Math 244 Notes

د. صدراوي

I, 1)

Matrices and operations on Matrices

Definition: A matrix is an array of numbers arranged in rows and columns. We write

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & \vdots \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

a_{ij} denotes the element in the i^{th} row and j^{th} column.

We say A is a matrix of order $m \times n$ and write $A = (a_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}}$

Example $A = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & -1 & 1 \\ 2 & 3 & 1 & 5 \end{pmatrix}$

$a_{21} = 0$

$a_{34} = 5$

I_2)

$$A = \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix}$$ is a column matrix

it is of order $m \times 1$

$B = (b_1, \ldots b_n)$ is a row matrix

of order $1 \times n$

if $m = n$ we say the matrix is a square matrix

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ 4 & -1 & 1 \end{pmatrix}$$ is square of order 3.

<u>Addition of matrices</u>

If $A = (a_{ij})_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}$ and $B = (b_{ij})_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}$

we define $A + B$ by

$$A + B = (a_{ij} + b_{ij})_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}$$

$I_3)$ Example

$$A=\begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & 2 & -4 \end{pmatrix} \qquad B=\begin{pmatrix} 0 & -1 & 3 & -2 \\ 5 & 3 & -1 & 0 \end{pmatrix}$$

$$A+B=\begin{pmatrix} 1 & 1 & 6 & -1 \\ 5 & 2 & 1 & -4 \end{pmatrix}$$

if $C=\begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 7 \\ 1 & 4 & -2 \end{pmatrix}$

$A+C$ is not defined since A is of order 2×4 and C of order 3×3

the zero matrix is $A=(a_{ij})_{\substack{1\le i\le m\\1\le j\le n}}$ where $a_{ij}=0$ for any i and j.

If $A=(a_{ij})_{\substack{1\le i\le m\\1\le j\le n}}$ and $\alpha\in\mathbb{R}$

we define $\alpha.A=(\alpha a_{ij})_{\substack{1\le i\le m\\1\le j\le n}}$

I4)

If $A = \begin{pmatrix} 4 & 3 & 1 \\ 2 & 2 & -1 \\ 0 & 1 & 2 \end{pmatrix}$ $B = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 3 \\ -2 & 2 & 2 \end{pmatrix}$

$$2A = \begin{pmatrix} 8 & 6 & 2 \\ 4 & 4 & -2 \\ 0 & 2 & 4 \end{pmatrix} \quad (-3)B = \begin{pmatrix} 3 & -3 & 0 \\ -3 & +3 & -9 \\ +6 & -6 & -6 \end{pmatrix}$$

$$2A + (-3)B = \begin{pmatrix} 11 & 3 & 2 \\ 1 & 7 & -11 \\ 6 & -4 & -2 \end{pmatrix}$$

obviously we define $A - B = A + (-1)B$ if A and B have the same order

<u>Theorem</u> if A, B, and C have the same order then

I5)

1) $A+B=B+A$

2) $A+(B+C)=(A+B)+C$

3) $A+0=0+A=A$

4) $A+(-A)=0$

5) $b(A+B)=bA+bB \quad b\in\mathbb{R}$

6) $(r+s)A=rA+sA \quad r,s\in\mathbb{R}$

7) $(r.s).A=r.(sA) \quad r,s\in\mathbb{R}$

8) $1.A=A.$

Example

of $A=\begin{pmatrix}1 & 2 & 0 & 1\\ -1 & -2 & 1 & 0\\ 1 & 1 & 0 & -1\end{pmatrix} \quad B=\begin{pmatrix}2 & 3 & 0 & 1\\ -1 & 2 & 2 & 2\\ 1 & 3 & 4 & -1\end{pmatrix}$

$2A-B=\begin{pmatrix}0 & 1 & 0 & 1\\ -1 & -6 & 0 & -2\\ 1 & -1 & -4 & -1\end{pmatrix}$

The transpose of a matrix

of $A=(a_{ij})_{\substack{1\le i\le m\\ 1\le j\le n}}$

I6)

the transpose of A is defined by $A^T = (a_{ji})_{\substack{1 \leq j \leq n \\ 1 \leq i \leq m}}$

in other words the rows of A are the columns of A^T and the columns of A are the rows of A^T.

<u>Example</u>

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & -1 \\ 0 & 1 & -1 & 1 & 0 \\ -3 & 1 & 2 & 1 & -2 \end{pmatrix}$$

$$A^T = \begin{pmatrix} 1 & 0 & -3 \\ 2 & 1 & 1 \\ 3 & -1 & 2 \\ 4 & 1 & 1 \\ -1 & 0 & -2 \end{pmatrix}$$

Properties of the transpose:

1) $(A^T)^T = A$ 2) $(bA)^T = bA^T$

3) $(A+B)^T = A^T + B^T$

I7)

Product of matrices

i) First case

$A = (a_1 \ldots a_n) \qquad B = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$

$AB = (a_1 b_1 + a_2 b_2 + \ldots a_n b_n)$

Example

$A = (-1 \ -2 \ 3) \qquad B = \begin{pmatrix} 0 \\ 3 \\ -2 \end{pmatrix}$

$AB = (-1 \cdot 0 + (-2) \cdot 3 + 3(-2))$

$= (-12)$

ii) general case

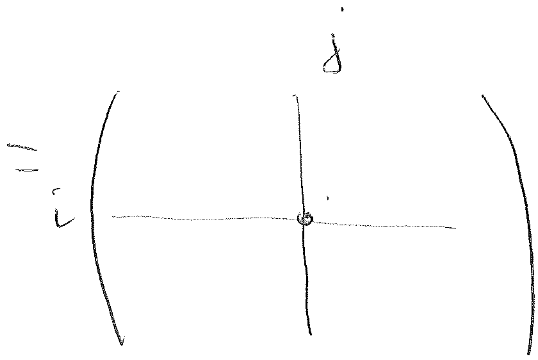
if $A = (a_{ij})_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}$

and $B = (b_{ij})_{\substack{1 \leq i \leq n \\ 1 \leq j \leq p}}$

We define AB as follows

I8)

$$A \cdot B = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \vdots \\ a_{i1} & \cdots & a_{in} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \cdots b_{1i} & b_{1p} \\ b_{21} & \vdots & \\ \vdots & \vdots & \\ b_{i} & \vdots & \\ b_{m1} & b_{ni} & b_{np} \end{pmatrix}$$



$$c_{ij} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots a_{in} b_{nj}$$

$$= \sum_{b=1}^{n} a_{ib} b_{bj}$$

<u>Example</u> $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \end{pmatrix}$

$$B = \begin{pmatrix} 1 & 1 \\ 0 & 2 \\ -1 & -2 \end{pmatrix}$$

Iq)

$$A.B = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 2 \\ -1 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 1.1+2.0+3(-1) & 1.1+2.2+3.(-2) \\ 0.1+1.0+(-1)(-1) & 0.1+1.2+(-1)(-2) \end{pmatrix}$$

$$= \begin{pmatrix} -2 & -1 \\ 1 & 4 \end{pmatrix}$$

2) If $A = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$ $B = \begin{pmatrix} 3 & 0 \\ -1 & 1 \\ 2 & 2 \end{pmatrix}$

$A \cdot B$ is not defined and

$$B.A = \begin{pmatrix} 3 & 0 \\ -1 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ -1 & 3 \\ 6 & 6 \end{pmatrix}$$

Exercice let $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 1 & +1 & -1 \end{pmatrix}$

$B = \begin{pmatrix} 1 & 2 & 3 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -2 & 2 & 1 \end{pmatrix}$ find AB, A^2B and ABA if possible.

1)

Elementary row operations

Definition: We call a row operation on a matrix one of the three following operations

i) We interchange two rows
ii) We multiply a row by a non zero number
iii) We add a multiple of a row to another row

Definition
We say that two matrices A and B are equivalent and write $A \sim B$ if one of the two is obtained from the other by a finite number of row operations

Notation
We denote by:
1) R_{ij} interchanging the rows i and j
2) αR_i multiplying the i^{th} row by α

3) αR_{ij} add to the row j the row αR_i.

Examples:

1)
$$\begin{pmatrix} 1 & 2 & 0 & 1 \\ -1 & 3 & 1 & 2 \\ 3 & 2 & -1 & 0 \end{pmatrix} \xrightarrow{R_{23}} \begin{pmatrix} 1 & 2 & 0 & 1 \\ 3 & 2 & -1 & 0 \\ -1 & 3 & 1 & 2 \end{pmatrix} \xrightarrow{2R_3}$$

$$\begin{pmatrix} 1 & 2 & 0 & 1 \\ 3 & 2 & -1 & 0 \\ -2 & 6 & 2 & 4 \end{pmatrix} \xrightarrow{-2R_{13}} \begin{pmatrix} 1 & 2 & 0 & 1 \\ 3 & 2 & -1 & 0 \\ -4 & 2 & 2 & 2 \end{pmatrix}$$

2)
$$\begin{pmatrix} 1 & 0 & 3 \\ -1 & 1 & -2 \\ 2 & 1 & 4 \end{pmatrix} \xrightarrow{1R_{12}} \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 2 & 1 & 4 \end{pmatrix} \xrightarrow{-2R_{13}}$$

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & +1 & -2 \end{pmatrix} \xrightarrow{-R_3} \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & -1 & 2 \end{pmatrix} \xrightarrow{+1R_{23}}$$

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{pmatrix}$$

3)

The row echelon form of a matrix

Definition

A matrix is in row echelon form if it satisfies the following properties

1) zero rows are below nonzero rows

2) The first nonzero element of a nonzero row is 1

3) the first nonzero element of a row is always in a column to the left of the nonzero element in a row below.

Examples

a) $A = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$ is in row echelon form

b) $B = \begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ is in row echelon form

c) $C = \begin{pmatrix} 1 & -2 & 3 & 0 \\ 0 & 1 & -3 & 2 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ is not in row echelon form.

4)

Theorem.

Every matrix is equivalent to a matrix in row echelon form.

Examples

i) reduce the matrix $A = \begin{pmatrix} 1 & 3 & 2 & 2 \\ 2 & -1 & -3 & -3 \\ 3 & -4 & -1 & 5 \end{pmatrix}$ to the row echelon form

$$\begin{pmatrix} 1 & 3 & 2 & 2 \\ 2 & -1 & -3 & -3 \\ 3 & -4 & -1 & 5 \end{pmatrix} \xrightarrow{-2R_{12}} \begin{pmatrix} 1 & 3 & 2 & 2 \\ 0 & -7 & -7 & -7 \\ 3 & -4 & -1 & 5 \end{pmatrix}$$

$$\xrightarrow{-3R_{13}} \begin{pmatrix} 1 & 3 & 2 & 2 \\ 0 & -7 & -7 & -7 \\ 0 & -13 & -7 & -1 \end{pmatrix} \xrightarrow{R_2/-7} \begin{pmatrix} 1 & 3 & 2 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & -13 & -7 & -1 \end{pmatrix}$$

$$\xrightarrow{13R_{23}} \begin{pmatrix} 1 & 3 & 2 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 6 & 12 \end{pmatrix} \xrightarrow{\frac{1}{6}R_3} \begin{pmatrix} 1 & 3 & 2 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

2) find the ~~echelon~~ echelon form of

$$A = \begin{pmatrix} 1 & -1 & 2 & 1 & -1 \\ 2 & 1 & 1 & -1 & 4 \\ 1 & 2 & -1 & -2 & 5 \\ 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

5)

$$\begin{pmatrix} 1 & -1 & 2 & 1 & -1 \\ 2 & 1 & 1 & -1 & 4 \\ 1 & 2 & -1 & -2 & 5 \\ 1 & 0 & 1 & 0 & 1 \end{pmatrix} \xrightarrow[\substack{-R_{13} \\ -R_{14}}]{-2R_{12}}$$

$$\begin{pmatrix} 1 & -1 & 2 & 1 & -1 \\ 0 & 3 & -3 & -3 & 6 \\ 0 & 3 & -3 & -3 & 6 \\ 0 & 1 & -1 & -1 & 2 \end{pmatrix} \xrightarrow{R_{24}} \begin{pmatrix} 1 & -1 & 2 & 1 & -1 \\ 0 & 1 & -1 & -1 & 2 \\ 0 & 3 & -3 & -3 & 6 \\ 0 & 3 & -3 & -3 & 0 \end{pmatrix}$$

$$\xrightarrow[-3R_{24}]{-3R_{23}} \begin{pmatrix} 1 & -1 & 2 & 1 & -1 \\ 0 & 1 & -1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The reduced row echelon form

Definition: a matrix is in the reduced row echelon form if it satisfies the following

a) It is in row echelon form

b) Each leading nonzero entry is the only nonzero entry in the column containing it.

6)

Ex: Find the reduced echelon form of the matrix
$$\begin{pmatrix} 1 & -1 & -1 & 1 & -3 & 2 \\ 1 & 0 & 1 & -2 & 2 & -1 \\ 3 & -2 & -2 & 1 & -3 & 0 \\ 3 & -2 & -1 & 1 & 2 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & -1 & 1 & -3 & 2 \\ 1 & 0 & 1 & -2 & 2 & -1 \\ 3 & -2 & -2 & 1 & -3 & 0 \\ 3 & -2 & -1 & 1 & 2 & 3 \end{pmatrix} \xrightarrow[\substack{-3R_{13} \\ -3R_{14}}]{-1R_{12}}$$

$$\begin{pmatrix} 1 & -1 & -1 & 1 & -3 & 2 \\ 0 & 1 & 2 & -3 & 5 & -3 \\ 0 & 0 & -1 & 1 & 1 & -3 \\ 0 & 0 & 0 & 1 & 6 & 0 \end{pmatrix} \xrightarrow{1R_{21}}$$

$$\begin{pmatrix} 1 & 0 & 1 & -2 & 2 & -1 \\ 0 & 1 & 2 & -3 & 5 & -3 \\ 0 & 0 & -1 & 1 & 1 & -3 \\ 0 & 0 & 0 & 1 & 6 & 0 \end{pmatrix} \xrightarrow{1R_{31}}$$

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 3 & -4 \\ 0 & 1 & 2 & -3 & 5 & -3 \\ 0 & 0 & -1 & 1 & 1 & -3 \\ 0 & 0 & 0 & 1 & 6 & 0 \end{pmatrix} \xrightarrow{2R_{32}}$$

7)

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 3 & -4 \\ 0 & 1 & 0 & -1 & 7 & -9 \\ 0 & 0 & -1 & 1 & 1 & -3 \\ 0 & 0 & 0 & 1 & 6 & 0 \end{pmatrix} \xrightarrow{-R_3}$$

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 3 & -4 \\ 0 & 1 & 0 & -1 & 7 & -9 \\ 0 & 0 & 1 & -1 & -1 & 3 \\ 0 & 0 & 0 & 1 & 6 & 0 \end{pmatrix} \xrightarrow[\substack{1R_{42} \\ 1R_{43}}]{1R_{41}}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 9 & -4 \\ 0 & 1 & 0 & 0 & 13 & -9 \\ 0 & 0 & 1 & 0 & 5 & 3 \\ 0 & 0 & 0 & 1 & 6 & 0 \end{pmatrix}$$

Definition:

we say that a matrix A is invertible if A is of order n and there exists a square matrix B of order n such that $AB = BA = I_n$

the inverse is denoted by A^{-1}.

8)

Theorem

1) The inverse is unique if it exists
2) $(A^{-1})^{-1} = A$
3) If A and B are invertible then
$$(AB)^{-1} = B^{-1}A^{-1}$$
4) In general If $A_1, \ldots, A_b$ are invertible
$$(A_1 \ldots A_b)^{-1} = A_b^{-1} \ldots A_1^{-1}$$
5) $(\alpha A)^{-1} = \frac{1}{\alpha} A^{-1}$ for any nonzero α
6) If A has an inverse
$$(A^T)^{-1} = (A^{-1})^T$$

Definition: An elementary matrix of order n is a matrix obtained from the identity matrix I_n by one row operation.

Examples.

1) $I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$E_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

is an elementary matrix and

So is $E_2 = \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

E_1 is obtained by the operation R_{12}
and E_2 is obtained by performing $3R_{41}$

put $A = \begin{pmatrix} 1 & -1 & 1 & 2 \\ 3 & 0 & 1 & -1 \\ 1 & 1 & 2 & -2 \\ 0 & 0 & 2 & -3 \end{pmatrix}$

$A_1 = \begin{pmatrix} 3 & 0 & 1 & -1 \\ 1 & -1 & 1 & 2 \\ 1 & 1 & 2 & -2 \\ 0 & 0 & 2 & -3 \end{pmatrix}$ $\quad A_2 = \begin{pmatrix} 1 & -1 & 7 & -7 \\ 3 & 0 & 1 & -1 \\ 1 & 1 & 2 & -2 \\ 0 & 0 & 2 & -3 \end{pmatrix}$

we have $A_1 = E_1 A$, $A_2 = E_2 A$.
and ~~$A_1 = R_{12} A$~~ $A_1 = R_{12} A \quad A_2 = 3R_{41} A$

theorem

if E is an elementary matrix then E is invertible and its inverse is an elementary matrix.

