FINAL EXAMINATION, SEMESTER I, 2024

DEPT. MATH., COLLEGE OF SCIENCE, KSU

MATH: 107 FULL MARK: 40 TIME: 3 HOURS

Q1. [3+2+4=9]

(a) Solve the system of linear equations by Gaussian elimination:

$$x + 2y - z = 2$$
$$x - 2y + 3z = 1$$
$$x + 2y - z = 2$$

(b) Explain why the above system of linear equations cannot be solved by Cramer's rule.

(c) Find the inverse of the matrix A by using $adj(A)$, where

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix}$$

Q2. [3+3=6]

(a) Find the volume of a box having adjacent sides AB, AC and AD where $A(2,1,-1)$, $B(3,0,2)$, $C(4,-2,1)$ and $D(5,-3,0)$.

(b) Find the velocity, speed and acceleration of a particle that moves along the plane curve

$$\mathbf{r}(t) = \langle t\sin(2t); t\cos(2t)\rangle \text{ at } t = 0.$$

Q3. [3+2+3=8]

(a) The position vector of a moving point at time t is $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + \mathbf{k}$. Find the tangential and normal components of acceleration, and curvature.

(b) Find the domain of the function defined by $f(x,y) = \sqrt{1+y-x} + \sqrt{x+y-1} + \ln(1-x^2-y)$.

(c) Prove that the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^4+y^3}{y^2+(y^2+x^2)^2}$$

does not exit.

Q4. [3+2+3=8]

(a) Use Chain rule to find $\frac{\partial p}{\partial r}$ and $\frac{\partial p}{\partial s}$ if $p = u^2 + 3v^2 - 4w^2$ with $u = 2r - s$, $v = -r + 2s$, $w = r + s$.

(b) If $w = f(x^2 + y^2)$, show that $y\frac{\partial w}{\partial x} - x\frac{\partial w}{\partial y} = 0$.

(c) Let $f(x,y,z) = xy^2e^z$. Find the directional derivative of f at the point $P(2,-1,0)$ in the direction of vector $\mathbf{a} = 4\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$.

Q5 [3+4+2=9]

(a) Identify the surface $z = x^2 + y^2$, and find an equation for the tangent plane and parametric equations for the normal line to the given surface at the point $(-1,1,2)$.

(b) Find the local extrema and saddle points of the function defined by

$$f(x,y) = 9x^2y - 6x^3 + y^3 - 12y.$$

(c) Find the three positive numbers whose sum is 1 and whose product is a maximum.

1

Model Answer of Final Exam
S1/1446

Q_1: [3+2+4] 9

(a) The augmented Mx is

$$\begin{bmatrix} 1 & 2 & -1 & 2 \\ 1 & -2 & 3 & 1 \\ 1 & 2 & -1 & 2 \end{bmatrix}$$

$$\xrightarrow{-R_1+R_2,\ -R_1+R_3} \begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & -4 & 4 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{-\frac{1}{4}R_2} \begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 1 & -1 & 1/4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

REF

So, x and y are leading variables but z is free variable. Let $z=t$, $t\in\mathbb{R}$ then $y-t=1/4$ i.e. $y=\frac{1}{4}+t$, and

$x+2(\frac{1}{4}+t)-t=2$

i.e. $x=\frac{3}{2}-t$. There are infinite many solutions for the system.

(b)

$$\therefore \det(A) = \begin{vmatrix} 1 & 2 & -1 \\ 1 & -2 & 3 \\ 1 & 2 & -1 \end{vmatrix} = 0$$

$\therefore$ the system cannot be solved by Cramer's rule. the given system has infinite many solutions as shown in part (a).

(c)

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix}$$

$\det A = 3(3-2)+2(4-1)=9$

The matrix of Cofactors is

$$C = \begin{bmatrix} 1 & 4 & -2 \\ -5 & 7 & 1 \\ 3 & -6 & 3 \end{bmatrix}$$

$\operatorname{adj}(A) = C^T$, $A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$

$$\therefore A^{-1} = \frac{1}{9}\begin{bmatrix} 1 & -5 & 3 \\ 4 & 7 & -6 \\ -2 & 1 & 3 \end{bmatrix}$$

Q_2: [3+3] 6

(a)

$\vec{AB} = \langle 1,-1,3\rangle$, $\vec{AC} = \langle 2,-3,2\rangle$ and $\vec{AD} = \langle 3,-4,1\rangle$

The volume of the box is

$V = |(\vec{AB}\times\vec{AC})\cdot\vec{AD}|$,

$$(\vec{AB}\times\vec{AC})\cdot\vec{AD} = \begin{vmatrix} 1 & -1 & 3 \\ 2 & -3 & 2 \\ 3 & -4 & 1 \end{vmatrix} = 4$$

$\therefore V = 4$ cubic unit.

