-b)(d-c)$$

Q. 4: [2+2+1]

Let $A = \begin{bmatrix} 2 & 0 & 3 \\ 1 & 3 & 2 \\ -2 & 1 & -4 \end{bmatrix}$

Find $adj(A)$ (the adjoint of A). Use determinant to check whether the matrix A is invertible or not.

Ans:

The cofactor matrix is

$$C = \begin{bmatrix} -14 & 0 & 7 \\ 3 & -2 & -2 \\ -9 & -1 & 6 \end{bmatrix}$$

$$adj(A) = C^T = \begin{bmatrix} -14 & 3 & -9 \\ 0 & -2 & -1 \\ 7 & -2 & 6 \end{bmatrix}$$

$\because \det(A) = |A| = -7$

$\therefore A$ is invertible matrix.

Q. 5: [5]

Use Cramer's Rule to solve the following linear system

$$\begin{aligned} x - y + z &= 1 \\ 2x + 3z &= 0 \\ x + y - z &= 1 \end{aligned}$$

Ans:

$$\det(A) = \begin{vmatrix} 1 & -1 & 1 \\ 2 & 0 & 3 \\ 1 & 1 & -1 \end{vmatrix} = -6, \ \det(A_1) = \begin{vmatrix} 1 & -1 & 1 \\ 0 & 0 & 3 \\ 1 & 1 & -1 \end{vmatrix} = -6$$

$$\det(A_2) = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 0 & 3 \\ 1 & 1 & -1 \end{vmatrix} = 4, \ \det(A_3) = \begin{vmatrix} 1 & -1 & 1 \\ 2 & 0 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 4$$

$$x = \frac{\det(A_1)}{\det(A)} = 1, \ \mathrm{y} = \frac{\det(A_2)}{\det(A)} = -\frac{2}{3}, \ \mathrm{z} = \frac{\det(A_3)}{\det(A)} = -\frac{2}{3}$$

Hence, the solution is $x = 1, \ \mathrm{y} = -\frac{2}{3}, \ \mathrm{z} = -\frac{2}{3}$