Example

$$E = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 2R_{31} I_3$$

$$E^{-1} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

theorem

If A is a square matrix of order n the following are equivalent

1) A is invertible
2) A is a product of elementary matrices
3) The reduced echelon form of A is I_n

Algorithm for finding the inverse of a matrix A of order n

1) reduce $[A|I_n]$ to the reduced echelon form $[B|C]$

2) If $B = I_n$ then $C = A^{-1}$

3) if $B \neq I_n$ then A is not invertible

Example find the inverse of the matrix $A = \begin{pmatrix} 2 & 4 & 3 \\ 0 & 1 & -1 \\ 3 & 3 & 7 \end{pmatrix}$

$$\left(\begin{array}{ccc|ccc} 2 & 4 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 3 & 3 & 7 & 0 & 0 & 1 \end{array}\right) \xrightarrow{-1R_{31}}$$

$$\left(\begin{array}{ccc|ccc} -1 & 1 & -4 & 1 & 0 & -1 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 3 & 3 & 7 & 0 & 0 & 1 \end{array}\right) \xrightarrow{-R_1} \left(\begin{array}{ccc|ccc} 1 & -1 & 4 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 3 & 3 & 7 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{-3R_{13}} \left(\begin{array}{ccc|ccc} 1 & -1 & 4 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 6 & -5 & 3 & 0 & -2 \end{array}\right) \xrightarrow{-6R_{23}}$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 4 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 3 & -6 & -2 \end{array}\right) \xrightarrow{1 R_{21}}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 3 & -1 & 1 & 1 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 3 & -6 & -2 \end{array}\right) \xrightarrow[-3 R_{31}]{1 R_{32}}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -10 & 19 & 7 \\ 0 & 1 & 0 & 3 & -5 & -2 \\ 0 & 0 & 1 & 3 & -6 & -2 \end{array}\right)$$

thus $A^{-1} = \begin{pmatrix} -10 & 19 & 7 \\ 3 & -5 & -2 \\ 3 & -6 & -2 \end{pmatrix}$

we verify

$$A \cdot A^{-1} = \begin{pmatrix} 2 & 4 & 3 \\ 0 & 1 & -1 \\ 3 & 3 & 7 \end{pmatrix} \begin{pmatrix} -10 & 19 & 7 \\ 3 & -5 & -2 \\ 3 & -6 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Exercice : show that the inverse

13) of the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix}$

is given by $A^{-1} = \begin{pmatrix} 0 & 1 & -1 \\ 2 & -2 & -1 \\ -1 & 1 & 1 \end{pmatrix}$

<u>Exercice</u>

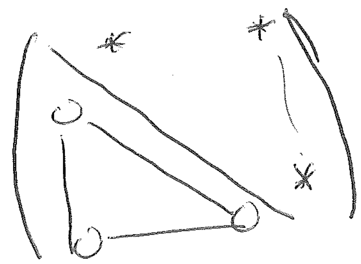
show that the inverse of the matrix

$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & -1 & 1 \end{pmatrix}$ is $A^{-1} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{3}{2} & -1 \end{pmatrix}$

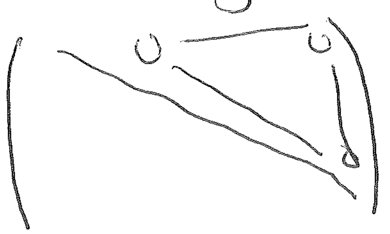
<u>Special matrices</u>

- upper triangular matrices

$a_{ij} = 0$ if $i > j$



- lower triangular $a_{ij} = 0$ if $i < j$



14) Symmetric matrices:

$$A = A^T$$

Example $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & -1 \\ 3 & -1 & 4 \end{pmatrix}$

antisymmetric matrices

$$A^T = -A$$

Example $\begin{pmatrix} 0 & -1 & 2 & -3 \\ 1 & 0 & 4 & 2 \\ -2 & -4 & 0 & 0 \\ 3 & -2 & 0 & 0 \end{pmatrix}$

Theorem:

1) AA^T and $A^T A$ are symmetric
2) $A + A^T$ is symmetric
3) $A - A^T$ is antisymmetric
4) αA is symmetric if A is symmetric and it is antisymmetric if A is antisymmetric.
5) Every matrix A can be written $A = B + C$ where B is symmetric and C is antisymmetric

15)

the conjugate matrix of A.
if $A=(a_{ij})$ $\bar{A}=(\bar{a}_{ij})$ is called
the conjugate of A

$A=\bar{A}$ if all elements of A are real.
we always have

1) $\bar{\bar{A}}=A$
2) $\overline{A+B}=\bar{A}+\bar{B}$
3) $\overline{A \cdot B}=\bar{A}\cdot\bar{B}$

we set $A^{*}=\bar{A}^{T}$ called the conjugate transpose

<u>theorem</u>

1) $A^{**}=A$
2) $(A+B)^{*}=A^{*}+B^{*}$
3) $(bA)^{*}=\bar{b}A^{*}$
4) $(AB)^{*}=B^{*}A^{*}$

we say a matrix A is unitary if
A is invertible and $A^{-1}=A^{*}$
If A is unitary with real elements
we say A is orthogonal.

We say A is Hermitian if $A = A^*$ and ~~a~~ Skew Hermitian if $A^* = -A$.

Examples

1) $A = \begin{bmatrix} 1 & 2 & i \\ 1+i & 3 & -i \\ 0 & 0 & 1 \end{bmatrix}$ $\quad \bar{A} = \begin{bmatrix} 1 & 2 & -i \\ 1-i & 3 & i \\ 0 & 0 & 1 \end{bmatrix}$

$A^* = \begin{bmatrix} 1 & 1-i & 0 \\ 2 & 3 & 0 \\ -i & i & 1 \end{bmatrix}$

2) $B = \begin{bmatrix} \frac{4}{5}i & \frac{3}{5} \\ \frac{3}{5}i & -\frac{4}{5} \end{bmatrix}$ is unitary

because $B \cdot B^* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$

$B^* = \begin{bmatrix} -\frac{4}{5}i & -\frac{3}{5}i \\ \frac{3}{5} & -\frac{4}{5} \end{bmatrix}$

3) $A = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is orthogonal

since $A \cdot A^T = I_3$.

17)

(i) $A = \begin{bmatrix} -1 & 1+i & -1-i \\ 1-i & 1 & -2 \\ -1+i & -2 & 0 \end{bmatrix}$

is Hermitian

Determinants

We define determinants as follows

1) If $A = [a_1]$ $\quad \det A = a_1$

2) If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ $\quad \det A = a_{11} a_{22} - a_{21} a_{12}$

3) If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + a_{13} \det A_{13}$$

where A_{ij} is the matrix obtained from A by deleting the i^{th} row and j^{th} column.

Examples

1) $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ $\quad \det A = 4 - 6 = -2$

2) $A = \begin{pmatrix} 2 & 4 \\ 3 & 6 \end{pmatrix}$ $\quad \det A = 12 - 12 = 0$.

3) $A = \begin{pmatrix} 1 & 2 & 2 \\ 4 & 5 & 6 \\ 3 & 2 & 1 \end{pmatrix}$

$$\det A = 1 \cdot \det A_{11} - 2 \det A_{12} + 2 \det A_{13}$$
$$= 1 \cdot (5-12) - 2(4-18) + 2(8-15)$$
$$= -7 + 28 - 14$$
$$= 7.$$

<u>Notation</u>: we write

$$\det \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix}$$

and use similar notation for other determinants.

In general if $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ & & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$

we define

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + a_{13} \det A_{13} + \cdots + (-1)^{n+1} a_{1n} \det A_{1n}$$

20)

$$\det A = \sum_{j=1}^{j=n} (-1)^{1+j} a_{ij} \det A_{ij}$$

<u>Example</u>

$$A = \begin{pmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 2 & 3 \\ 4 & 1 & 1 & 0 \\ 2 & 0 & 1 & 1 \end{pmatrix}$$

$$\det A = 1 \cdot \begin{vmatrix} -1 & 2 & 3 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} + 2 \begin{vmatrix} 2 & -1 & 3 \\ 4 & 1 & 0 \\ 2 & 0 & 1 \end{vmatrix}$$

$$-3 \begin{vmatrix} 2 & -1 & 2 \\ 4 & 1 & 1 \\ 2 & 0 & 1 \end{vmatrix}$$

$$= 1 \cdot 0 + 2 \cdot 0 - 3 \cdot 0 = 0$$

Sarrus method for determinants of order 3

Example

$$\text{find } \Delta = \begin{vmatrix} -1 & 2 & 3 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix}$$

$$\begin{array}{ccccc} -1 & 2 & 3 & -1 & 2 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{array}$$

$$= -1 + 3 - 2 = 0$$

21

Now for determinants in general

if $A = \begin{pmatrix} a_{11} & & a_{1n} \\ & & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$

if A_{ij} denotes the matrix obtained by deleting the ith row and the jth column then we call $\det A_{ij}$ the minor of a_{ij} and the cofactor of a_{ij} is defined by $C_{ij} = (-1)^{i+j} \det A_{ij}$

So we can write

$$\det A = a_{11} C_{11} + a_{12} C_{12} + \cdots + a_{1n} C_{1n}$$

<u>Theorem</u>: We have

$$\det A = a_{i1} C_{i1} + a_{i2} C_{i2} + \cdots + a_{in} C_{in}$$

for any i and

$$\det A = a_{1j} C_{1j} + a_{2j} C_{2j} + \cdots + a_{nj} C_{nj}$$

for any j

Examples

1) If $A = \begin{pmatrix} 2 & 3 & -1 \\ 3 & 0 & 0 \\ 5 & -1 & 4 \end{pmatrix}$

$$\det A = 3\,C_{21} = 3(-1)^{2+1} \det A_{21}$$

$$= -3 \begin{vmatrix} 3 & -1 \\ -1 & 4 \end{vmatrix} = -3 \cdot 11 = -33$$

Also

$$\det A = 2\,C_{11} + 3\,C_{21} + 5\,C_{31}$$

$$= 2 \begin{vmatrix} 0 & 0 \\ -1 & 4 \end{vmatrix} - 3 \begin{vmatrix} 3 & -1 \\ -1 & 4 \end{vmatrix} + 5 \begin{vmatrix} 3 & -1 \\ 0 & 0 \end{vmatrix}$$

$$= -3 \cdot 11 = -33$$

2) $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 0 & -1 \\ 1 & -1 & 0 & 3 \\ 1 & 1 & 0 & 1 \end{pmatrix}$

$$\det A = 3\,C_{13} = 3 \begin{vmatrix} 2 & 0 & -1 \\ 1 & -1 & 3 \\ 1 & 1 & 1 \end{vmatrix} = 3(-10) = -30$$

Remark:

if $A = \begin{pmatrix} a_{11} & & \\ 0 & \ddots & \\ 0 & 0 & a_{nn} \end{pmatrix}$ is upper triangular

$\det A = a_{11} \cdots a_{nn}$

the same holds for lower triangular matrices

Another example for the Sarrus method:

find $\Delta = \begin{vmatrix} 1 & -1 & 2 \\ 3 & 4 & -2 \\ 1 & 2 & 4 \end{vmatrix}$

$$\begin{array}{ccccc} 1 & -1 & 2 & 1 & -1 \\ 3 & 4 & -2 & 3 & 4 \\ 1 & 2 & 4 & 1 & 2 \end{array}$$

$$\Delta = 4.4 + (-1)(-2) + 2.3.2 - (4.2 + 2(-2) + 4.3(-1))$$
$$= 16 + 2 + 12 - (8 - 4 - 12)$$
$$= 30 - (-8) = 38.$$

Properties of determinants

24) 1) The determinant ~~is~~ ~~un~~changes sign if we permute two rows or two columns

2) If we multiply a row (or a column) by a constant the determinant is multiplied by the same constant.

3) The determinant is unchanged if we add to a row (or a column) a multiple of another row (column)

<u>Example</u>

find $\Delta = \begin{vmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 1 & 1 & -1 & 1 \\ 2 & 1 & 1 & 0 \end{vmatrix}$

$$\Delta = \begin{vmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 1 & 1 & -1 & 1 \\ 2 & 1 & 1 & 0 \end{vmatrix} \overset{-1R_{13}}{\underset{-2R_{14}}{=}} \begin{vmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & -2 & 0 \\ 0 & 3 & -1 & -2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 2 & -2 & 0 \\ 3 & -1 & -2 \end{vmatrix}$$

$$= \left|\begin{array}{ccc|cc} 1 & 2 & 3 & 1 & 2 \\ 2 & -2 & 0 & 2 & -2 \\ 3 & -1 & -2 & 3 & -1 \end{array}\right| = 4 - 6 - (-18 - 8) = 24$$

Remarks

1) If a row (or a column) is a multiple of another row (or column) the determinant is zero

2) If a row (column) is a linear combination of two other rows (columns) the determinant is zero.

Examples

1) $\begin{vmatrix} 1 & -1 & 2 & 3 \\ 2 & 0 & 4 & 1 \\ 3 & 1 & 6 & -1 \\ 4 & 3 & 8 & 1 \end{vmatrix} = 0$ since $C_3 = 2C_1$

2) $\begin{vmatrix} 1 & 1 & 2 & 1 \\ -2 & 3 & 1 & 1 \\ 0 & -1 & -1 & 1 \\ 4 & 0 & 4 & 1 \end{vmatrix} = 0$ since $C_3 = C_1 + C_2$

3) $\begin{vmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \\ 2 & 6 & 10 \end{vmatrix} = 0$ since $R_3 = 3R_1 + R_2$

26)

Example

find $\Delta = \begin{vmatrix} 1 & 3 & 4 & 3 \\ 2 & -1 & 2 & 0 \\ -2 & 1 & 3 & 0 \\ 1 & 1 & 4 & 2 \end{vmatrix}$

$$\Delta = \begin{vmatrix} 1 & 3 & 4 & 3 \\ 0 & -7 & -6 & -6 \\ 0 & 7 & 11 & 6 \\ 0 & -2 & 0 & -1 \end{vmatrix} = \begin{vmatrix} -7 & -6 & -6 \\ 7 & 11 & 6 \\ -2 & 0 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} -7 & -6 & -6 \\ 0 & 5 & 0 \\ -2 & 0 & -1 \end{vmatrix} = 5\, C_{22} = 5 \begin{vmatrix} -7 & -6 \\ -2 & -1 \end{vmatrix}$$

$$= -25.$$

Theorem: let $A \in M_n(\mathbb{R})$

1) A is invertible if and only if $\det A \neq 0$
2) $\det(A \cdot B) = \det A \cdot \det B$ for any A, B in $M_n(\mathbb{R})$
3) $\det(A^T) = \det A$.
4) if A is invertible $\det A^{-1} = \frac{1}{\det A}$

27)

Exercise

show that $\begin{vmatrix} a & b & 0 & a \\ 0 & a & a & b \\ b & a & a & 0 \\ a & 0 & b & a \end{vmatrix} = b^2(4a^2-b^2)$

Solution

$$\begin{vmatrix} a & b & 0 & a \\ 0 & a & a & b \\ b & a & a & 0 \\ a & 0 & b & a \end{vmatrix} = \begin{vmatrix} 2a+b & b & 0 & 0 \\ 2a+b & a & a & b \\ 2a+b & a & a & 0 \\ 2a+b & 0 & b & a \end{vmatrix}$$

$$= (2a+b) \begin{vmatrix} 1 & b & 0 & 0 \\ 1 & a & a & b \\ 1 & a & a & 0 \\ 1 & 0 & b & a \end{vmatrix} = (2a+b) \begin{vmatrix} 1 & b & 0 & 0 \\ 0 & 0 & 0 & b \\ 1 & a & a & 0 \\ 1 & 0 & b & a \end{vmatrix}$$

$$= (2a+b) \cdot b \, C_{24}$$

$$= b(2a+b) \begin{vmatrix} 1 & b & 0 \\ 1 & a & a \\ 1 & 0 & b \end{vmatrix} = b(2a+b) \begin{vmatrix} 1 & b & 0 \\ 0 & a-b & a \\ 0 & -b & b \end{vmatrix}$$

$$= b^2(2a+b) \begin{vmatrix} a-b & a \\ -1 & 1 \end{vmatrix}$$

$$= b^2(2a+b)(2a-b)$$

$$= b^2(4a^2-b^2).$$

28)

Exercise

Show that $\begin{vmatrix} a & a+1 & a+2 \\ b & b+1 & b+2 \\ c & c+1 & c+2 \end{vmatrix} = 0$

for any real numbers a, b and c

solution

$$\Delta = \begin{vmatrix} a & a+1 & a+2 \\ b & b+1 & b+2 \\ c & c+1 & c+2 \end{vmatrix} = \begin{vmatrix} a & 1 & 2 \\ b & 1 & 2 \\ c & 1 & 2 \end{vmatrix} = 0$$

(since $C_3 = 2C_2$)

~~Ae~~

Definition of the adjoint matrix

Let A be a square matrix. Call C the matrix of cofactors ie: $C = (C_{ij})$. The adjoint matrix is defined by $\operatorname{adj} A = C^T$

