FINAL EXAMINATION, SEMESTER II, 2025

DEPT. MATH., COLLEGE OF SCIENCE, KSU

MATH: 107 FULL MARK: 40 TIME: 3 HOURS

Q1. [3+3+3=9]

(a) Find the real numbers x, y, and z such that the matrix

$$A = \begin{bmatrix} 2 & x-y+z & x+y+z \\ 0 & 5 & 0 \\ 0 & 4x+2y+z & 7 \end{bmatrix}$$

is symmetric.

(b) Determine the invertibility of the matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

and then find the matrix B satisfying the equation $BA = A^2 + 3A$.

(c) Let

$$\begin{bmatrix} 3 & 1 & \lambda^2-6 & \lambda-3 \\ 1 & 2 & 1 & 0 \\ 2 & 3 & 1 & -1 \end{bmatrix}$$

be the augmented matrix of a system of linear equations. Find the value of λ that makes the linear system inconsistent.

Q2. [3+3+3=9]

Let $A(1,1,0)$, $B(-1,2,1)$, $C(0,2,-1)$, and $D(3,1,1)$ be four points in the space.

(a) Find the equation of the plane passing through A, B, and C.

(b) Find the area of the triangle ABC.

(c) Find the volume of the box $ABCD$.

Q3. [3+3+3=9]

(a) Let C be the curve with parametric equations: $x = \sin t + \cos t$, $y = (\sin t)e^t$, $z = \cos t$. Find parametric equations for the tangent line to C at $P(1,0,1)$.

(b) If $\mathbf{r}(t) = \langle (t-1)^2, 2t, \ln t\rangle$, $(\frac{1}{2} \le t \le 2)$ is the position vector of a moving point P, then find its velocity, speed and acceleration at $t = 1$.

(c) Find the tangential and normal components of acceleration for the position vector given by $\mathbf{r}(t) = \langle \cos t, \sin t, e^{-t}\rangle$ at $t = 0$.

Q4. [2+2+3+3+3=13]

(a) If $\omega = e^{-x}\cos y + e^{-y}\cos x$, then show that $\frac{\partial^2\omega}{\partial x^2} + \frac{\partial^2\omega}{\partial y^2} = 0$.

(b) If $u = \ln(x+y)$ with $x = e^{-2t}$ and $y = t^3 - t^2 + 5$, use chain rule to find $\frac{du}{dt}$.

(c) Find the directional derivative of $f(x,y) = x^2 - 5xy + 3y^2$ at the point $P(3,-1)$ in the direction of the vector $\mathbf{a} = \mathbf{i} + \mathbf{j}$.

(d) If $f(x,y) = x^3 + 3xy - y^3$, find the local extrema and saddle points of f.

(e) Use Lagrange multipliers to find the minimum value of f where $f(x,y,z) = x^2 + y^2 + z^2$ subject to the constraint $x - y + z = 1$.

Q1. 3+3+3=9 (a) Since the given matrix is symmetric, we have: $x - y + z = 0$; $x + y + z = 0$; $4x + 2y + z = 0$. Solving these homogeneous linear system, one obtains: $x = y = z = 0$.
(b) The $|A| = 6 \neq 0$, the matrix is invertible. Now,

$$B = A + 3I = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

(c) By applying elementary row operations on the augmented matrix, we get

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & \lambda^2 - 4 & \lambda + 2 \end{bmatrix}$$