(b) Given

$\vec{r}(t) = t\sin 2t\,\vec{i} + t\cos 2t\,\vec{j}$

Then

$\vec{v}(t) = \vec{r}\,'(t)$

$= (\sin 2t + 2t\cos 2t)\vec{i} + (\cos 2t - 2t\sin 2t)\vec{j}$

Speed $= \|\vec{v}(t)\|$

$= \sqrt{1+4t^2}$

and the acceleration is

$\vec{a}(t) = \vec{v}\,'(t) = (2\cos 2t + 2\cos 2t - 4t\sin 2t)\vec{i} + (-2\sin 2t - 2\sin 2t - 4t\cos 2t)\vec{j}$

$\therefore \vec{a}(t) = (4\cos 2t - 4t\sin 2t)\vec{i} - (4\sin 2t + 4t\cos 2t)\vec{j}$

$= \langle 4\cos 2t - 4t\sin 2t, -4\sin 2t - 4t\cos 2t\rangle$

• At $t=0$, $\vec{v}(0) = \langle 0,1\rangle$, $\|\vec{v}(0)\| = 1$ and $\vec{a}(0) = \langle 4,0\rangle$.

$\frac{2}{}$

Q_3: [3 + 2 + 3] 8

(a) $\vec{r}(t) = \cos t\,\vec{i} + \sin t\,\vec{j} + \vec{k}$

The tangential component of the acceleration $\vec{a}(t)$ is

$$a_T = \frac{\vec{r}'(t) \cdot \vec{r}''(t)}{\|\vec{r}'(t)\|}$$

$$\vec{r}'(t) = -\sin t\,\vec{i} + \cos t\,\vec{j} + 0\,\vec{k}$$

$$\vec{r}''(t) = -\cos t\,\vec{i} - \sin t\,\vec{j} + 0\,\vec{k}$$

$$\therefore a_T = \frac{\sin t \cos t - \cos t \sin t}{\sqrt{\sin^2 t + \cos^2 t}} = 0$$

The Normal component of $\vec{a}(t)$ is

$$a_N = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|}$$

$$\vec{r}'(t) \times \vec{r}''(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin t & \cos t & 0 \\ -\cos t & -\sin t & 0 \end{vmatrix}$$

$$= (\sin^2 t + \cos^2 t)\,\vec{k} = \vec{k}$$

$\therefore a_N = 1$ where $\|\vec{r}'(t) \times \vec{r}''(t)\| = 1$ and $\|\vec{r}'(t)\| = 1$.

The curvature is

$$K = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3} = 1.$$

(b)

$$f(x,y) = \sqrt{1+y-x} + \sqrt{x+y-1} + \ln(1-x^2-y)$$

The domain of the function f is

$$D_f = \{(x,y) \in \mathbb{R}^2 \mid 1+y-x \geq 0,\ x+y-1 \geq 0,\ 1-x^2-y > 0\}$$

$$\therefore D_f = \{(x,y) \in \mathbb{R}^2 \mid y \geq x-1,\ y \geq 1-x,\ y < 1-x^2\}$$

(c) To prove that

$$\lim_{(x,y)\to(0,0)} \frac{x^4+y^3}{y^2+(y^2+x^2)^2} \text{ doesn't exist}$$

We can use the two-path rule as follows.

- For the path $y=0$ (along x-axis), $x \to 0$ we get

$$\lim_{(x,0)\to(0,0)} \frac{x^4+0^3}{0^2+(0^2+x^2)^2}$$

$$= \lim_{(x,0)\to(0,0)} \frac{x^4}{x^4} = 1 \quad ①$$

- For the path $y=x$, $x \to 0$, we get

$$\lim_{x\to 0} \frac{x^4+x^3}{x^2+(2x^2)^2}$$

$$= \lim_{x\to 0} \frac{x^4+x^3}{x^2+4x^4}$$

$$= \lim_{x\to 0} \frac{x^2+x}{1+4x^2} = \frac{0}{1} = 0 \quad ②$$

$\therefore$ From ① and ②, the limit does not exist. #

Note that:

Also, we can use the path $x=0$ (along y-axis), $y \to 0$ to get

$$\lim_{(0,y)\to(0,0)} \frac{0^4+y^3}{y^2+y^4}$$

$$= \lim_{(0,y)\to(0,0)} \frac{y}{1+y^2} = 0 \quad ②'$$

لا يكتب في
هذا الهامش

3/

Q_4 [3+2+3] 8

(a)

$P = u^2 + 3v^2 - 4w^2$, $u = 2r - s$,
$v = -r + 2s$, $w = r + s$

$$\frac{\partial P}{\partial r} = \frac{\partial P}{\partial u}\frac{\partial u}{\partial r} + \frac{\partial P}{\partial v}\frac{\partial v}{\partial r} + \frac{\partial P}{\partial w}\frac{\partial w}{\partial r}$$

$$= 4u - 6v - 8w = 6r - 24s$$

$$\frac{\partial P}{\partial s} = \frac{\partial P}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial P}{\partial v}\frac{\partial v}{\partial s} + \frac{\partial P}{\partial w}\frac{\partial w}{\partial s}$$

$$= -2u + 12v - 8w$$
$$= -24r + 18s$$

(b)

$w = f(x^2+y^2)$, $u = x^2+y^2$

$y\frac{\partial w}{\partial x} - x\frac{\partial w}{\partial y}$??