Examples

1) $$A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \\ 3 & -1 & 1 \end{pmatrix}$$

find $\operatorname{adj} A$

We have $C_{ij} = (-1)^{i+j} \det A_{ij}$

We get $C_{11} = 2 \quad C_{12} = 7 \quad C_{13} = 1$

$C_{21} = -5 \quad C_{22} = -8 \quad C_{23} = 7$

$C_{31} = 4 \quad C_{32} = -5 \quad C_{33} = 2$

So $$C = \begin{pmatrix} 2 & 7 & 1 \\ -5 & -8 & 7 \\ 4 & -5 & 2 \end{pmatrix}$$

thus $$\operatorname{adj} A = C^T = \begin{pmatrix} 2 & -5 & 4 \\ 7 & -8 & -5 \\ 1 & 7 & 2 \end{pmatrix}$$

2) $A = \begin{pmatrix} 1 & 2 & 3 & 1 \\ -1 & 1 & 0 & 1 \\ 2 & 0 & 1 & -1 \\ 3 & 1 & -1 & 2 \end{pmatrix}$

we get

$$\operatorname{adj} A = \begin{pmatrix} 0 & 2 & -2 & -2 \\ 6 & -25 & -17 & 1 \\ -6 & 11 & 7 & 1 \\ -6 & 15 & 15 & -3 \end{pmatrix}$$

the importance of the adjoint matrix is due to the following theorem

<u>Theorem</u> for any $A \in M_n(\mathbb{R})$

$$\operatorname{adj} A \cdot A = A \cdot \operatorname{adj} A = \det A \cdot I_n$$

<u>Corollary</u>: If $\det A \neq 0$ then A is invertible and $A^{-1} = \frac{1}{\det A} \cdot \operatorname{adj} A$

<u>Examples</u>

1) $A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 3 & 1 & -1 \end{pmatrix}$ $\quad$ $\cancel{A} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 3 & 1 & -1 \end{pmatrix}$

3ii)

$$C = \begin{pmatrix} -1 & 4 & 1 \\ 0 & 2 & 2 \\ 1 & -2 & -1 \end{pmatrix}$$

$$\operatorname{adj} A = \begin{pmatrix} -1 & 0 & 1 \\ 4 & 2 & -2 \\ 1 & 2 & -1 \end{pmatrix}$$

$\det A = 2$ so

$$A^{-1} = \begin{pmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ 2 & 1 & -1 \\ \frac{1}{2} & 1 & -\frac{1}{2} \end{pmatrix}$$

2) $A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ -1 & 0 & 2 & 1 \\ 1 & 1 & -3 & 1 \end{pmatrix}$ $\quad C = \begin{pmatrix} 3 & -6 & 0 & 3 \\ 5 & 5 & 3 & -1 \\ -3 & -3 & 0 & 6 \\ -2 & -2 & -3 & 4 \end{pmatrix}$

$\det A = 9$ so

$$A^{-1} = \begin{pmatrix} \frac{1}{3} & 5/9 & -\frac{1}{3} & -\frac{2}{9} \\ -\frac{2}{3} & \frac{5}{9} & -\frac{1}{3} & -\frac{2}{9} \\ 0 & \frac{1}{3} & 0 & -1/3 \\ \frac{1}{3} & -\frac{1}{9} & \frac{2}{3} & \frac{4}{9} \end{pmatrix}$$

chapter 3

Linear systems of equations

A linear system of equations takes the form

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

we call this a system of m equations and n unknowns $x_1, \ldots x_n$.

$(s_1, \ldots s_n)$ is a solution of this system if the equations hold when $x_i = s_i$ $1 \le i \le n$. This linear system can be written in matrix form as follows $AX = b$ where $A = (a_{ij})_{\substack{1\le i\le m \\ 1\le j\le n}}$ and $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$

Examples

1) $\begin{cases} x+y=2 \\ x-y=0 \end{cases}$

is a system of two equations and two unknowns. $(1,1)$ is a solution of this system

2) $\begin{cases} x_1+x_2-x_3=0 \\ 2x_1+x_2+x_3=1 \\ \quad x_2+x_3=-1 \end{cases}$

is a system of 3 equations and three unknowns

3) $\begin{cases} 2x+y-z+t=0 \\ 3x-y+2z+t=1 \\ x+y+z-3t=4 \end{cases}$

is a system of 3 equations and 4 unknowns.

the matrix

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} & b_1 \\ a_{21} & & a_{2n} & b_2 \\ \vdots & & & \\ a_{m1} & \cdots & a_{mn} & b_m \end{pmatrix}$$

is called the augmented matrix of the system. It can also be written

$$\left(\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ & & & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right) = (A|b)$$

The Gauss-method

It consists of reducing $(A|b)$ to the row echelon form. We get a simpler system to resolve.

Remark: the system obtained by the Gauss method is equivalent to the initial system ie it has the same solutions as the original system.

Examples

1) Solve the system using the Gauss method

$$\begin{cases} x+3y-z=7 \\ x+2y+4z=5 \\ 4y-3z=8 \end{cases}$$

solution

$$\begin{pmatrix} 1 & 3 & -1 & 7 \\ 1 & 2 & 4 & 5 \\ 0 & 4 & -3 & 8 \end{pmatrix} \xrightarrow{-1R_{12}} \begin{pmatrix} 1 & 3 & -1 & 7 \\ 0 & -1 & 5 & -2 \\ 0 & 4 & -3 & 8 \end{pmatrix} \xrightarrow{4R_{23}}$$

$$\begin{pmatrix} 1 & 3 & -1 & 7 \\ 0 & -1 & 5 & -2 \\ 0 & 0 & 17 & 0 \end{pmatrix} \xrightarrow[\frac{1}{17}R_3]{-R_2} \begin{pmatrix} 1 & 3 & -1 & 7 \\ 0 & 1 & -5 & 2 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

we get the system $\begin{cases} x+3y-z=7 \\ y-5z=2 \\ z=0 \end{cases}$

36)

so $\begin{cases} x=1 \\ y=2 \\ z=0 \end{cases}$

$S=\{(1,2,0)\}$

2) Solve the system $\begin{cases} x_1-3x_2+2x_3+x_4=2 \\ 3x_1-9x_2+10x_3+2x_4=9 \\ 2x_1-6x_2+4x_3+2x_4=4 \\ 2x_1-6x_2+8x_3+x_4=7 \end{cases}$

$$\begin{pmatrix} 1 & -3 & 2 & 1 & 2 \\ 3 & -9 & 10 & 2 & 9 \\ 2 & -6 & 4 & 2 & 4 \\ 2 & -6 & 8 & 1 & 7 \end{pmatrix} \xrightarrow[\substack{-2R_{13} \\ -2R_{14}}]{-3R_{12}} \begin{pmatrix} 1 & -3 & 2 & 1 & 2 \\ 0 & 0 & 4 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & -1 & 3 \end{pmatrix}$$

$$\xrightarrow{\frac{1}{4}R_2} \begin{pmatrix} 1 & -3 & 2 & 1 & 2 \\ 0 & 0 & 1 & -\frac{1}{4} & \frac{3}{4} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & -1 & 3 \end{pmatrix} \xrightarrow{-4R_{24}} \begin{pmatrix} 1 & -3 & 2 & 1 & 2 \\ 0 & 0 & 1 & -\frac{1}{4} & \frac{3}{4} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

we get the system $\begin{cases} x_1-3x_2+2x_3+x_4=2 \\ x_3-\frac{1}{4}x_4=\frac{3}{4} \end{cases}$

if we put $x_4=t$ we get $x_3=\frac{1}{4}t+\frac{3}{4}$

If we set $x_2 = s$ we get

$$x_1 = 3s - 2\left(\frac{1}{4}t + \frac{3}{4}\right) - t + 2$$

$$= 3s - \frac{3}{2}t + \frac{1}{2}$$

thus $S = \left\{ \left(3s - \frac{3}{2}t + \frac{1}{2}, s, \frac{1}{4}t + \frac{3}{4}, t\right) \quad s, t \in \mathbb{R} \right\}$

<u>Definition</u>

We say a linear system is consistent if it has a solution otherwise we say it is inconsistent

<u>Example</u>: Solve the system

$$\begin{cases} x - y + z = 1 \\ 2x + y + z = 3 \\ 4x - y + 3z = 6 \end{cases}$$

$$\begin{pmatrix} 1 & -1 & 1 & 1 \\ 2 & 1 & 1 & 3 \\ 4 & -1 & 3 & 6 \end{pmatrix} \xrightarrow[-4R_{13}]{-2R_{12}} \begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 3 & -1 & 1 \\ 0 & 3 & -1 & 2 \end{pmatrix} \xrightarrow{-1R_{23}}$$

38)

$$\begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 3 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$ we get

$$\begin{cases} x - y + z = 1 \\ 3y - z = 1 \\ 0 = 1 \end{cases}$$ which is impossible

the system is inconsistent and has no solutions

The Gauss-Jordan method

In this case we reduce $(A|b)$ to the reduced row echelon form.

Example: solve the system

$$\begin{cases} x + y + 2z = 1 \\ 2x - y + t = -2 \\ x - y - z - 2t = 4 \\ 2x - y + 2z - t = 0 \end{cases}$$

$$\begin{pmatrix} 1 & 1 & 2 & 0 & 1 \\ 2 & -1 & 0 & 1 & -2 \\ 1 & -1 & -1 & -2 & 4 \\ 2 & -1 & 2 & -1 & 0 \end{pmatrix} \xrightarrow[\substack{-1R_{13} \\ -2R_{14}}]{-2R_{12}} \begin{pmatrix} 1 & 1 & 2 & 0 & 1 \\ 0 & -3 & -4 & 1 & -4 \\ 0 & -2 & -3 & -2 & 3 \\ 0 & -3 & -2 & -1 & -2 \end{pmatrix} \xrightarrow{-1R_{24}}$$

39)

$$\begin{pmatrix} 1 & 1 & 2 & 0 & 1 \\ 0 & -3 & -4 & 1 & -4 \\ 0 & -2 & -3 & -2 & 3 \\ 0 & 0 & 2 & -2 & 2 \end{pmatrix} \xrightarrow[\frac{R_4}{2}]{-R_2} \begin{pmatrix} 1 & 1 & 2 & 0 & 1 \\ 0 & 3 & 4 & -1 & 4 \\ 0 & -2 & -3 & -2 & 3 \\ 0 & 0 & 1 & -1 & 1 \end{pmatrix}$$

$$\xrightarrow{1R_{32}} \begin{pmatrix} 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & -3 & 7 \\ 0 & -2 & -3 & -2 & 3 \\ 0 & 0 & 1 & -1 & 1 \end{pmatrix} \xrightarrow{2R_{23}} \begin{pmatrix} 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & -3 & 7 \\ 0 & 0 & -1 & -8 & 17 \\ 0 & 0 & 1 & -1 & 1 \end{pmatrix}$$

$$\xrightarrow{1R_{34}} \begin{pmatrix} 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & -3 & 7 \\ 0 & 0 & -1 & -8 & 17 \\ 0 & 0 & 0 & -9 & 18 \end{pmatrix} \xrightarrow[-R_3]{\frac{R_4}{-9}} \begin{pmatrix} 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & -3 & 7 \\ 0 & 0 & 1 & 8 & -17 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix}$$

$$\xrightarrow{-8R_{43}} \begin{pmatrix} 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & -3 & 7 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix} \xrightarrow{3R_{42}}$$

$$\begin{pmatrix} 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix} \xrightarrow{-1R_{32}} \begin{pmatrix} 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix}$$

$$\xrightarrow{-1R_{21}} \begin{pmatrix} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix} \xrightarrow{-2R_{31}} \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix}$$

40/

$$\Rightarrow \begin{cases} x = 1 \\ y = 2 \\ z = -1 \\ t = -2 \end{cases}$$

Example

find the condition on α so that the system
is consistent

$$\begin{cases} \alpha x + y + z = 1 \\ x + \alpha y + z = 1 \\ x + y + \alpha z = 1 \end{cases}$$

Solution

the matrix of the system is

$$A = \begin{pmatrix} \alpha & 1 & 1 & 1 \\ 1 & \alpha & 1 & 1 \\ 1 & 1 & \alpha & 1 \end{pmatrix}, \quad A \xrightarrow{R_{13}} \begin{pmatrix} 1 & 1 & \alpha & 1 \\ 1 & \alpha & 1 & 1 \\ \alpha & 1 & 1 & 1 \end{pmatrix} \longrightarrow$$

$$\begin{pmatrix} 1 & 1 & \alpha & 1 \\ 0 & \alpha-1 & 1-\alpha & 0 \\ \alpha-1 & 0 & 1-\alpha & 0 \end{pmatrix}$$

so if $\alpha = 1$ we get one equation

$$x + y + z = 1$$

$$S = \{ (1-r-s, s, r) \quad s, r \in \mathbb{R}$$

If $\alpha \neq 1$ $\quad A \sim \begin{pmatrix} 1 & 1 & \alpha & 1 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & -1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & \alpha & 1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & -1-\alpha & -1 \end{pmatrix}$

42)

$$\longrightarrow \begin{pmatrix} 1 & 1 & \alpha & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -2-\alpha & -1 \end{pmatrix}$$

If $\alpha = -2$ the system is inconsistent

if $\alpha \neq -2$ the system is consistent

In conclusion the system is consistent if and only if $\alpha \neq -2$

Example:

find the values of a and b for which the system

$$\begin{cases} ax + bz = 2 \\ ax + \cancel{a}y + 4z = 4 \\ ay + 2z = b \end{cases}$$

has i) one solution

ii) infinitely many solutions

iii) No solution

solution $\begin{pmatrix} a & 0 & b & 2 \\ a & a & 4 & 4 \\ 0 & a & 2 & b \end{pmatrix} \xrightarrow{-1R_{12}} \begin{pmatrix} a & 0 & b & 2 \\ 0 & a & 4-b & 2 \\ 0 & a & 2 & b \end{pmatrix}$

43) $\xrightarrow{-1R_{23}}$ $\begin{pmatrix} a & 0 & b & 2 \\ 0 & a & 4-b & 2 \\ 0 & 0 & b-2 & b-2 \end{pmatrix} = R$

if $\underline{b=2}$ we get

$\begin{pmatrix} a & 0 & 2 & 2 \\ 0 & a & 2 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{cases} ax + 2z = 2 \\ ay + 2z = 2 \end{cases}$

if $a = 0$ the solution is

$S = \{(x, y, 1) \quad x, y \in \mathbb{R}\}$

We have infinitely many solutions

if $a \neq 0$ $\quad x = \frac{2-2z}{a} = y \quad z \in \mathbb{R}$

We have infinitely many solutions

if $\underline{b \neq 2 \text{ and } a = 0}$ from R we get

$\begin{pmatrix} a & 0 & b & 2 \\ 0 & a & 4-b & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix}$ $\quad \begin{cases} bz = 2 \\ (4-b)z = 2 \\ z = 1 \end{cases}$ ie $b = 2$ and $b \neq 2$

The system is inconsistent.

