FINAL EXAMINATION, SEMESTER II, 1445

Department of Mathematics, College of Science

King Saud University

MATH 107 FULL MARK 40 TIME 3 HOURS

Q1. [Marks: $3+4=7$]

(a) Find the value of λ for which the following system of linear equations

$$3x+y+(\lambda^2-6)z=\lambda-3$$
$$2x+3y+z=-1$$
$$x+2y+z=0$$

has: (i) no solution (ii) unique solution (iii) infinitely many solutions.

(b) Use adjoint method to find the ***inverse*** of the coefficient matrix of the following system, and hence using this ***inverse*** find the solution of the system

$$x+2y+3z=1$$
$$2x-y=0$$
$$3x+3z=2$$

Q2: [Marks: $2+3+3+3=11$]

(a) If l has parametric equations $x=5-3t, y=-2+t, z=1+9t$, find parametric equations for the line through $P(-6,4,-3)$ that is parallel to l.

(b) Find an equation of the plane through the point $P(2,4,-5)$ with normal vector $\mathbf{a}=-\mathbf{i}+2\mathbf{j}-3\mathbf{k}$.

(c) Find the volume of a box having adjacent sides AB, AC, and AD where $A(2,1,-1), B(3,0,2), C(4,-2,1)$ and $D(5,-3,0)$.

(d) Sketch the graph of $4x^2-9y^2+z^2=36$ in an xyz - coordinate system, give traces and identify the surface.

Q3: [Marks: $3+4+3=10$]

(a) A moving particle starts at an initial position $\mathbf{r}(0)=\langle 1,0,0\rangle$ with initial vector $\mathbf{v}(0)=\mathbf{i}-\mathbf{j}+\mathbf{k}$. If its acceleration $\mathbf{a}(t)=4t\mathbf{i}+6t\,\mathbf{j}+\mathbf{k}$, find the velocity and position at time t.

(b) Find tangential component of acceleration and normal component of acceleration for the curve given by $\mathbf{r}(t)=2t\mathbf{i}+t^2\mathbf{j}-\frac{1}{3}t^3\mathbf{k}$. Also, find curvature.

(c) Let $f(x,y)=\dfrac{xy^2}{x^2+y^4}$. Show that $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Q4: [Marks: $3+3+3+3=12$]

(a) Find the equation of the tangent plane, and parametric equations for the normal line to the graph of $x^3-2xy+z^3+7y+6=0$ at the point $P(1,4,-3)$.

(b) Find the directional derivative of $f(x,y,z)=xy+yz+xz$ at the point $A(1,1,1)$ in the direction of the vector $\mathbf{v}=2\mathbf{i}+\mathbf{j}-\mathbf{k}$.

(c) If $g(x,y)=-x^3+4xy-2y^2+1$, find the local extrema and saddle points of g.

(d) Let $f(x,y,z)=2x^2+y^2+3z^2$. Use Lagrange multiplier to find minimum value of f subject to the constraint $2x-3y-4z=49$.

Solution M - 107 Final Exam 1445 P-1

Q_1 (a)

$$\left[\begin{array}{ccc|c} 3 & 1 & \lambda^2-6 & \lambda-3 \\ 2 & 3 & 1 & -1 \\ 1 & 2 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & \lambda^2-4 & \lambda+2 \end{array}\right] \quad (1)$$

(2)

(i) The system has no solution for $\lambda = 2$ since $r_3 \Rightarrow 0z = 4$, so no solution, (1)

(ii) The system has unique solution for $\lambda \neq 2, \lambda \neq -2$ (1)

(iii) The system has infinitely many solution for $\lambda = -2$ (1)

(b) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 0 \\ 3 & 0 & 3 \end{bmatrix}$

Matrix of the Cofactor is $\begin{bmatrix} -3 & -6 & 3 \\ -6 & -6 & 6 \\ 3 & 6 & -5 \end{bmatrix}$ (1)