$\because \frac{\partial w}{\partial x} = \frac{dw}{du}\frac{\partial u}{\partial x} = 2x\frac{dw}{du}$ ①

& $\frac{\partial w}{\partial y} = \frac{dw}{du}\frac{\partial u}{\partial y} = 2y\frac{dw}{du}$ ②

$\therefore$ from ① and ②,

$$y\frac{\partial w}{\partial x} - x\frac{\partial w}{\partial y} = 2xy\frac{dw}{du} - 2xy\frac{dw}{du} = 0$$

(c) $f(x,y,z) = xy^2e^z$

The directional derivative of f is

$D_{\vec{u}} f(x,y,z) = \nabla f(x,y,z) \cdot \vec{u}$

$\nabla f(x,y,z) = \langle y^2e^z, 2xye^z, xy^2e^z \rangle$,

$\vec{u} = \frac{\vec{a}}{\|\vec{a}\|} = \langle \frac{2}{3}, \frac{-1}{3}, \frac{2}{3} \rangle$

$\therefore D_{\vec{u}} f(2,-1,0) = \langle 1, -4, 2 \rangle \cdot \langle \frac{2}{3}, \frac{-1}{3}, \frac{2}{3} \rangle$
$= \frac{10}{3}$ #

Q_5 [3+4+2] 9

(a)

The surface $z = x^2+y^2$ is a paraboloid whose axis is z-axis. Let $F(x,y,z) = x^2+y^2-z = 0$

$\nabla F(x,y,z) = \langle 2x, 2y, -1 \rangle$

$\nabla F \langle -1,1,2 \rangle = \langle -2, 2, -1 \rangle$

which is the normal to the surface and the tangent plane at $P(-1,1,2)$

$\Rightarrow$ The Eqn of the tangent plane is

$-2(x+1) + 2(y-1) - (z-2) = 0$

i.e. $2x - 2y + z + 2 = 0$

and the parametric Eqns of the normal line are

$x = -1 - 2t$, $y = 1 + 2t$, $z = 2 - t$, $t \in \mathbb{R}$.

(b)

$f(x,y) = 9x^2y - 6x^3 + y^3 - 12y$

$\Rightarrow f_x = 18xy - 18x^2$
$= 18x(y - x) = 0$ ①,

$f_y = 9x^2 + 3y^2 - 12 = 0$ ②

Solving Eqns ①, ②

① $\Rightarrow x = 0$ or $y = x$

for $x = 0$ ② $\Rightarrow y^2 = 4$ i.e. $y = \pm 2$

for $y = x$ ② $\Rightarrow x^2 = 1$ i.e. $x = \pm 1$

- So, the critical points are $(0,2)$, $(0,-2)$, $(1,1)$ and $(-1,-1)$
- The Discriminant D of f is

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - (f_{xy})^2$$

$$\therefore D = \begin{vmatrix} 18y - 36x & 18x \\ 18x & 6y \end{vmatrix}$$

$$= \underbrace{(18y - 36x)}_{f_{xx}}(6y) - (18x)^2$$

4.

$\because D(0,2) = 432 > 0$, $f_{xx}(0,2) = 36 > 0$

$\therefore$ f has a local minimum at $(0,2)$,

$\because D(0,-2) = 432 > 0$, $f_{xx}(0,-2) = -36 < 0$

$\therefore$ f has a local maximum at $(0,-2)$

$\because D(1,1) < 0$ and $D(-1,-1) < 0$

$\therefore$ Both $(1,1)$ and $(-1,-1)$ are saddle points.

---

c)

Here, Lagrange pb can be written as:

$$\max f(x,y,z) = xyz \quad (1)$$

$$\text{s.t } g(x,y,z) = x+y+z-1 = 0 \quad (2)$$

where x, y and z are three positive numbers.

we have $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$ (3)

Substitute (1) and (2) in (3), we get

$$\langle yz, xz, xy \rangle = \lambda \langle 1,1,1 \rangle$$

$$yz = \lambda, \; xz = \lambda, \; xy = \lambda$$

$$\therefore x = y = z$$

$$(2) \Rightarrow x = y = z = \frac{1}{3}. \quad \#$$

i.e. the three positive numbers are $\frac{1}{3}$, $\frac{1}{3}$ and $\frac{1}{3}$.