If $\underline{b \neq 2 \text{ and } a \neq 0}$

R becomes

$$\begin{pmatrix} a & 0 & b & 2 \\ 0 & a & 4-b & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix} \Rightarrow \begin{cases} ax + bz = 2 \\ ay + (4-b)z = 2 \\ z = 1 \end{cases}$$

we get one solution

$$S = \left\{ \left(\frac{2-b}{a}, \frac{b-2}{a}, 1 \right) \right\}$$

<u>Homogenuous systems of equations</u>

these systems take the form

$$\begin{cases} a_{11} x_1 + \cdots + a_{1n} x_n = 0 \\ a_{21} x_1 + \cdots + a_{2n} x_n = 0 \\ \\ a_{m1} x_1 + \cdots + a_{mn} x_n = 0 \end{cases} \quad (S_1)$$

<u>Example</u>: $\begin{cases} x + y - 2z = 0 \\ 3x + y + z = 0 \end{cases}$

<u>theorem</u>: i) If X is a solution of the homogenuous system and Y is

any other solution of (S_1)
then $X+Y$ is also a solution of
the system

ii) if X is a solution of a homogeneous linear system then αX is also a solution for any $\alpha \in \mathbb{R}$

iii) if A is a square matrix and A is invertible the solution of $AX=0$ is $\cancel{X}=0$ and it is unique

<u>Theorem.</u> if X_0 is a solution of the system $AX=b$ then any solution takes the form $Y=X_0+X_1$ where X_1 is a solution of $AX=0$

<u>Example</u>: find m so that the system

46)

$$\begin{cases} mx + y + z = 0 \\ x + y - z = 0 \\ x + my + z = 0 \end{cases}$$

has non zero solutions

We compute $\Delta = \begin{vmatrix} m & 1 & 1 \\ 1 & 1 & -1 \\ 1 & m & 1 \end{vmatrix} \xrightarrow[-1R_{13}]{1R_{12}}$

$$\begin{vmatrix} m & 1 & 1 \\ m+1 & 2 & 0 \\ 1-m & m-1 & 0 \end{vmatrix} = \begin{vmatrix} m+1 & 2 \\ 1-m & m-1 \end{vmatrix} = (m-1) \begin{vmatrix} m+1 & 2 \\ -1 & 1 \end{vmatrix}$$

$$= (m-1)(m+3)$$

if $m \neq 1$ and $m \neq -3$ the matrix

$A = \begin{pmatrix} m & 1 & 1 \\ 1 & 1 & -1 \\ 1 & m & 1 \end{pmatrix}$ is invertible

and we have a unique solution

$S = \{(0,0,0)\}$

If $m = -3$ we get $\begin{cases} -3x + y + z = 0 \\ x + y - z = 0 \\ x - 3y + z = 0 \end{cases}$

47) we use the Gauss method

$$\begin{pmatrix} -3 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -3 & 1 & 0 \end{pmatrix} \xrightarrow{R_{12}} \begin{pmatrix} 1 & 1 & -1 & 0 \\ -3 & 1 & 1 & 0 \\ 1 & -3 & 1 & 0 \end{pmatrix} \xrightarrow[3R_{12}]{-1R_{13}}$$

$$\begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 4 & -2 & 0 \\ 0 & -4 & 2 & 0 \end{pmatrix} \xrightarrow{+1R_{23}} \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 4 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

we get $\begin{cases} x+y-z=0 \\ y-\frac{1}{2}z=0 \end{cases} \Rightarrow \begin{cases} x=\frac{1}{2}z \\ y=\frac{1}{2}z \end{cases}$

$$S=\left\{\left(\tfrac{1}{2}z, \tfrac{1}{2}z, z\right),\ z\in\mathbb{R}\right\}$$

If $m=1$ we get $\begin{cases} x+y+z=0 \\ x+y-z=0 \\ x+y+z=0 \end{cases}$

we solve using the Gauss method

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow \begin{cases} x=-y \\ z=0 \end{cases}$ $\quad S=\{(t,-t,0),\ t\in\mathbb{R}\}$

48

Solve the system

$$\begin{cases} x_1 + x_2 + x_3 = 0 \\ -2x_1 - 3x_2 = 0 \\ x_1 + 2x_2 - x_3 = 0 \end{cases}$$

we use the Gauss method

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ -2 & -3 & 0 & 0 \\ 1 & 2 & -1 & 0 \end{pmatrix} \xrightarrow[-R_{13}]{2R_{12}} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 1 & -2 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

we get $\begin{cases} x_1 + x_2 + x_3 = 0 \\ x_2 - 2x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = -3x_3 \\ x_2 = 2x_3 \end{cases}$

$$S = \{ (-3x_3, 2x_3, x_3), \ x_3 \in \mathbb{R} \}$$

we write

$$S = \{ x_3(-3, 2, 1), \ x_3 \in \mathbb{R} \}$$

Definition: if X_1 and X_2 are solutions we call $\alpha_1 X_1 + \alpha_2 X_2$ a linear combination. In general if $X_1, \dots X_p$ are solutions we call

$\alpha_1 X_1 + \alpha_2 X_2 + \dots + \alpha_p X_p$ a linear combination of the X_i's.

Examples

1) if the solutions of a system are given by

$$S = \{(t, 2t-s, s+t, s) \quad s, t \in \mathbb{R}\}$$

we write

$$S = \{t(1,2,1,0) + s(0,-1,1,1), \ s,t \in \mathbb{R}\}$$

2) If the solutions are given by

$$S = \left\{ \begin{array}{l} (x_1, x_3 - 4x_1, x_3, 2x_3, x_5) \\ x_1, x_3, x_5 \in \mathbb{R} \end{array} \right\}$$

we write

$$S = \left\{ \begin{array}{l} x_1(1,-4,0,0,0) + x_3(0,1,1,2,0) \\ + x_5(0,0,0,0,1) \quad x_1, x_3, x_5 \in \mathbb{R} \end{array} \right\}$$

The Cramer method

It is based on the following

theorem

let A be a square matrix of order n

the system
$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

has a unique solution if $\det A \neq 0$
and it is given by

$$x_1 = \frac{\det A_1}{\det A}, \quad x_2 = \frac{\det A_2}{\det A}, \quad \cdots, \quad x_n = \frac{\det A_n}{\det A}$$

where A_i is the matrix obtained from A by replacing the ith column with the column $b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$

Example: Solve the system
$$\begin{cases} x+y+z=2 \\ x-y+2z=0 \\ 2x+z=2 \end{cases}$$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 2 & 0 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$$

51)

$$\det A = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & -2 & -1 \end{vmatrix} = 4 \neq 0$$

So $$x = \frac{\begin{vmatrix} 2 & 1 & 1 \\ 0 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix}}{4} = \frac{4}{4} = 1$$

$$y = \frac{\begin{vmatrix} 1 & 2 & 1 \\ 1 & 0 & 2 \\ 2 & 2 & 1 \end{vmatrix}}{4} = 1$$

$$z = \frac{\begin{vmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \\ 2 & 0 & 2 \end{vmatrix}}{4} = 0$$

$$S = \{(1,1,0)\}$$

Example Solve the system

$$\begin{cases} x_1 + x_2 + x_3 = a \\ x_1 + (1+a)x_2 + x_3 = 2a \\ x_1 + x_2 + (1+a)x_3 = 0 \end{cases}$$

52)

The matrix is $A=\begin{pmatrix}1&1&1\\1&1+a&1\\1&1&1+a\end{pmatrix}$

$$\det A=\begin{vmatrix}1&1&1\\1&1+a&1\\1&1&1+a\end{vmatrix}=\begin{vmatrix}1&1&1\\0&a&0\\0&0&a\end{vmatrix}$$

$$=a^2$$

If $a\neq 0$ we get by the Cramer method:

$$x_1=\frac{\begin{vmatrix}a&1&1\\2a&1+a&1\\0&1&1+a\end{vmatrix}}{a^2}=\frac{a}{a^2}\begin{vmatrix}1&1&1\\2&1+a&1\\0&1&1+a\end{vmatrix}$$

$$=\frac{1}{a}\begin{vmatrix}1&1&1\\0&a-1&-1\\0&1&1+a\end{vmatrix}=\frac{a^2}{a}=a.$$

$$x_2=\frac{\begin{vmatrix}1&a&1\\1&2a&1\\1&0&1+a\end{vmatrix}}{a^2}=\frac{1}{a}\begin{vmatrix}1&1&1\\1&2&1\\1&0&1+a\end{vmatrix}$$

$$=\frac{1}{a}\begin{vmatrix}1&1&1\\0&1&0\\0&-1&a\end{vmatrix}=1$$

$$x_3 = \frac{\begin{vmatrix} 1 & 1 & a \\ 1 & 1+a & 2a \\ 1 & 1 & 0 \end{vmatrix}}{a^2} = \frac{1}{a}\begin{vmatrix} 1 & 1 & 1 \\ 0 & a & 1 \\ 0 & 0 & -1 \end{vmatrix} = -1$$

$$S = \{(a, 1, -1)\}$$

<u>If $a = 0$</u> : we get the system

$$\begin{cases} x_1 + x_2 + x_3 = 0 \\ x_1 + x_2 + x_3 = 0 \\ x_1 + x_2 + x_3 = 0 \end{cases}$$

$$S = \{(-s-t, s, t),\ s, t \in \mathbb{R}\}$$

<u>Example</u> Solve the system

$$\begin{cases} ax + bz = 2 \\ ax + ay + 4z = 4 \\ ay + 2z = b \end{cases}$$

we compute $\begin{vmatrix} a & 0 & b \\ a & a & 4 \\ 0 & a & 2 \end{vmatrix} = \begin{vmatrix} a & 0 & b \\ 0 & a & 4-b \\ 0 & a & 2 \end{vmatrix}$

$$= a^2(b-2)$$

54)

If $a \neq 0$ and $b \neq 2$ we have a unique solution given by

$$X = \frac{\begin{vmatrix} 2 & 0 & b \\ 4 & a & 4 \\ b & a & 2 \end{vmatrix}}{a^2(b-2)} = \frac{\begin{vmatrix} 2 & 0 & b \\ 4 & a & 4 \\ b-4 & 0 & -2 \end{vmatrix}}{a^2(b-2)} = \frac{a(-4-b^2+4b)}{a^2(b-2)}$$

$$= \frac{-1}{a}\,\frac{(b^2-4b+4)}{b-2} = -\frac{(b-2)}{a} = \frac{2-b}{a}$$

$$Y = \frac{\begin{vmatrix} a & 2 & b \\ a & 4 & 4 \\ 0 & b & 2 \end{vmatrix}}{a^2(b-2)} = \frac{\begin{vmatrix} a & 2 & b \\ 0 & 2 & 4-b \\ 0 & b & 2 \end{vmatrix}}{a^2(b-2)} = \frac{1}{a}\,\frac{(b^2-4b+4)}{\cancel{a}(b-2)}$$

$$= \frac{b-2}{a}$$

$$Z = \frac{\begin{vmatrix} a & 0 & 2 \\ a & a & 4 \\ 0 & a & b \end{vmatrix}}{a^2(b-2)} = \frac{\begin{vmatrix} a & 0 & 2 \\ 0 & a & 2 \\ 0 & a & b \end{vmatrix}}{a^2(b-2)} = \frac{ab-2a}{a(b-2)} = 1\ \cancel{2}$$

if $a = 0$ we get the system

$$\begin{cases} bz = 2 \\ 4z = 4 \\ 2z = b \end{cases}$$

5] if $b=2$ we have the solution

$$S=\{(x,y,1)\quad x,y\in\mathbb{R}\}$$

if $b\neq 2$ the system is inconsistent.

If $a\neq 0$ and $b=2$ we get

$$\begin{cases} ax+2z=2 \\ ax+ay+4z=4 \\ ay+2z=2 \end{cases}$$

$$\begin{pmatrix} a & 0 & 2 & 2 \\ a & a & 4 & 4 \\ 0 & a & 2 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} a & 0 & 2 & 2 \\ 0 & a & 2 & 2 \\ 0 & a & 2 & 2 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} a & 0 & 2 & 2 \\ 0 & a & 2 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{cases} ax+2z=2 \\ ay+2z=2 \end{cases} \Longrightarrow S=\left\{\left(\frac{2-2z}{a},\frac{2-2z}{a},z\right)\quad z\in\mathbb{R}\right\}$$

Theorem

If A is invertible the system $AX=b$ has a unique solution $X=A^{-1}b$

Example

Solve the system $\begin{cases} x+y+z=2 \\ x-y+2z=0 \\ 2x+z=2 \end{cases}$

we use the algorithm after checking $\det A = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = 4 \neq 0$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 2 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow[-2R_{13}]{-1R_{12}} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & -1 & 1 & 0 \\ 0 & -2 & -1 & -2 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{-1R_{23}} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & -1 & 1 & 0 \\ 0 & 0 & -2 & -1 & -1 & 1 \end{array}\right) \xrightarrow[-\frac{1}{2}R_3]{-\frac{1}{2}R_2}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array}\right) \xrightarrow{-1R_{21}} \left(\begin{array}{ccc|ccc} 1 & 0 & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array}\right)$$

57)

$$\xrightarrow[\frac{1}{2} R_{32}]{-\frac{3}{2} R_{31}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \\ 0 & 1 & 0 & \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array}\right) = (I_3 | A^{-1})$$

the solution is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1} \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \\ \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \qquad S = \{(1,1,0)\}$$

58)

Vector spaces

$\mathbb{R}^3$ is defined as

$\mathbb{R}^3 = \{ (x, y, z) , x, y, z \in \mathbb{R} \}$

We may call the elements of $\mathbb{R}^3$ vectors. If $u = (x, y, z)$ and $v = (x_1, y_1, z_1)$ we can define

$u + v = (x + x_1, y + y_1, z + z_1)$

If $\alpha \in \mathbb{R}$ and $u = (x, y, z)$

$\alpha u = (\alpha x, \alpha y, \alpha z)$ and $(0, 0, 0)$ is the zero vector

the opposite of $u = (x, y, z)$ is $-u = (-x, -y, -z)$

and $u - v = (x - x_1, y - y_1, z - z_1)$ if $v = (x_1, y_1, z_1)$

These operations can be defined for

$\mathbb{R}^n = \{ (a_1, a_2, \ldots a_n) \quad a_i \in \mathbb{R} \quad 1 \leq i \leq n \}$

For $u = (u_1, u_2, \ldots, u_n)$ $v = (v_1, v_2 \ldots, v_n)$

$u + v = (u_1 + v_1, \quad \ldots, u_n + v_n)$

we put $0 = (0, \ldots, 0)$

59)

$\alpha u = (\alpha u_1, \ldots, \alpha u_n)$

$-u = (-u_1, \ldots, -u_n)$

$u - v = (u_1 - v_1, \ldots, u_n - v_n)$

<u>Theorem:</u> on $\mathbb{R}^n$ we have

1) $u + v = v + u$
2) $u + (v + w) = (u + v) + w$
3) $u + 0 = 0 + u = u$
4) $u + (-u) = (-u) + u = 0$
5) $\alpha(u + v) = \alpha u + \alpha v$ $\quad \alpha \in \mathbb{R}$
6) $(\alpha + \beta) u = \alpha u + \beta u$
7) $(\alpha \cdot \beta) u = \alpha(\beta \cdot u)$
8) $1 \cdot u = u$

These operations with their properties are verified on other sets where the operations are defined.

<u>Definition:</u> A vector space on $\mathbb{R}$ is a set V non empty with two operations defined : addition and

multiplication and satisfying the following properties

1) $u \in V, v \in V \quad u+v \in V$

2) $(u+v)+w = u+(v+w)$ (associativity)

3) There exists $0 \in V$ satisfying $v+0 = 0+v = v$ (neutral element)

4) For any $v \in V$ there exists $-v \in V$ so that $v+(-v) = (-v)+(v) = 0$

5) $u+v = v+u$ (commutativity)

6) If $\alpha \in \mathbb{R}$ and $v \in V \quad \alpha v \in V$

7) $\alpha(u+v) = \alpha u + \alpha v$ for any $\alpha \in \mathbb{R}, u, v \in V$

8) $\alpha.(\beta v) = (\alpha\beta).v = \beta(\alpha v) \quad \alpha \in \mathbb{R}, \beta \in \mathbb{R}\ v \in V$

9) $1.v = v$ for any v

<u>Examples</u>: 1) $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^n$ are vector spaces with the natural operations

2) Functions defined on an interval I with the usual operations form a vector space

3) the space of polynomials of degree less than or equal to n.

$$\mathcal{P}_n = \{ P,\ P \text{ polynomial with } d^\circ P \leq n \}$$

4) the space of matrices $M_{m,n}(\mathbb{R})$ with the usual addition of matrices and multiplication by a scalar

<u>Definition:</u>

Let V be a vector space and $W \subset V$ a subset of V. We say W is a subspace of V If W is a vector space with the operations defined on V

<u>theorem:</u>

W is a subspace of V if and only if
1) $0 \in W$
2) If $u \in W$ and $v \in W$ then $u+v \in W$

2)

3) If $\alpha \in \mathbb{R}$ and $u \in W$ then $\alpha u \in W$

Examples

1) Show that $W = \{(a, b, a-2b), a, b \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^3$

We have for $a = b = 0$, $0 = (0, 0, 0) \in W$

if $(a, b, a-2b) \in W$
$(a_1, b_1, a_1 - 2b_1) \in W$

$(a, b, a-2b) + (a_1, b_1, a_1 - 2b_1) = (a + a_1, b + b_1, a + a_1 - 2(b + b_1)) \in W$

if $\alpha \in \mathbb{R}$ and $(a, b, a-2b) \in W$
$\alpha(a, b, a-2b) = (\alpha a, \alpha b, \alpha a - 2\alpha b) \in W$

2) Show that $W = \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \; a, b, c \in \mathbb{R} \right\}$ is a subspace of $M_2(\mathbb{R})$

first $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in W$

If $\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \in W$ and $\begin{pmatrix} a_1 & 0 \\ b_1 & c_1 \end{pmatrix} \in W$

63)

then $\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} + \begin{pmatrix} a_1 & 0 \\ b_1 & c_1 \end{pmatrix} = \begin{pmatrix} a+a_1 & 0 \\ b+b_1 & c+c_1 \end{pmatrix}$

is in W.

last if $\alpha \in \mathbb{R}$ $\quad \alpha \cdot \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} = \begin{pmatrix} \alpha a & 0 \\ \alpha b & \alpha c \end{pmatrix} \in W$

3) $W = \{ A \in M_2(\mathbb{R}),\ \det A = 0 \}$

is not a subspace of $M_2(\mathbb{R})$.

since if $A_1 = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}$ $\quad A_1 \in W$

$A_2 = \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix}$ $\quad A_2 \in W$

but $A_1 + A_2 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ $\quad \det(A_1 + A_2) = -3 \neq 0$

so $A_1 + A_2 \notin W$.