$\det(A) = -6$

$Adj(A) = \begin{bmatrix} -3 & -6 & 3 \\ -6 & -6 & 6 \\ 3 & 6 & -5 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

$$A^{-1} = \frac{1}{6}\begin{bmatrix} 3 & 6 & -3 \\ 6 & 6 & -6 \\ -3 & -6 & 5 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 1 & -\frac{1}{2} \\ 1 & 1 & -1 \\ -\frac{1}{2} & -1 & \frac{5}{6} \end{bmatrix} \quad (2)$$

$$A\underline{x} = B$$

$$\Rightarrow \underline{x} = A^{-1}B = \begin{bmatrix} \frac{1}{2} & 1 & -\frac{1}{2} \\ 1 & 1 & -1 \\ -\frac{1}{2} & -1 & 5/6 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -1 \\ 7/6 \end{bmatrix} \Rightarrow x = -\frac{1}{2},\ y = -1,\ z = 7/6 \quad (1)$$

Q_2 (a) The direction vector $\underline{a} = \langle -3, 1, 9 \rangle$ for the line l that passes through $P(-6,4,-3)$. The parametric equations for this line is given by

$$x = -6-3t,\ y = 4+t,\ z = -3+9t,\quad t \in \mathbb{R}.$$

(2)

(b) The equation of the required plane is

$$-1(x-2) + 2(y-4) - 3(z+5) = 0$$

$$\Rightarrow x - 2y + 3z + 21 = 0$$

(3)

(c)
$$\left.\begin{aligned} \underline{a} &= \overrightarrow{AB} = \langle 1,-1,3 \rangle \\ \underline{b} &= \overrightarrow{AC} = \langle 2,-3,2 \rangle \\ \underline{c} &= \overrightarrow{AD} = \langle 3,-4,1 \rangle \end{aligned}\right\} \ (1)$$

$$\therefore V = |\underline{a} \times \underline{b} \cdot \underline{c}| = \left| \begin{vmatrix} 1 & -1 & 3 \\ 2 & -3 & 2 \\ 3 & -4 & 1 \end{vmatrix} \right| \cdot \quad (2)$$

$\therefore V = 4$ unit of volume.

(d)
$$4x^2 - 9y^2 + z^2 = 36$$

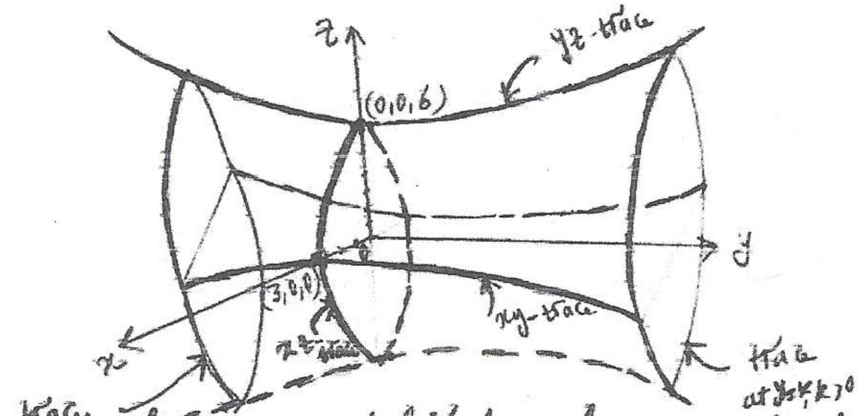
$$\Rightarrow \frac{x^2}{9} - \frac{y^2}{4} + \frac{z^2}{36} = 1$$

It is hyperboloid of one sheet and its axis is y-axis

(1)

[Please see next Page]

Trace	Equation of trace	Description
on xy-plane $(z=0)$	$\frac{x^2}{9}-\frac{y^2}{4}=1$	Hyperbola
on yz-plane $(x=0)$	$\frac{z^2}{36}-\frac{y^2}{4}=1$	Hyperbola
on xz-plane $(y=0)$	$\frac{x^2}{9}+\frac{z^2}{36}=1$	Ellipse