Hence, $\lambda = 2$ makes the given system inconsistent.
Q2. 3+3+3=9 (a) $\mathbf{AB} = \langle -2, 1, 1\rangle$, and $\mathbf{AC} = \langle -1, 1, -1\rangle$. So, $\mathbf{AB} \times \mathbf{AC} = \langle -2, -3, -1\rangle$. Hence, the equation of the plane is $-2(x-1) - 3(y-1) - 1(z-0) = 0$, i.e., $2x + 3y + z = 5$.
(b) The area of the triangle ABC is given by $\frac{1}{2}\|\mathbf{AB} \times \mathbf{AC}\| = \frac{1}{2}\sqrt{14}$.
(c) The volume of the box $ABCD$ is given by $|(\mathbf{AB}\times\mathbf{AC})\cdot\mathbf{AD}| = |\langle -2, -3, -1\rangle\cdot\langle 2, 0, 1\rangle| = |-5| = 5$.
Q3. 3+3+3=9 (a) Given C: $\mathbf{r}(t) = \langle \sin t + \cos t, (\sin t)e^t, \cos t\rangle$. Then $\mathbf{r}'(t) = \langle \cos t - \sin t, (\cos t)e^t + (\sin t)e^t, -\sin t\rangle$. The point $P(1, 0, 1)$ corresponds to $t = 0$, and $\mathbf{r}'(0) = \langle 1, 1, 0\rangle$. So, the equation of the tangent line to C at $P(1, 0, 1)$ has parametric equations: $x = 1 + t$, $y = t$, $z = 1$, $t \in \Re$.
(b) We get $\mathbf{r}'(t) = \langle 2(t-1), 2, \frac{1}{t}\rangle$ and $\mathbf{r}''(t) = \langle 2, 0, -\frac{1}{t^2}\rangle$.
Then $\mathbf{v}(1) = \mathbf{r}'(1) = \langle 0, 2, 1\rangle$, $v(1) = \sqrt{5}$ and $\mathbf{a}(1) = \mathbf{r}''(1) = \langle 2, 0, -1\rangle$.
(c) We have $\mathbf{r}'(t) = \langle -\sin t, \cos t, -e^{-t}\rangle$, and $\mathbf{r}''(t) = \langle -\cos t, -\sin t, e^{-t}\rangle$. Then $\mathbf{r}'(0) = \langle 0, 1, -1\rangle$, $\mathbf{r}''(0) = \langle -1, 0, 1\rangle$, and $\|\mathbf{r}'(0)\| = \sqrt{2}$. So, $a_T(0) = \frac{\mathbf{r}'(0)\cdot\mathbf{r}''(0)}{\|\mathbf{r}'(0)\|} = -\frac{1}{\sqrt{2}}$. Next $\mathbf{r}'(0) \times \mathbf{r}''(0) = \langle 1, 1, 1\rangle$, and so, $a_N(0) = \frac{\|\mathbf{r}'(0)\times\mathbf{r}''(0)\|}{\|\mathbf{r}'(0)\|} = \sqrt{\frac{3}{2}}$.
Q4. 2+2+3+3+3=13 (a) We have $\frac{\partial\omega}{\partial x} = -e^{-x}\cos y - e^{-y}\sin x$, and $\frac{\partial^2\omega}{\partial x^2} = e^{-x}\cos y - e^{-y}\cos x$. $\frac{\partial\omega}{\partial y} = -e^{-x}\sin y - e^{-y}\cos x$, and $\frac{\partial^2\omega}{\partial y^2} = -e^{-x}\cos y + e^{-y}\cos x$. Hence we obtain $\frac{\partial^2\omega}{\partial x^2} + \frac{\partial^2\omega}{\partial y^2} = 0$.
(b) By applying chain rule, we get $\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} = \frac{-2e^{-2t}}{x+y} + \frac{3t^2-2t}{x+y} = \frac{-2e^{-2t}+3t^2-2t}{e^{-2t}+t^3-t^2+5}$.
(c) We get $\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{i} = (2x - 5y)\mathbf{i} + (6y - 5x)\mathbf{j}$. $\nabla f|_P = 11\mathbf{i} - 21\mathbf{j}$, and $\mathbf{u} = \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j})$. Hence, $D_{\mathbf{u}}f|_P = \nabla f|_p \cdot \mathbf{u} = \frac{-10}{\sqrt{2}} \approx -7.07$
(d) We have $f_x(x, y) = 3x^2 + 3y$, and $f_y(x, y) = 3x - 3y^2$. By solving $x^2 + y = 0$ and $x - y^2 = 0$, we obtain the critical points $(0, 0)$ and $(1, -1)$. Now $D(x, y) = -36xy - 9$. Thus, we get $D(0, 0) = -9 < 0$ implies $(0, 0, 0)$ is a saddle point. $D(1, -1) = 27 > 0$ and $f_{xx}(1, -1) = 6 > 0$ implying $f(1, -1) = -1$ is a local minimum.
(e) Applying Lagrange multiplier, we get $\nabla f = \lambda\nabla g \implies \langle 2x, 2y, 2z\rangle = \lambda\langle 1, -1, 1\rangle$. Thus, $x = \frac{\lambda}{2}, y = \frac{-\lambda}{2}$ and $z = \frac{\lambda}{2}$; using $x - y + z = 1$, we get $\lambda = \frac{2}{3}$, and hence, $x = \frac{1}{3}, y = \frac{-1}{3}$ and $z = \frac{1}{3}$. Therefore, the minimal value of f is $f(\frac{1}{3}, \frac{-1}{3}, \frac{1}{3}) = \frac{1}{3}$.