4) $W = \{ A \in M_2(\mathbb{R}),\ A^2 = A \}$

is not a subspace of $M_2(\mathbb{R})$

'64)

take $A_1 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

$A_1^2 = A_1$ so $A_1 \in W$

take $A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $\quad A_2^2 = A_2$

so $A_2 \in W$

$A_1 + A_2 = \begin{pmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ satisfies

$$(A_1 + A_2)^2 = \begin{pmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{5}{2} & 1 \\ 1 & \frac{1}{2} \end{pmatrix} \neq A_1 + A_2$$

so $A_1 + A_2 \notin W$.

5) $W = \{ ax^2 + bx + c \,,\, a + b + c = 0 \}$
is a subspace of P_2 (exercice!)

65)

6) $W = \{ (x, y, z) \in \mathbb{R}^3 ,\ x + 2y - z = 1 \}$

is not a subspace of $\mathbb{R}^3$ since

$(0, 0, 0) \notin W$.

7) $W = \left\{ \begin{pmatrix} 3 & a & 3 \\ a & b & a \end{pmatrix} \quad a, b \in \mathbb{R} \right\}$

is not a subspace of $M_{2,3}(\mathbb{R})$

since $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \notin W$.

8) $W = \{ (x, y, z) \in \mathbb{R}^3 \quad xy = 0 \}$

is not a subspace of $\mathbb{R}^3$

take $u = (1, 0, 1) \in W$

$v = (0, 1, 1) \in W$

$u + v = (1, 1, 2) \notin W$

Linear combination and spanning sets

Let $v_1, \dots v_n$ be vectors in V

We call $\alpha_1 v_1 + \dots + \alpha_n v_n$ a linear combination if $\alpha_1, \dots \alpha_n \in \mathbb{R}$.

Examples

1) in $\mathbb{R}^2$ $\quad a=(2,-1)$, $b=(1,3)$

$$\alpha a + \beta b = \alpha(2,-1) + \beta(1,3) = (2\alpha, -\alpha) + (\beta, 3\beta)$$
$$= (2\alpha+\beta, -\alpha+3\beta)$$

2) Let $f(x) = 3x^2+2x+1$ $\quad g(x) = x^2 - x$
$h(x) = x+1$, $k(x) = 1$. Is f a linear combination of g, h and k?

This would lead to

$$f(x) = \alpha g(x) + \beta h(x) + \gamma k(x)$$
$$3x^2+2x+1 = \alpha(x^2-x) + \beta(x+1) + \gamma$$
$$= \alpha x^2 + (\beta - \alpha)x + \beta + \gamma$$

thus $\begin{cases} \alpha = 3 \\ \beta - \alpha = 2 \\ \beta + \gamma = 1 \end{cases} \Longrightarrow \begin{cases} \alpha = 3 \\ \beta = 5 \\ \gamma = -4 \end{cases}$

So f is a linear combination of g, h, and k

3) Is the vector $(1,-7,5)$ a linear combination of the vectors $(1,1,0)$ $(1,-1,1)$ and $(1,0,1)$

We write $(1,-7,5) = \alpha(1,1,0) + \beta(1,-1,1) + \gamma(1,0,1)$

So $(1,-7,5) = (\alpha+\beta+\gamma, \alpha-\beta, \beta+\gamma)$

$$\begin{cases} \alpha+\beta+\gamma = 1 \\ \alpha-\beta = -7 \\ \beta+\gamma = 5 \end{cases} \quad \text{so} \quad \begin{cases} \alpha = -4 \\ \beta = 3 \\ \gamma = 2. \end{cases}$$

So $(1,-7,5)$ is a linear combination of the vectors $(1,1,0)$, $(1,-1,1)$ and $(1,0,1)$.

Theorem:
the linear system $AX=b$ is Consistent if and only if b is a linear combination of the columns of A.

Definition let $S = \{v_1, \dots v_n\}$ be a set of vectors of V. We say is a spanning set of V or that S spans V if every element of V is a linear combination of the vectors $v_1, \dots, v_n$

Examples

1) Do $(1,-1)$ and $(2,1)$ span $\mathbb{R}^2$?

let $(x,y) \in \mathbb{R}^2$

$(x,y) = \alpha(2,1) + \beta(1,-1)$ gives

$(x,y) = (2\alpha+\beta, \alpha-\beta)$

so $\begin{cases} 2\alpha+\beta = x \\ \alpha-\beta = y \end{cases}$ thus $\begin{cases} \alpha = \frac{1}{3}(x+y) \\ \beta = \frac{1}{3}(x-2y) \end{cases}$

2) Does the set $S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \right\}$ span $M_2(\mathbb{R})$

we write $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \alpha \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \beta \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$
$+ \delta \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$

to get $\begin{cases} \alpha = a \\ \beta = c \\ \gamma+\delta = b \\ \gamma-\delta = d \end{cases} \Rightarrow \begin{cases} \alpha = a \\ \beta = c \\ \gamma = \frac{1}{2}(b+d) \\ \delta = \frac{1}{2}(b-d) \end{cases}$

So S spans $M_2(\mathbb{R})$.

<u>Theorem.</u> Let $S = \{v_1, \dots, v_n\}$ be a subset of $\mathbb{R}^n$ and A the matrix whose columns are the vectors $v_1 \dots v_n$. S spans $\mathbb{R}^n$ if and only if $\det A \neq 0$

<u>Example</u>

Do the vectors $(0,1,1)$, $(1,-1,0)$ and $(3,-5,-2)$ span $\mathbb{R}^3$?

We put $A = \begin{pmatrix} 1 & 0 & 3 \\ -1 & 1 & -5 \\ 0 & 1 & -2 \end{pmatrix}$

$$\det A = \begin{vmatrix} 1 & 0 & 3 \\ -1 & 1 & -5 \\ 0 & 1 & -2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \end{vmatrix} = 0$$

So the vectors do not span $\mathbb{R}^3$

<u>Theorem:</u> Let $S = \{v_1, \dots, v_n\}$ be a set of vectors of a vector space V. the set of all linear combinations of $v_1, \dots, v_n$ is a subspace W and is

is the smallest subspace of V containing S. We say W is the span of S and write $W = \langle S \rangle$

Examples

1) Let $W = \{ (x,y,z) \in \mathbb{R}^3 , 2x+3y-z=0 \}$ find a spanning set of W.

$$W = \{ (x, y, 2x+3y)) , x, y \in \mathbb{R} \}$$

$$= \{ x(1,0,2) + y(0,1,3) \quad x, y \in \mathbb{R} \}$$

So $W = \langle (1,0,2), (0,1,3) \rangle$

2) Let $S = \left\{ A_1 = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} , A_2 = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \right\}$

Does $A = \begin{pmatrix} 0 & 5 \\ -4 & -1 \end{pmatrix} \in \langle S \rangle$.

We write

$$\begin{pmatrix} 0 & 5 \\ -4 & -1 \end{pmatrix} = \alpha \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix} + \beta \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha + 3\beta & 2\alpha + \beta \\ -\alpha + \beta & \beta \end{pmatrix}$$

Fi)

to get

$$\begin{cases} \alpha + 3\beta = 0 \\ 2\alpha + \beta = 5 \\ -\alpha + \beta = -4 \\ \beta = -1 \end{cases} \Rightarrow \begin{cases} \alpha = +3 \\ \beta = -1 \end{cases}$$

so $A \in \langle S \rangle$.

Definition

We say that the vectors $v_1, \dots, v_n$ are linearly independent if the equation $\alpha_1 v_1 + \dots + \alpha_n v_n = 0$ implies $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$

Examples:

1) Show that $(1,1)$ and $(2,-1)$ are linearly independent in $\mathbb{R}^2$

suppose $\alpha_1 (1,1) + \alpha_2 (2,-1) = (0,0)$

We get $\begin{cases} \alpha_1 + 2\alpha_2 = 0 \\ \alpha_1 - \alpha_2 = 0 \end{cases}$ so $\begin{cases} \alpha_1 = 0 \\ \alpha_2 = 0 \end{cases}$

2) Show that $f(x) = 2x+1$, $g(x) = x^2$, $h(x) = 2$ are linearly independent in P_2

if we have $\alpha(2x+1) + \beta x^2 + \gamma . 2 = 0$ for any x then

$$\beta x^2 + 2\alpha x + \alpha + 2\gamma = 0 \text{ for any } x$$

this leads to $\begin{cases} \beta = 0 \\ 2\alpha = 0 \\ \alpha + 2\gamma = 0 \end{cases}$

So $\alpha = \beta = \gamma = 0$

<u>Definition</u>: we say that a set $S = \{v_1 \dots v_n\}$ is linearly dependent if it is not linearly independent.

Theorem

Let $S = \{v_1, \dots, v_n\} \subset \mathbb{R}^m$ and let A be the matrix whose columns are the vectors $v_1, \dots, v_n$

S is linearly independent if and only if the system $AX = 0$ has a unique solution $X = 0$ and if $m = n$ S is linearly independent if and only if A is invertible ie if and only if $\det A \neq 0$.

Corollary: if $m < n$ S is linearly dependent.

Examples:

1) is $S = \{(1,-1,0), (1,2,3), (1,1,1), (2,-1,1)\}$ independent in $\mathbb{R}^3$?

No since $m = 3$ and $n = 4$.

2) Show that

$$S = \{(1,1,0),\ (0,3,1),\ (2,1,-1)\}$$

is linearly independent in $\mathbb{R}^3$

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 3 & 1 \\ 0 & 1 & -1 \end{pmatrix} \qquad \det A = \begin{vmatrix} 1 & 0 & 2 \\ 1 & 3 & 1 \\ 0 & 1 & -1 \end{vmatrix}.$$

$$= \begin{vmatrix} 1 & 0 & 2 \\ 0 & 3 & -1 \\ 0 & 1 & -1 \end{vmatrix} = -2 \neq 0$$

so S is linearly independent.

3) Show that $\{e^x, xe^x, x^2e^x\}$ is linearly independent in $V = \{f\ \mathbb{R} \to \mathbb{R}\}$

suppose $\alpha e^x + \beta x e^x + \gamma x^2 e^x = 0$ for any x

put $x = 0$ to get $\alpha = 0$

So $\beta x e^x + \gamma x^2 e^x = 0$ for any x

So $\beta x + \gamma x^2 = 0$ for any x

differentiate to get $\beta + 2\gamma x = 0$ for any x

set $x = 0$ to get $\beta = 0$, then $\gamma x^2 = 0$ for any x

for $x = 1$ we get $\gamma = 0$.

Theorem:

Let $S = \{v_1, \ldots, v_n\}$ with $n \geq 2$

S is linearly dependent if and only if one of the vectors is a linear combination of the other vectors.

Example: find a and b so that

$(1, a, 2, b)$, $(1, -1, 2, 0)$ and $(1, 2, 1, 3)$

are linearly dependent.

we write $(1, a, 2, b) = \alpha(1, -1, 2, 0) + \beta(1, 2, 1, 3)$

so $\begin{cases} 1 = \alpha + \beta \\ a = 2\beta - \alpha \\ 2 = 2\alpha + \beta \\ b = 3\beta \end{cases}$ so $\begin{cases} \alpha = 1 \\ a = 2\beta - \alpha \\ \beta = \frac{b}{3} \\ \beta = 0 \end{cases}$

thus $a = -1$, $b = 0$

Basis and dimension

Definition:
Let $S = \{v_1, \dots, v_n\}$ be a subset of vector space V. We say S is a basis of V if

i) S spans V
ii) S is linearly independent

Examples

1) $\{(1,0,0), (0,1,0), (0,0,1)\}$ is a basis of $\mathbb{R}^3$

2) $S = \{1, x, x^2\}$ is a basis of
$\mathcal{P}_2 = \{ P = ax^2 + bx + c \ , \ a, b, c \in \mathbb{R} \}$

3) $S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$
is a basis of $M_2(\mathbb{R})$.

4) show that $S = \{2x^2 + 1, x + 1, 3x\}$ is a basis of $\mathcal{P}_2$.

We show that S spans P_2

Let $f = ax^2+bx+c$ be in P_2

$$ax^2+bx+c = \alpha(2x^2+1)+\beta(x+1)+\gamma.3x$$
$$= 2\alpha x^2+(\beta+3\gamma)x+\alpha+\beta$$

we get $\begin{cases} 2\alpha = a \\ \beta+3\gamma = b \\ \alpha+\beta = c \end{cases} \Rightarrow \begin{cases} \alpha = \frac{a}{2} \\ \beta = c-\frac{a}{2} \\ \gamma = \frac{1}{3}(b-(c-\frac{a}{2})) \end{cases}$

So $\{2x^2+1, x+1, 3x\}$ spans P_2.

We show $\{2x^2+1, x+1, 3x\}$ is linearly independent:

$$\alpha(2x^2+1)+\beta(x+1)+\gamma.3x = 0$$
$$2\alpha x^2+(\alpha+\beta)+(\beta+3\gamma)x = 0$$

so $\begin{cases} 2\alpha = 0 \\ \alpha+\beta = 0 \\ \beta+3\gamma = 0 \end{cases}$ and $\begin{cases} \alpha = 0 \\ \beta = 0 \\ \gamma = 0 \end{cases}$

<u>theorem</u> Let $S = \{v_1, \dots, v_n\}$ be a basis of V and $B = \{u_1, \dots, u_m\}$ a subset of V. If $m > n$ B is linearly

dependent

Corollary:
Let $S=\{v_1,\dots,v_n\}$ be a basis of the vector space V and $B=\{u_1,\dots,u_m\}$ a set of linearly independent vectors then $m \leq n$.

Corollary: If $S=\{v_1,\dots,v_n\}$ and $B=\{u_1,\dots,u_m\}$ are two bases of V then $m=n$.

Since all bases of V have the same number of vectors we call it the dimension of V and write

$$\dim V = n$$

Examples

1) $\dim \mathbb{R}^3 = 3$, $\dim \mathbb{R}^n = n$
2) $\dim M_2(\mathbb{R}) = 4$, $\dim M_{m,n}(\mathbb{R}) = m \cdot n$

If $P_n = \{ a_n x^n + \cdots + a_1 x + a_0 ,\ a_i \in \mathbb{R},\ 0 \le i \le n \}$

is the vector space of polynomials of degree $\le n$ then $\dim P_n = n+1$ and a basis of P_n is

$$\{ x^n, x^{n-1}, \cdots, x, 1 \}$$

<u>Theorem</u>

Let V be a vector space with $\dim V = n$ and $S = \{ v_1, \cdots, v_n \}$ a set of vectors of V.

1) If S is linearly independent then S is a basis of V.

2) If S spans V then it is a basis of V.

<u>Examples</u> 1) show that $\{ (-1,1,1), (1,1,1), (1,2,3) \}$ is a basis of $\mathbb{R}^3$.

8c)

It is enough to check S is linearly independent

$$\begin{vmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & -1 \\ 0 & 1 & +2 \\ 0 & 2 & 2 \end{vmatrix} = -2 \neq 0$$

so S is independent.

2) Show that $\{2x+1, 3x, x^2-1\}$ is a basis of P_2

We show it is linearly independent

$\alpha(2x+1) + \beta \cdot 3x + \gamma(x^2-1) = 0$

$\gamma x^2 + (2\alpha + 3\beta)x + \alpha - \gamma = 0$

$$\begin{cases} \gamma = 0 \\ 2\alpha + 3\beta = 0 \\ \alpha - \gamma = 0 \end{cases} \implies \begin{cases} \alpha = 0 \\ \beta = 0 \\ \gamma = 0 \end{cases}$$

Theorem:

Let V be a vector space with $\dim V = n$ and W a subspace of V

1) W has finite dimension and $\dim W \leq n$

2) $\dim W = n \iff V = W$

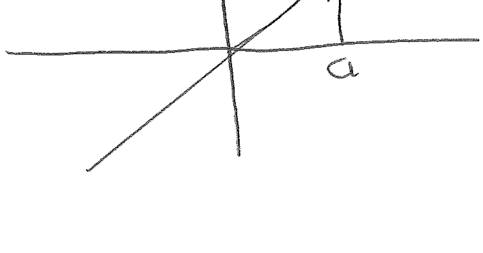
Notation: W a subspace of V is sometimes denoted by $W \leq V$

Examples:

1) If $W \subset \mathbb{R}^2$ is a subspace of $\mathbb{R}^2$ then either $\dim W = 2$ or $\dim W = 1$ or $\dim W = 0$

i) $\dim W = 0 \Rightarrow W = \{(0,0)\}$

ii) $\dim W = 1 \Rightarrow W = \langle (a,b) \rangle$



iii) $\dim |X| = 2 \Rightarrow |X| = \mathbb{R}^2$

Theorem

If $B = \{v_1, \ldots, v_n\}$ is a subset of V and B spans V then B contains a basis of V.

Algorithm A

Let B be a subset of $\mathbb{R}^n$. To get a basis of $\langle B \rangle$ we consider the matrix A whose rows are the vectors of B and we reduce A to the row echelon form (or reduced row echelon form). The basis of $\langle B \rangle$ consists of the non zero rows of the row echelon form.

Example: Let $B = \{(1,1,-1), (0,1,1), (2,1,3)\}$ find a basis of $\langle B \rangle$.