Hyperboloid of one sheet, its axis is y-axis

Q_3 (a) $\underline{r}(0) = \langle 1, 0, 0 \rangle$

since $\underline{a}(t) = \underline{v}'(t)$

1.5 $\underline{v}(t) = \int \underline{a}(t)\,dt = \int (4t\underline{i} + 6t\underline{j} + \underline{k})\,dt$

$= 2t^2\underline{i} + 3t^2\underline{j} + t\underline{k} + \underline{c}$

As $\underline{v}(0) = \underline{i} - \underline{j} + \underline{k} = \underline{c}$

$\Rightarrow \underline{c} = \underline{i} - \underline{j} + \underline{k}$ $\Big\}$ $\underline{v}(t) = (2t^2+1)\underline{i} + (3t^2-1)\underline{j} + (t+1)\underline{k}$

since $\underline{v}(t) = \underline{r}'(t)$

$\underline{r}(t) = \int \underline{v}(t)\,dt = \int \left((2t^2+1)\underline{i} + (3t^2-1)\underline{j} + (t+1)\underline{k}\right) dt$

$= \left(\frac{2}{3}t^3 + t\right)\underline{i} + (t^3 - t)\underline{j} + \left(\frac{t^2}{2} + t\right)\underline{k} + \underline{c}$

As $\underline{r}(0) = \underline{i} \Rightarrow \underline{c} = \underline{i}$

$\therefore \underline{r}(t) = \left(\frac{2}{3}t^3 + t + 1\right)\underline{i} + (t^3 - t)\underline{j} + \left(\frac{t^2}{2} + t\right)\underline{k}$

1.5

(b) $\underline{r}'(t) = 2\underline{i} + 2t\underline{j} - t^2\underline{k} \Rightarrow \underline{r}''(t) = 2\underline{j} - 2t\underline{k}$

$\|\underline{r}'(t)\| = \sqrt{4 + 4t^2 + t^4} = t^2 + 2$

$\underline{r}'(t) \cdot \underline{r}''(t) = \langle 2, 2t, -t^2 \rangle \cdot \langle 0, 2, -2t \rangle$

$= 4t + 2t^3 = 2t(t^2 + 2)$

$\therefore a_T = \dfrac{\underline{r}'(t) \cdot \underline{r}''(t)}{\|\underline{r}'(t)\|} = \dfrac{2t(t^2+2)}{t^2+2} = 2t$

1.5

Q_3 (b) $\underline{r}'(t) \times \underline{r}''(t)$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 2t & -t^2 \\ 0 & 2 & -2t \end{vmatrix}$$

$$= (-4t^2 + 2t^2)\underline{i} - (-4t - 0)\underline{j} + (4 - 0)\underline{k}$$

$$= -2t^2\underline{i} + 4t\underline{j} + 4\underline{k}$$

$\| \underline{r}'(t) \times \underline{r}''(t) \|$

$$= \sqrt{4t^4 + 16t^2 + 16}$$

$$= 2\sqrt{t^4 + 4t^2 + 4}$$

$$= 2(t^2 + 2)$$

$$\therefore a_N = \frac{\| \underline{r}'(t) \times \underline{r}''(t) \|}{\| \underline{r}'(t) \|}$$

$$= \frac{2(t^2+2)}{t^2+2} = 2$$

1.5

$$k \text{ (curvature)} = a_N \frac{1}{\| \underline{r}'(t) \|^2}$$

$$= 2/(t^2+2)^2$$

1

Q3 (c) Take $y = mx$ then

$$\lim_{(x,y)\to(0,0)} f(x,y)$$

$$= \lim_{x\to 0} f(x, mx)$$

$$= \lim_{x\to 0} \frac{m^2 x^3}{x^2 + m^4 x^4}$$

1.5

$$= \lim_{x\to 0} \frac{m^2 x}{1 + m^4 x^2} = 0$$

Now use $x = y^2$, then

$$\lim_{(x,y)\to(0,0)} f(x,y)$$

$$= \lim_{y\to 0} f(y^2, y)$$

1.5

$$= \lim_{y\to 0} \frac{y^4}{2y^4} = \frac{1}{2}$$

Since different paths lead to different limits, by Two-path Rule, limit does not exit!

Q4 (a) Let $F = x^3 - 2xy + z^3 + 7y + 6 = 0$

$\therefore \nabla F = \langle 3x^2 - 2y, -2x + 7, 3z^2 \rangle$

$\Rightarrow \nabla F(1, 4, -3) = \langle -5, 5, 27 \rangle$

Eq. of tangent plane is 1.5

$5x - 5y - 27z = 66$

The parametric eqs for the normal line are:

$x = 1 - 5t,\ y = 4 + 5t,\ z = -3 + 27t$

1.5

Q4 (b) $f(x,y,z) = xy + yz + xz$

$$\nabla f = \langle y+z,\ x+z,\ x+y \rangle$$

$$\nabla f(1,1,1) = \langle 2, 2, 2 \rangle \qquad 1$$

$$\underline{u} = \frac{v}{\|v\|} = \frac{\langle 2, 1, -1 \rangle}{\sqrt{4+1+1}} = \left\langle \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-1}{\sqrt{6}} \right\rangle$$

$$D_{\underline{u}} f(1,1,1) = \nabla f(1,1,1) \cdot \underline{u} \qquad 1$$

$$= \langle 2,2,2 \rangle \cdot \left\langle \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-1}{\sqrt{6}} \right\rangle$$

$$= \frac{4}{\sqrt{6}} + \frac{2}{\sqrt{6}} - \frac{2}{\sqrt{6}}$$

$$= \frac{4}{\sqrt{6}} = \frac{2\sqrt{6}}{3} \qquad 1$$

(c) $g(x,y) = -x^3 + 4xy - 2y^2 + 1$

$$g_x = -3x^2 + 4y$$

$$\left.\begin{aligned} g_{xx} &= -6x \\ g_y &= 4x - 4y \\ g_{yy} &= -4 \\ g_{xy} &= 4 \end{aligned}\right\} \qquad 1$$

Q_4 (c) with $g_x = 0$ and $g_y = 0$,

$$-3x^2 + 4y = 0, \quad 4x - 4y = 0$$

$$\Rightarrow \quad y = x = 0, \quad y = x = \frac{4}{3}$$

$$D = g_{xx}(0,0)\, g_{yy}(0,0) - \left[g_{xy}(0,0)\right]^2 \quad 1$$

$= 0 - 16 < 0$, by Second Partial Test, at $(0,0)$ we get saddle point while at $(4/3, 4/3)$,

$$D = g_{xx}\left(\frac{4}{3},\frac{4}{3}\right) g_{yy}\left(\frac{4}{3},\frac{4}{3}\right) - \left[g_{xy}\left(\frac{4}{3},\frac{4}{3}\right)\right]^2$$

$$= -8(-4) - 16 = 16 > 0 \quad 1$$

and because $g_{xx}\left(\frac{4}{3},\frac{4}{3}\right) = -8 < 0$

g has rel. max at $\left(\frac{4}{3},\frac{4}{3}\right)$.

(d) Let $g(x,y,z) = 2x - 3y - 4z = 49$

Then, because

1 $\nabla f(x,y,z) = 4x\,\underline{i} + 2y\,\underline{j} + 6z\,\underline{k}$

and $\lambda \nabla g(x,y,z) = 2\lambda\,\underline{i} - 3\lambda\,\underline{j} - 4\lambda\,\underline{k}$

1 We obtain:

$$\left.\begin{aligned} 4x &= 2\lambda \\ 2y &= -3\lambda \\ 6z &= -4\lambda \end{aligned}\right\} \text{ and } 2x - 3y - 4z = 49$$

The solution of the system is $x = 3$, $y = -9$, $z = -4$.

1 $\therefore$ the minimum value of f is

$$f(3,-9,-4) = 147.$$