83)

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 2 & 1 & 3 \end{pmatrix} \xrightarrow{-2R_{13}} \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & 5 \end{pmatrix}$$

$$\xrightarrow{R_{23}} \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 6 \end{pmatrix} \xrightarrow{\frac{1}{6}R_3} \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

a basis of $\langle B \rangle$ is

$\{(1,1,-1), (0,1,1), (0,0,1)\}$

<u>Example</u>

Let $B = \{(1,1,3,-1), (1,1,-1,-1), (2,2,-1,-1), (3,3,1,-3)\}$

find a basis of $\langle B \rangle$

$$\begin{pmatrix} 1 & 1 & 3 & -1 \\ 1 & 1 & -1 & -1 \\ 2 & 2 & -1 & -1 \\ 3 & 3 & 1 & -3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 3 & -1 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & -7 & 1 \\ 0 & 0 & -8 & 0 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 1 & 3 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$ so a basis of $\langle B \rangle$

is $\{(1,1,3,-1), (0,0,1,0), (0,0,0,1)\}$

84)

Algorithm B

Let B be a subset of $\mathbb{R}^n$. To obtain a basis of $\langle B \rangle$ consider the matrix A whose columns are the vectors of B. Reduce A to the row echelon form (or reduced row echelon form) Then consider the columns containing the leading nonzero entries The columns of A with the same order as the columns containing the nonzero leading entries form a basis of $\langle B \rangle$.

Example: Let $B = \{(1,1,-1), (2,1,1), (1,0,1)$
$(-1,0,0)\}$

find a basis of $\langle B \rangle$

85)

$$A = \begin{pmatrix} 1 & 2 & 1 & -1 \\ 1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 & -1 \\ 0 & -1 & -1 & 1 \\ 0 & 3 & 2 & -1 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 2 & 1 & -1 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -2 \end{pmatrix}$$

So $S = \{(1,1,-1), (2,1,1), (1,0,1)\}$

is a basis of $\langle B \rangle$.

<u>Theorem</u>

If S is linearly independent in V there exists a basis T of V containing S

<u>Examples</u>.

1) find a basis of $\mathbb{R}^3$ containing $(-1,1,1)$.

$u_1 = (-1,1,1)$ $u_2 = (0,1,0)$, $u_3 = (0,0,1)$

is a basis of $\mathbb{R}^3$ since

$$\begin{vmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = -1 \neq 0$$

2) Find a basis of P_2 containing $2x^2+x-1$

$S = \{2x^2+x-1, x, 1\}$ is a basis of P_2.

3) Find a basis of P_2 containing $\{x^2+x+1, x-1\}$

$S = \{x^2+x+1, x-1, 1\}$ is a basis of P_2

Algorithm:

Let $\{v_1, \dots, v_k\}$ a linearly independent, in $\mathbb{R}^n$, set of vectors. To obtain a basis of $\mathbb{R}^n$ apply the following

1) choose a basis $\{u_1, \dots, u_n\}$ of $\mathbb{R}^n$

2) Apply algorithm B to $\{v_1, \dots, v_k, u_1, \dots, u_n\}$

Example

find a basis of $\mathbb{R}^3$ containing $\{(1,1,1),(1,-1,0)\}$.

We have $\{(1,0,0),(0,1,0),(0,0,1)\}$ is the usual basis of $\mathbb{R}^3$

write $A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}$

$$A \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & -2 & -1 & 1 & 0 \\ 0 & -1 & -1 & 0 & 1 \end{pmatrix} \xrightarrow{-2R_{32}} \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & -1 & -1 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{R_{23}} \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 & -2 \end{pmatrix} \xrightarrow{-R_2} \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & -2 \end{pmatrix}$$

so a basis of $\mathbb{R}^3$ is $\{(1,1,1),(1,-1,0),(1,0,0)\}$

Definition: Let $S = \{v_1 \ldots v_n\}$ be a basis of V, if $v = \alpha_1 v_1 + \ldots + \alpha_n v_n$ we call $\alpha_1, \ldots \alpha_n$ the coordinates of v w.r.t to the basis S and we write

$$[v]_S = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$$

and we call $[v]_S$ the coordinate vector with respect to the basis S

Examples

1) Let $S = \{(1,0,0), (0,1,0), (0,0,1)\}$

$S_1 = \{(1,1,0), (-1,1,0), (1,1,1)\}$

and $v = (1,2,3)$

find $[v]_S$ and $[v]_{S_1}$

clearly $[v]_S = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

write $(1,2,3) = \alpha(1,1,0) + \beta(-1,1,0) + \gamma(1,1,1)$

to get $\begin{cases} \alpha - \beta + \gamma = 1 \\ \alpha + \beta + \gamma = 2 \\ \gamma = 3 \end{cases}$ so $\begin{cases} \alpha - \beta = -2 \\ \alpha + \beta = -1 \end{cases}$

so $\alpha = \frac{-3}{2}$ $\beta = \frac{1}{2}$ $[v]_{S_1} = \begin{pmatrix} -3/2 \\ 1/2 \\ 3 \end{pmatrix}$

8915

2) Let $C = \{1, x, x^2\}$

$B = \{1+x, x+x^2, 1+x^2\}$

$P = 2+3x+4x^2$

find $[P]_C$ and $[P]_B$

clearly $[P]_C = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$

$$2+3x+4x^2 = \alpha(1+x) + \beta(x+x^2) + \gamma(1+x^2)$$

$$2+3x+4x^2 = \alpha+\gamma + (\alpha+\beta)x + (\beta+\gamma)x^2$$

so $\begin{cases} \alpha+\gamma = 2 \\ \alpha+\beta = 3 \\ \beta+\gamma = 4 \end{cases} \Longrightarrow \begin{cases} \alpha = \frac{1}{2} \\ \beta = \frac{5}{2} \\ \gamma = 3/2 \end{cases}$

and $[P]_B = \begin{pmatrix} \frac{1}{2} \\ 5/2 \\ \frac{3}{2} \end{pmatrix}$

Theorem

Let $B = \{v_1, \ldots, v_n\}$ and $C = \{u_1, \ldots, u_n\}$ be bases of a vector space V. Let ${}_C P_B$ be the matrix of order n whose columns are the vectors $[v_1]_C, \ldots, [v_n]_C$ then

$$[v]_C = {}_C P_B \, [v]_B \text{ for any vector } v.$$

and the matrix ${}_C P_B$ has an inverse and its inverse is ${}_B P_C$ whose columns are $[u_1]_B, \ldots, [u_n]_B$

${}_C P_B$ is called the transition matrix from B to C.

Example Let $B = \{(1,0,0), (0,1,0), (0,0,1)\}$ be the usual basis of $\mathbb{R}^3$.

$C = \{(1,-1,2), (1,1,-1), (0,0,1)\}$ and $v = (1,2,3)$. find $[v]_B$, $[v]_C$.

obviously $[v]_B = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and

$$[v]_C = {}_CP_B\,[v]_B$$

$$= {}_BP_C^{-1}\,[v]_B = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 2 & -1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & 3 \\ 0 & 0 & 2 \end{pmatrix}^{T} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{3}{2} & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ \frac{3}{2} \\ \frac{11}{2} \end{pmatrix}$$

obviously we can also find $[v]_C$ as follows

$$(1,2,3) = \alpha(1,-1,2) + \beta(1,1,-1) + \gamma(0,0,1)$$

$$\begin{cases} \alpha + \beta = 1 \\ \beta - \alpha = 2 \\ 2\alpha - \beta + \gamma = 3 \end{cases} \Longrightarrow \begin{cases} \beta = 3/2 \\ \alpha = -1/2 \\ \gamma = \frac{11}{2} \end{cases} \Rightarrow [v_C] = \begin{pmatrix} -\frac{1}{2} \\ 3/2 \\ 11/2 \end{pmatrix}$$

<u>Example</u> let $B = \{1, x, x^2\}$

$C = \{x^2, x+1, x+2\}$ be bases of P_2

and $v = 2x^2 + x - 1$ find $[v]_B$ and $[v]_C$

92)

Clearly $[v]_B = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$

$$[v]_C = {}_C P_B [v]_B$$
$$= {}_B P_C^{-1} [v]_B$$
$$= \begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}^{-1} \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$$
$$= - \begin{pmatrix} 0 & 1 & -1 \\ 0 & -2 & 1 \\ -1 & 0 & 0 \end{pmatrix}^{T} \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & 1 \\ -1 & 2 & 0 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}$$

We can verify that

$$2x^2 + x - 1 = 2 \cdot x^2 + 3(x+1) - 2(x+2)$$

<u>Example</u> Let $C = \{(-1,1,0), (1,1,1), (1,1,0)\}$ and $B = \{(1,1,0), (0,1,1), (1,0,1)\}$ and $v = (3,0,-7)$ find $[v]_B$, $[v]_C$ and ${}_C P_B$ and ${}_B P_C$.

93) We compute ${}_C P_B$

$$(1,1,0) = \alpha(-1,1,0) + \beta(1,1,1) + \gamma(1,1,0)$$

So $\begin{cases} -\alpha+\beta+\gamma=1 \\ \alpha+\beta+\gamma=1 \\ \beta=0 \end{cases} \Rightarrow \begin{cases} \alpha=0 \\ \beta=0 \\ \gamma=1 \end{cases}$ (this was obvious)

$$(0,1,1) = a(-1,1,0) + b(1,1,1) + c(1,1,0)$$

$$\begin{cases} -a+b+c=0 \\ a+b+c=1 \\ b=1 \end{cases} \Rightarrow \begin{cases} a=1/2 \\ b=1 \\ c=-\frac{1}{2} \end{cases}$$

Finally

$$(1,0,1) = b_1(-1,1,0) + b_2(1,1,1) + b_3(1,1,0)$$

$$\begin{cases} -b_1+b_2+b_3=1 \\ b_1+b_2+b_3=0 \\ b_2=1 \end{cases} \Rightarrow \begin{cases} b_1=-1/2 \\ b_2=1 \\ b_3=-1/2 \end{cases}$$

So $${}_C P_B = \begin{pmatrix} 0 & \frac{1}{2} & -1/2 \\ 0 & 1 & 1 \\ 1 & -1/2 & -1/2 \end{pmatrix}$$

94)

$$_{B}P_{C} = {}_{C}P_{B}^{-1}$$

$$= \begin{pmatrix} 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 1 \\ 1 & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 0 & \frac{1}{2} & 1 \\ 1 & \frac{1}{2} & 0 \\ -1 & \frac{1}{2} & 0 \end{pmatrix}$$

and

$$[N]_C = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \quad \text{where}$$

$$(3,0,-7) = \alpha(-1,1,0) + \beta(1,1,1) + \gamma(1,1,0)$$

$$\begin{cases} -\alpha+\beta+\gamma = 3 \\ \alpha+\beta+\gamma = 0 \\ \beta = -7 \end{cases} \Longrightarrow \begin{cases} \alpha = -3/2 \\ \beta = -7 \\ \gamma = \frac{17}{2} \end{cases}$$

So $[N]_C = \begin{pmatrix} -3/2 \\ -7 \\ 17/2 \end{pmatrix}$ and

$$[v]_B = {}_B P_C [v]_C$$

$$= \begin{pmatrix} 0 & \frac{1}{2} & 1 \\ 1 & \frac{1}{2} & 0 \\ -1 & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} -3/2 \\ -7 \\ \frac{17}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 5 \\ -5 \\ -2 \end{pmatrix}$$

Rank of a matrix

Let A be a matrix of order $m \times n$.
We call the subspace spanned by the rows of A the row space of A and denote it by row A.
Similarly the subspace spanned by the columns of A is called the column space of A and is denoted by col$A$

Proposition: if B is obtained from A by applying a row operation then row A = row B

96)

Corollary

If R is the row echelon form of a matrix A then $\text{row} A = \text{row} R$

Definition

We call the rank of A and denote by $\text{rank} A$ the number $\dim(\text{row} A)$ which is equal to the number of nonzero rows in R, the row echelon form of A.

Remark: the nonzero rows of R form a basis of row A.

Example: find the rank of the matrix $A = \begin{pmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{pmatrix}$

$$A \longrightarrow \begin{pmatrix} 1 & 4 & 5 & 2 \\ 0 & -7 & -7 & -4 \\ 0 & 7 & 7 & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 4 & 5 & 2 \\ 0 & -7 & -7 & -4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 4 & 5 & 2 \\ 0 & 1 & 1 & \frac{4}{7} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

97)

So ranb $A=2$ and a basis of row A is

$$\{(1,4,5,2),(0,1,1,\tfrac{4}{7})\}.$$

Theorem

If A is a matrix of order $m \times n$ then $\dim(\text{row}A) = \dim(\text{col}A)$
$= \text{rank}A$

If R is the reduced echelon form of A to get a basis of $\text{col}A$ we take the columns in A that correspond to columns in R containing the leading non zero entries.

Example

find a basis for $\text{row}A$ and a basis for $\text{col}A$ and $\text{rank}A$ for the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 1 & 3 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

99)

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 1 & 3 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix} \xrightarrow[-1R_{14}]{-1R_{13}} \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 0 & 3 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 \end{pmatrix}$$

$$\xrightarrow{-R_{23}} \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & -1 & 1 & 0 \end{pmatrix} \xrightarrow{-2R_{32}} \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & -2 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & -1 & 1 & 0 \end{pmatrix}$$

$$\xrightarrow{2R_{42}} \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 5 & -2 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & -1 & 1 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & +1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -\frac{2}{5} \end{pmatrix}$$

So rank $A = 4$ and a basis for row A is $\{(1,0,1,0,0), (0,1,-1,-1,1), (0,0,1,-1,0), (0,0,0,1,-2/5)\}$

a basis for col A is

$\{(1,0,1,1), (0,2,3,0), (1,0,0,0), (0,1,0,1)\}$

we also write this basis as

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Corollary

If A is of order $m \times n$ we have

1) $\operatorname{rank} A \leq n$ and $\operatorname{rank} A \leq m$
2) A is invertible if and only if $\operatorname{rank} A = n$
3) $\operatorname{rank} A = \operatorname{rank} A^T$
4) If P is invertible of order m and Q is invertible of order n then $\operatorname{rank}(PAQ) = \operatorname{rank} A$ where A is of order $m \times n$.

Theorem if A is of order $m \times n$ the following are equivalent

1) $AX = 0$ has only the zero solution
2) the columns of A are linearly independent

3) $\operatorname{rank} A = n$

4) $A^T A$ is invertible

<u>Theorem</u> If A is of order $m \times n$ the following are equivalent

1) $AX = b$ is consistent for any $b \in \mathbb{R}^m$

2) the columns of A span $\mathbb{R}^m$

3) $\operatorname{rank} A = m$

4) $A A^T$ is invertible.

<u>Definition</u>:

If A is of order $m \times n$ we define

$N(A) = \{ X \in \mathbb{R}^n ,\ AX = 0 \}$

$N(A)$ is a subspace of $\mathbb{R}^n$ called

102)

the zero space of A

We put nullity$(A) = \dim(N(A))$

<u>Definition</u>

For A of order $m \times n$

$\operatorname{Im} A = \{ Ax, \ x \in \mathbb{R}^m \}$

$\operatorname{Im} A$ is a subspace of $\mathbb{R}^m$

called the image space of A.

<u>Proposition</u>: If A is of order $m \times n$

$\dim \operatorname{Im} A = \dim \operatorname{col} A$ and $\operatorname{Im} A = \operatorname{col} A$

$= \operatorname{rank} A$

and nullity A

103)

Example

Let $A = \begin{bmatrix} 2 & -4 & 6 & 8 \\ 2 & -1 & 1 & 2 \\ 4 & -5 & 9 & 10 \\ 0 & -1 & 1 & 2 \end{bmatrix}$

find rank A, nullity A

$N(A)$ and $\operatorname{Im} A$.

We reduce A to

$$\begin{pmatrix} 2 & -4 & 6 & 8 \\ 0 & 3 & -5 & -6 \\ 0 & 3 & -3 & -6 \\ 0 & -1 & 1 & 2 \end{pmatrix} \xrightarrow[3R_{43}]{3R_{42}} \begin{pmatrix} 2 & -4 & 6 & 8 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 2 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & -2 & 3 & 4 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Longleftrightarrow \begin{cases} x-2y+3z+4t=0 \\ y-z-2t=0 \\ z=0 \end{cases}$$

$x=0$, $z=0$, $y=2t$

$N(A) = \{(0,2t,0,t),\ t\in\mathbb{R}\}$

$= \{t(0,2,0,1)\ \ t\in\mathbb{R}\}$

104)

$\dim N(A) = 1$, rank $A = 3$

$\operatorname{Im} A = \langle (2,2,4,0), (-4,-1,-5,-1)$
$(6,1,9,1) \rangle$

<u>Proposition</u>

If A is of order $m \times n$ then

1) rank A + nullity $A = n$

2) If rank $A = r$

nullity $A = n - r$ and

nullity $A^T = m - r$

3) the system $Ax = b$ is consistent if and only if

rank A = rank $[A|b]$

Example: is the system

$$\begin{cases} x+2y-3z=4 \\ 3x-y+5z=2 \\ 4x+y+2z=-2 \end{cases}$$

Consistent?

We reduce $\left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & 2 & -2 \end{array}\right)$ to

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & \frac{10}{7} \\ 0 & 0 & 0 & 1 \end{array}\right)$$

rank $A=2$ and rank $(A|b)=3$

the system is inconsistent.

__Remark__

If X_0 is a solution of the system $AX=b$ then any solution has the form

$$X = X_0 + \alpha_1 v_1 + \cdots + \alpha_k v_k$$

where $\{v_1, \dots, v_k\}$ is a basis of $N(A)$

Example

if A is of order 3×8 and $\operatorname{rank} A = 3$ find nullity A, dim row A, $\operatorname{rank}(A^T)$

$\operatorname{rank} A + \operatorname{nullity} A = 8$

so $\operatorname{nullity} A = 5$

$\dim \operatorname{row} A = \operatorname{rank} A = \operatorname{rank}(A^T) = 3$

107

Example

find m so that the nullity of

$$A = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -1 \\ 3 & 4 & -2 & m \end{pmatrix}$$

is equal to 1

we reduce the matrix

$$\begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -1 \\ 3 & 4 & -2 & m \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & m-6 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & m-5 \end{pmatrix}$$

nullity $A = 1 \iff \text{rank } A = 3$

so we must have $m \neq 5$

Inner product spaces

Definition:
Let V be a vector space. We say that the map $\langle\, , \rangle$ $V \times V \to \mathbb{R}$ is an inner product if it satisfies the following for u, v, w in V and α in $\mathbb{R}$ arbitrary.

1) $\langle u, v\rangle = \langle v, u\rangle$

2) $\langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$

3) $\langle \alpha u, v\rangle = \alpha \langle u, v\rangle$

4) $\langle v, v\rangle \geq 0$

5) $\langle v, v\rangle = 0 \iff v = 0$

Examples
1) in $\mathbb{R}^n$ for $u = (u_1, \dots, u_n)$ and $v = (v_1, \dots, v_n)$ define

$$\langle u, v\rangle = u_1 v_1 + \dots + u_n v_n$$

this is an inner product

and in general for $\alpha_1, \dots \alpha_n > 0$

$\langle u, v \rangle = \alpha_1 u_1 v_1 + \dots + \alpha_n u_n v_n$

is also an inner product.

2) $V = C([a,b])$ the space of continuous functions with real values

define $\langle f, g \rangle = \int_a^b f(x) g(x) dx$

3) $V = P_2(x)$ the set of polynomials of degree less than or equal to 2

V is a vector space and if

$P = a_0 + a_1 x + a_2 x^2$

$Q = b_0 + b_1 x + b_2 x^2$

$\langle P, Q \rangle = a_0 b_0 + a_1 b_1 + a_2 b_2$ is

an inner product.

4) On $M_2(\mathbb{R}) = V$ if $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ we define

$\langle A, B\rangle = a_{11}b_{11} + a_{21}b_{21} + a_{12}b_{12} + a_{22}b_{22}$

this defines an inner product on $M_2(\mathbb{R})$.

Proposition

if V is an inner product space then for all u, v, w in V we have

1) $\langle 0, v\rangle = \langle v, 0\rangle = 0$

2) $\langle u, v+w\rangle = \langle u, v\rangle + \langle u, w\rangle$

3) $\langle u, \alpha v\rangle = \alpha\langle u, v\rangle$

4) $\langle u-v, w\rangle = \langle u, w\rangle - \langle v, w\rangle$

5) $\langle u, v-w\rangle = \langle u, v\rangle - \langle u, w\rangle$

Orthogonality.

If $v \in V$ we define the norm of v (or length) as $\|v\| = \sqrt{\langle v, v\rangle}$ and the distance between u and v as $\|u-v\| = d(u, v)$

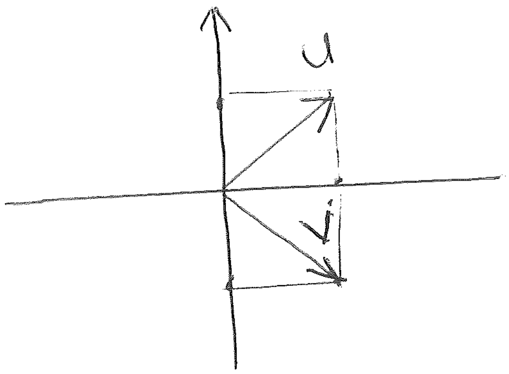
111

Examples

1) in $\mathbb{R}^2$ $\quad u=(1,1)$ with the usual inner product
$\quad v=(1,-1)$

$u-v=(0,2)$

$$d(u,v)=\|u-v\|=\sqrt{\langle u-v,u-v\rangle}$$
$$=\sqrt{0^2+2^2}=2$$



2) In P_2 $\quad P=1+x-x^2$
$\quad Q=2x^2+x-1$

with the usual inner product

$P-Q=2-3x^2$

$$\|P-Q\|=\sqrt{2^2+(-3)^2}=\sqrt{13}$$

3) In $\mathbb{R}^n$ with $u=(a_1,\dots a_n)$
$v=(b_1,\dots b_n)$

$$\|u-v\|=\sqrt{(b_1-a_1)^2+\dots+(b_n-a_n)^2}$$

is the euclidean distance

112/

4) In $C[0,\pi]$

$$\langle f,g\rangle = \int_0^{\pi} f(x)g(x)\,dx$$

defines an inner product

if $f_1 = \cos x$, $g_1 = \sin x$

$$d(f_1,g_1) = \|f_1-g_1\| = \sqrt{\langle f_1-g_1, f_1-g_1\rangle}$$

$$= \left(\int_0^{\pi} (\cos x - \sin x)^2\,dx\right)^{1/2}$$

$$= \left(\int_0^{\pi} (1-2\sin x\cos x)\,dx\right)^{1/2}$$

$$= \sqrt{\pi}.$$

Cauchy schwarz inequality

If V is an inner product space we have for all u,v in V

1) $|\langle u,v\rangle| \leq \|u\|\cdot\|v\|$

2) $|\langle u,v\rangle| = \|u\|\cdot\|v\|$ if and only if u and v are linearly dependent ie there exists $\alpha \in \mathbb{R}$ such that $v=\alpha u$.

113)

Examples

1) in $\mathbb{R}^n$ if $u=(a_1, \dots, a_n)$
$v=(b_1, \dots, b_n)$

with the usual inner product

$\langle u,v\rangle = a_1b_1 + \dots + a_nb_n$

$\|u\| = \sqrt{\langle u,u\rangle} = \sqrt{a_1^2 + \dots + a_n^2}$

$\|v\| = \sqrt{b_1^2 + \dots + b_n^2}$ so

$|a_1b_1 + \dots + a_nb_n| \leq (a_1^2 + \dots + a_n^2)^{1/2}(b_1^2 + \dots + b_n^2)^{\frac{1}{2}}$

2) In $C[a,b]$ with the inner product $\langle f,g\rangle = \int_a^b f(x)g(x)\,dx$

we get

$\left|\int_a^b f(x)g(x)\,dx\right| \leq \left(\int_a^b f^2(x)\,dx\right)^{1/2}\left(\int_a^b g^2(x)\,dx\right)^{\frac{1}{2}}$

Proposition: if V is an inner product space

1) $\|u\| \geq 0$
2) $u=0 \Leftrightarrow \|u\|=0$
3) $\|\alpha u\| = |\alpha|\,\|u\|$
4) $\|u+v\| \leq \|u\| + \|v\|$

114)

Corollary

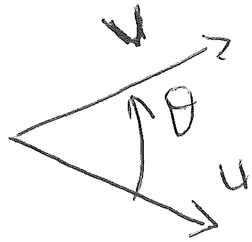
if $d(u,v) = \|u-v\|$ then

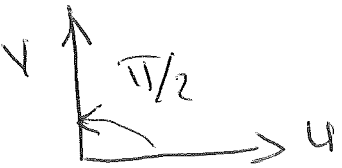
1) $d(u,v) \geq 0$
2) $d(u,v) = 0 \iff u = v$
3) $d(u,v) = d(v,u)$
4) $d(u,v) \leq d(u,w) + d(w,v)$

Definition Let u, v be non zero vectors in V we define

$$\cos\theta = \frac{\langle u,v\rangle}{\|u\| \cdot \|v\|} \qquad 0 \leq \theta \leq \pi$$

and call θ the angle between the two vectors u and v.





We say u and v are orthogonal if $\langle u,v\rangle = 0$

115)

Examples

1) in $\mathbb{R}^3$ $\quad u=(1,2,3) \quad v=(-1,-1,0)$

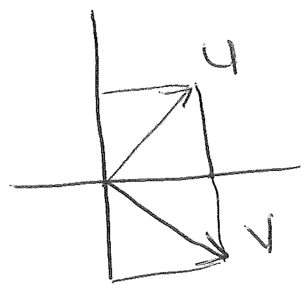
find $\cos\theta$ if the inner product is the usual inner product

$$\cos\theta = \frac{1.(-1)+2(-1)+3(0)}{\sqrt{1+4+9}\,\sqrt{1+1}} = \frac{-3}{\sqrt{28}}$$

$$\theta = \cos^{-1}\left(\frac{-3}{28}\right)$$

2) in $\mathbb{R}^2$ $\quad u=(1,1)\,,\ v=(-1,1)$

$$\cos\theta = \frac{-1+1}{\sqrt{2}\cdot\sqrt{2}} = 0$$



Remark: the orthogonality of vectors depends on the inner product

Example: $V = M_2(\mathbb{R})$

with the inner product

$$\left\langle \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \right\rangle = 2aa_1 + bb_1 + 2cc_1 + dd_1$$

11b)

put $C = \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}$ $D = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$

the angle between C and D

is given by

$$\cos\theta = \frac{4-1-4+1}{\sqrt{2+8+4+1}\sqrt{8+1+2+1}} = 0$$

so C and D are orthogonal

ie $\theta = \pi/2$

if we define

$$\left\langle \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \right\rangle = aa_1 + bb_1 + cc_1 + 4dd_1$$

we get

$$\left\langle \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \right\rangle = 2-2-1+4 = 3 \neq 0$$

so C and D are not orthogonal for this inner product.

Example

Define on $M_2(\mathbb{R})$ the product for $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

and $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$

$$\langle A, B \rangle = a_{11}b_{11} + a_{12}b_{21} + 3a_{22}b_{22}$$

Does this define an inner product on $M_2(\mathbb{R})$?

Let $C = \begin{pmatrix} 1 & 5 \\ -1 & 0 \end{pmatrix}$

$$\langle C, C \rangle = 1 - 5 + 0 = -4 < 0$$

So This does not define an inner product on $M_2(\mathbb{R})$

Example

on $\mathbb{R}^3$ we define

118

for $x = (x_1, x_2, x_3)$
$y = (y_1, y_2, y_3)$

$$\langle x, y\rangle = x_1 y_1 + x_2 y_1 + y_1 x_2 + x_3 y_3$$

Does this define an inner product?

clearly we have

$$\langle x, y\rangle = \langle y, x\rangle$$

$$\langle x+z, y\rangle = \langle x, y\rangle + \langle z, y\rangle$$

$$\langle \alpha x, y\rangle = \alpha \langle x, y\rangle$$

we show $\langle x, x\rangle = x_1^2 + 2x_1x_2 + x_3^2$ is not necessarily non negative

$$\langle x, x\rangle = (x_1 + x_2)^2 - x_2^2 + x_3^2$$

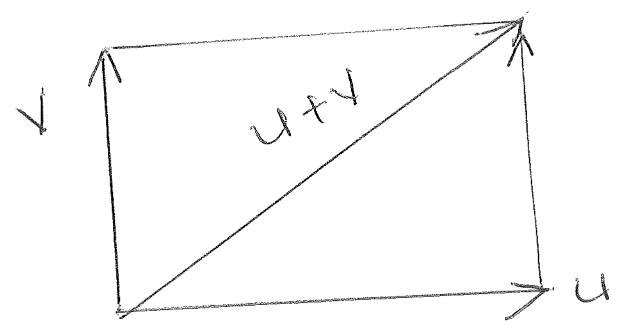
take $x = (-1, +1, 0)$

$\langle x, x\rangle = -1 < 0$ so it is not a scalar product.

Pythagorean theorem

If u and v are orthogonal in the vector space V then

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2$$



Orthonormal bases

Let $S \subset V$ be a subset of V.

We say S is orthogonal if any two different vectors of S are orthogonal.

We say S is orthonormal if it is orthogonal and for every $u \in S$ we have $\|u\| = 1$.

Example In $\mathbb{R}^3$ let $v_1 = \frac{1}{\sqrt{3}}(1,1,1)$, $v_2 = \frac{1}{\sqrt{2}}(-1,0,1)$ and $v_3 = \frac{1}{\sqrt{6}}(1,-2,1)$

For the usual inner product on $\mathbb{R}^3$

120) Show that $\{v_1, v_2, v_3\}$ is orthonormal.

We have

$$\langle v_1, v_2\rangle = \left\langle \left(\tfrac{1}{\sqrt3}, \tfrac{1}{\sqrt3}, \tfrac{1}{\sqrt3}\right), \left(\tfrac{-1}{\sqrt2}, 0, \tfrac{1}{\sqrt2}\right)\right\rangle$$

$$= -\frac{1}{\sqrt6} + \frac{1}{\sqrt6} = 0$$

$$\langle v_2, v_3\rangle = \left\langle \left(-\tfrac{1}{\sqrt2}, 0, \tfrac{1}{\sqrt2}\right), \left(\tfrac{1}{\sqrt6}, \tfrac{-2}{\sqrt6}, \tfrac{1}{\sqrt6}\right)\right\rangle$$

$$= -\frac{1}{\sqrt{12}} + \frac{1}{\sqrt{12}} = 0$$

Similarly $\langle v_1, v_3\rangle = 0$

Also $\langle v_1, v_1\rangle = \left\langle \left(\tfrac{1}{\sqrt3}, \tfrac{1}{\sqrt3}, \tfrac{1}{\sqrt3}\right), \left(\tfrac{1}{\sqrt3}, \tfrac{1}{\sqrt3}, \tfrac{1}{\sqrt3}\right)\right\rangle$

$$= \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$$

$$\|v_1\| = \sqrt{\langle v_1, v_1\rangle} = 1$$

Similarly $\|v_2\| = 1$, $\|v_3\| = 1$

So $\{v_1, v_2, v_3\}$ is an orthonormal set.

121)

Theorem:

1) if $S=\{u_1, \ldots u_n\}$ is orthogonal then S is linearly independant

2) If $S=\{u_1, \ldots u_n\}$ is an orthonormal basis of V then for any $u \in V$

$$u = \langle u, u_1\rangle u_1 + \ldots + \langle u, u_n\rangle u_n.$$

Gram schmidt process

Algorithm:

Let $\{v_1, \ldots v_n\}$ be a basis of V

To obtain an orthonormal basis of V we proceed as follows

put $u_1 = v_1$

$$u_2 = v_2 - \frac{\langle v_2, u_1\rangle u_1}{\|u_1\|^2}$$

$$u_3 = v_3 - \frac{\langle v_3, u_2\rangle u_2}{\|u_2\|^2} - \frac{\langle v_3, u_1\rangle u_1}{\|u_1\|^2}$$

$$\vdots$$

$$u_n = v_n - \frac{\langle v_n, u_{n-1}\rangle}{\|u_{n-1}\|^2} - \ldots - \frac{\langle v_n, u_1\rangle u_1}{\|u_1\|^2}$$

then $\{u_1, \ldots, u_n\}$ is an orthogonal basis and

$$\left\{\frac{u_1}{\|u_1\|}, \ldots, \frac{u_n}{\|u_n\|}\right\}$$ is an orthonormal basis of V.

Example

Let $v_1 = (1,1,0)$, $v_2 = (1,1,1)$, $v_3 = (-1,1,1)$

use the Gram Schmidt algorithm to get an orthogonal basis $\{u_1, u_2, u_3\}$ of $\mathbb{R}^3$ (usual inner product)

take $u_1 = v_1 = (1,1,0)$

$$u_2 = v_2 - \frac{\langle v_2, u_1\rangle}{\|u_1\|^2} u_1$$

$$= (1,1,1) - \frac{2}{2}(1,1,0) = (1,1,1) - (1,1,0) = (0,0,1)$$

$$u_3 = v_3 - \frac{\langle v_3, u_2\rangle}{\|u_2\|^2} u_2 - \frac{\langle v_3, u_1\rangle}{\|u_1\|^2} \cdot u_1$$

$$= (-1,1,1) - \frac{1}{1}(0,0,1) - 0\cdot(1,1,0) = (-1,1,0)$$

123)

So $\{u_1, u_2, u_3\} = \{(1,1,0), (0,0,1), (-1,1,0)\}$ is an orthogonal basis of $\mathbb{R}^3$.

and $\left\{\frac{u_1}{\|u_1\|}, \frac{u_2}{\|u_2\|}, \frac{u_3}{\|u_3\|}\right\}$ is an orthonormal basis of $\mathbb{R}^3$.

We get $\left\{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), (0,0,1), \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)\right\}$.

Example

Let $B = \{(1,1,1), (-1,1,0), (1,2,1)\}$

$= \{v_1, v_2, v_3\}$.

On $\mathbb{R}^3$ we define for $u = (a_1, a_2, a_3)$ and $v = (b_1, b_2, b_3)$

$\langle u, v\rangle = a_1 b_1 + 2a_2 b_2 + 3a_3 b_3$.

Use the Gram-Schmidt algorithm to obtain an orthonormal basis of $\mathbb{R}^3$.

124) We take $u_1 = v_1 = (1,1,1)$

$$\|u_1\|^2 = \langle u_1, u_1 \rangle = 1+2+3 = 6$$

$$\langle v_2, u_1 \rangle = \langle (-1,1,0), (1,1,1) \rangle$$

$$= -1 + 2 + 0 = 1$$

$$u_2 = v_2 - \frac{\langle v_2, u_1 \rangle}{\|u_1\|^2} u_1$$

$$= v_2 - \frac{1}{\|u_1\|^2} u_1$$

$$= (-1,1,0) - \frac{1}{6}(1,1,1)$$

$$= \left(-\frac{7}{6}, \frac{5}{6}, -\frac{1}{6}\right)$$

$$u_3 = v_3 - \frac{\langle v_3, u_2 \rangle}{\|u_2\|^2} u_2 - \frac{\langle v_3, u_1 \rangle}{\|u_1\|^2} u_1$$

$$\langle v_3, u_2 \rangle = \langle (1,2,1), \left(-\frac{7}{6}, \frac{5}{6}, -\frac{1}{6}\right) \rangle$$

$$= -\frac{7}{6} + \frac{20}{6} - \frac{3}{6} = \frac{10}{6} = \frac{5}{3}$$

$$\|u_2\|^2 = \frac{49}{36} + \frac{50}{36} + \frac{3}{36} = \frac{102}{36} = \frac{17}{6}.$$

$$\langle v_3, u_1 \rangle = \langle (1,2,1), (1,1,1) \rangle = 1+4+3 = 8$$

125)

$$u_3 = v_3 - \frac{\frac{5}{3}}{\frac{17}{6}} u_2 - \frac{8}{6} u_1$$

$$= (1,2,1) - \frac{10}{17}\left(-\frac{7}{6}, \frac{5}{6}, -\frac{1}{6}\right) - \left(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}\right)$$

$$= \left(\frac{49}{153}, \frac{37}{153}, -\frac{41}{153}\right)$$

We get an orthonormal basis

$$\left\{ \frac{u_1}{\|u_1\|}, \frac{u_2}{\|u_2\|}, \frac{u_3}{\|u_3\|} \right\}.$$

Example:

On $C[0,1]$ we consider the usual inner product

$$\langle f, g \rangle = \int_0^1 f(x) g(x) dx$$

Let W be the subspace spanned by

$$\{ p_1 = 1, \ p_2 = 2x - 1, \ p_3 = 12x^2 \}.$$

find an orthogonal basis of W.

We take $q_1 = 1 = p_1$

126)

$$q_2 = p_2 - \frac{\langle p_2, q_1 \rangle}{\|q_1\|^2} q_1$$

$$\langle p_2, q_1 \rangle = \int_0^1 (2x-1) \cdot 1 \, dx = \left[x^2 - x\right]_0^1 = 0$$

so

$$q_2 = 2x - 1 = p_2$$

$$q_3 = p_3 - \frac{\langle p_3, q_2 \rangle}{\|q_2\|^2} q_2 - \frac{\langle p_3, q_1 \rangle}{\|q_1\|^2} q_1$$

$$\langle p_3, q_2 \rangle = \int_0^1 12x^2(2x-1)\, dx$$

$$= \left[6x^4 - 4x^3\right]_0^1 = 2$$

$$\|q_2\|^2 = \int_0^1 (2x-1)^2 dx = \int_0^1 4x^2 - 4x + 1 \, dx$$

$$= \frac{4}{3} - 2 + 1 = \frac{1}{3}$$

$$\langle p_3, q_1 \rangle = \int_0^1 12x^2 dx = 4$$

$$\|q_1\|^2 = \int_0^1 1 \, dx = 1$$

127) We get

$$q_3 = p_3 - \frac{2}{\frac{1}{3}} p_2 - 4$$

$$= 12x^2 - 6(2x-1) - 4$$

$$= 12x^2 - 12x + 2$$

thus $\{q_1, q_2, q_3\} = \{1, 2x-1, 12x^2 - 12x + 2\}$

is an orthogonal basis of W.

Example

let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$

and $C = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

set $W = \langle A, B, C \rangle$ the subspace of $M_2(\mathbb{R})$ spanned by $\{A, B, C\}$. Find an orthonormal basis for W when the inner product on $M_2(\mathbb{R})$ is defined by

$$\left\langle \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \right\rangle$$

$$= a_{11} b_{11} + 2 a_{12} b_{12} + 2 a_{21} b_{21} + a_{22} b_{22}$$

we use the Gram-Schmidt process

$$A_1 = A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$B_1 = B - \frac{\langle B, A_1 \rangle}{\| A_1 \|^2} A_1$$

$$C_1 = C - \frac{\langle C, B_1 \rangle}{\| B_1 \|^2} B_1 - \frac{\langle C, A_1 \rangle}{\| A_1 \|^2} A_1$$

129)

$$\|A_1\|^2 = \|A\|^2 = 1$$

$$\langle B, A_1\rangle = \langle B, A\rangle = 1$$

so $B_1 = B - A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$$\langle C, B_1\rangle = \left\langle \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\rangle$$

$$= 2$$

$$\|B_1\|^2 = \left\langle \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\rangle = 2$$

$$\langle C, A_1\rangle = \left\langle \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\rangle = 1$$

so

$$C_1 = C - \frac{2}{2} B_1 - \frac{1}{1} A_1$$

$$= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Now $\{A_1, B_1, C_1\} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

is an orthogonal basis of W

$\left\{ \frac{A_1}{\|A_1\|}, \frac{B_1}{\|B_1\|}, \frac{C_1}{\|C_1\|} \right\} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

is an orthonormal basis of W.

Chapter 6

Linear transformations

Definition

Let V and W be vector spaces and $T\ V \longrightarrow W$ a map

We say T is a linear transformation if it satisfies

1) $T(u+v) = T(u) + T(v)$ for any u, v in V

2) $T(\alpha u) = \alpha T(u)$ for any $\alpha \in \mathbb{R}$ and $u \in V$

If $V = W$ we call T a linear operator.

Proposition If $T\ V \longrightarrow W$ is linear then

1) $T(0) = 0$ 2) $T(-u) = -T(u)$

3) $T(u-v) = T(u) - T(v)$.

131)

T is linear if and only if

$$T(\alpha u+\beta v)=\alpha T(u)+\beta Tv$$

for any u, v in V and α, β in $\mathbb{R}$.

<u>Examples</u>

1) $T \quad V \longrightarrow V$

$\qquad\quad v \longrightarrow v$

is linear, it is the identity map.

2) $T \quad V \longrightarrow W$

$\qquad\quad v \longrightarrow 0$

the zero map is linear

3) $T \quad V \longrightarrow V$

$\qquad\quad v \longrightarrow \alpha v$

where α is a fixed number.

4) $T \quad \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

$\qquad (x, y, z) \longrightarrow (2x+y, y-3z)$

is linear: Let $u=(x, y, z)$, $v=(x_1, y_1, z_1)$

$$T(u+v)=T(x+x_1, y+y_1, z+z_1)$$

$$=(2(x+x_1)+y+y_1, y+y_1-3(z+z_1))$$

$$=(2x+y, y-3z)+(2x_1+y_1, y_1 3z_1)$$

$$= T(u) + T(v)$$

Also $T(\alpha u) = T(\alpha x, \alpha y, \alpha z)$

$$= (2\alpha x + \alpha y, \alpha y - 3\alpha z)$$

$$= \alpha (2x + y, y - 3z)$$

$$= \alpha T(u).$$

5) $M_{m,n}(\mathbb{R}) \xrightarrow{T} M_{n,m}(\mathbb{R})$

$$A \longrightarrow 3A^T$$

is linear since

$$T(A+B) = 3(A+B)^T$$

$$= 3A^T + 3B^T = T(A) + T(B)$$

$$T(\alpha A) = 3(\alpha A)^T = 3\alpha A^T = 3\alpha\, T(A)$$

6) $\mathcal{P}_2 \xrightarrow{T} \mathcal{P}_3$

$$P \longrightarrow \int_0^x P(t)\,dt$$

is linear:

$$T(P+Q)(x) = \int_0^x (P+Q)(t)\,dt = \int_0^x P(t)\,dt + \int_0^x Q(t)\,dt$$

$$= T(P)(x) + T(Q)(x)$$

133)

$$T(\alpha P)(x) = \int_0^x (\alpha P)(t)\,dt$$
$$= \alpha \int_0^x P(t)\,dt$$
$$= \alpha T(P)(x)$$

7) $T\ \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$$(x,y,z) \longrightarrow \begin{pmatrix} 1 & 2 & 3 \\ -1 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= (x+2y+3z,\ -x+y,\ y+2z)$$

is linear

we write $T = T_A$.

<u>Proposition</u>

Let $T\ \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation and $\{e_1, \dots e_n\}$, $\{e'_1, \dots, e'_m\}$ the usual bases of $\mathbb{R}^n$ and $\mathbb{R}^m$.

Then $Tv = T_A v$ where $A = (a_{ij})$ is the matrix of order $m \times n$ which columns are $Te_1, \dots Te_n$ written in the basis $\{e'_1, \dots e'_m\}$

134)

$$\begin{array}{c} \\ \\ \\ \\ \\ \end{array}\begin{array}{cccc} Te_1 & \cdots & Te_n & \\ \left(\begin{array}{c} a_{11} \\ \vdots \\ \vdots \\ a_{1m} \end{array}\right. & & \left.\begin{array}{c} a_{n1} \\ \vdots \\ \vdots \\ a_{nm} \end{array}\right) & \begin{array}{c} e'_1 \\ \vdots \\ \vdots \\ e'_m \end{array} \end{array}$$

$Tv = Av$. A is called the matrix of T in the bases $\{e_1, \cdots e_n\}$ $\{e'_1 \cdots e'_m\}$

Examples

1) Let $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be defined by $T(x,y,z) = (2x-y+z,\ x+3y+z)$ find the matrix of T in the usual basis of $\mathbb{R}^3$ and $\mathbb{R}^2$

$T(1,0,0) = (2,1)$
$T(0,1,0) = (-1,3)$
$T(0,0,1) = (1,1)$

So $A = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 3 & 1 \end{pmatrix}$

$$T(x,y,z) = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

2) Assume $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is linear and $T(1,0,0) = (1,1,-1)$, $T(0,1,0) = (2,1,0)$, $T(0,0,1) = (1,1,3)$. Find $T(x,y,z)$ and the matrix of T in the usual basis of $\mathbb{R}^3$.

135)

$$T(x,y,z) = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ -1 & 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \begin{pmatrix} x+2y+z \\ x+y+z \\ -x+3z \end{pmatrix}$$

$$T(x,y,z) = (x+2y+z, x+y+z, -x+3z)$$

and the matrix is

$$\begin{array}{cccc} Te_1 & Te_2 & Te_3 & \\ \left(\begin{matrix} 1 \\ 1 \\ -1 \end{matrix}\right. & \begin{matrix} 2 \\ 1 \\ 0 \end{matrix} & \left.\begin{matrix} 1 \\ 1 \\ 3 \end{matrix}\right) & \begin{matrix} e_1 \\ e_2 \\ e_3 \end{matrix} \end{array}$$

3) Define $\mathbb{R}^2 \xrightarrow{T} \mathbb{R}^3$

$(x,y) \longrightarrow (2x+y, |y|, 3y.)$

is T linear?

$T(1,1) = (3,1,3)$

$T(1,-1) = (1,1,-3)$

$T(1,1) + T(1,-1) = (4,2,0)$

$T((1,1)+(1,-1)) = T(2,0) = (4,0,0)$

So T is not linear.

4) Let T $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$(x,y) \longrightarrow (x+y, x-y+1)$

is T linear?

$T(0,0)=(0,1)\neq(0,0)$

So T is not linear.

5) Let $T \quad M_2(\mathbb{R}) \longrightarrow M_2(\mathbb{R})$

$$A \longrightarrow A^2$$

is T linear

$$T\begin{pmatrix}1&0\\0&1\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}1&0\\0&1\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

$$T\begin{pmatrix}-1&0\\0&-1\end{pmatrix}=\begin{pmatrix}-1&0\\0&-1\end{pmatrix}\begin{pmatrix}-1&0\\0&-1\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

$$T\left(\begin{pmatrix}1&0\\0&1\end{pmatrix}+\begin{pmatrix}-1&0\\0&-1\end{pmatrix}\right)=T\begin{pmatrix}0&0\\0&0\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

$$T\begin{pmatrix}1&0\\0&1\end{pmatrix}+T\begin{pmatrix}-1&0\\0&-1\end{pmatrix}\neq T\left(\begin{pmatrix}1&0\\0&1\end{pmatrix}+\begin{pmatrix}-1&0\\0&-1\end{pmatrix}\right)$$

T is not linear.

The Bernel and the image of a linear transformation

Definition Let $T \quad V \longrightarrow W$ be a linear transformation

137) We define

$$N(T) = \{ v \in V, \ Tv = 0 \} = \operatorname{Ker} T$$

$$\operatorname{Im} T = \{ Tu, \ u \in V \}$$

$N(T)$ is called the kernel of T

$\operatorname{Im} T$ is called the image of T

__Proposition__

$\operatorname{Ker} T$ is a subspace of V

$\operatorname{Im} T$ is a subspace of W.

__Definition__

$$\operatorname{nullity}(T) = \dim \operatorname{Ker} T$$

$$\operatorname{rank}(T) = \dim \operatorname{Im} T$$

__Proposition__ If $\{v_1, \ldots v_n\}$ is a basis of V then $\{Tv_1, \ldots Tv_n\}$ spans $\operatorname{Im} T$.

Example

Let $T: \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ be defined

by

$$T(x,y,z,t) = (2x+y-z+3t,\ x+3z+t,\ x+y-4z+2t)$$

find $\operatorname{Ker} T$ and $\operatorname{Im} T$.

$$(x,y,z,t) \in \operatorname{Ker} T \iff T(x,y,z,t) = (0,0,0)$$

$$\iff \begin{cases} 2x+y-z+3t=0 \\ x+3z+t=0 \\ x+y-4z+2t=0 \end{cases} \iff \begin{cases} x+3z+t=0 \\ x+y-4z+2t=0 \\ 2x+y-z+3t=0 \end{cases}$$

$$\iff \begin{cases} x+3z+t=0 \\ x+y-4z+2t=0 \\ -y+7z-t=0 \end{cases} \iff \begin{cases} x=-3z-t \\ \\ y=7z-t \end{cases}$$

$$\operatorname{Ker} T = \{(-3z-t,\ 7z-t,\ z,\ t) \ z,t \in \mathbb{R}\}$$

$$= \{z(-3,7,1,0) + t(-1,-1,0,1),\ z,t \in \mathbb{R}\}$$

$$\operatorname{Ker} T = \langle (-3,7,1,0),\ (-1,-1,0,1) \rangle$$

[39]

To get a basis of $\operatorname{Im} T$ we write the matrix of T

$$M_T = \begin{array}{cccc} T(e_1) & T(e_2) & T(e_3) & T(e_4) \\ \end{array}$$
$$\begin{pmatrix} 2 & 1 & -1 & 3 \\ 1 & 0 & 3 & 1 \\ 1 & 1 & -4 & 2 \end{pmatrix}$$

where $\{e_1, e_2, e_3, e_4\}$ is the usual basis of $\mathbb{R}^4$. We reduce M_T

$$M_T \longrightarrow \begin{pmatrix} 1 & 0 & 3 & 1 \\ 1 & 1 & -4 & 2 \\ 2 & 1 & -1 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & -7 & 1 \\ 0 & 1 & -7 & 1 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & -7 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

by algorithm B a basis for $\operatorname{Im} T$ is $\{(2,1,1), (1,0,1)\}$.

We note that $\dim \operatorname{Ker} T + \dim \operatorname{Im} T = 4$

This is is general

Theorem

If $T\ V \longrightarrow W$ is linear and $\dim V < +\infty$ then

$$\dim \operatorname{Ker} T + \dim \operatorname{Im} T = \dim V$$

$$\text{nullity } T + \text{rank } T = \dim V$$

Definition

Let $T\ V \longrightarrow W$ be linear

1) We say T is one to one if $Tu = Tv$ implies $u = v$

2) We say T is onto if $\operatorname{Im} T = TV = W$

Proposition.

1) T is one to one if and only if $\operatorname{Ker} T = \{0\}$

2) if $T\ V \longrightarrow W$ is linear and $\dim V = \dim W$

then T is one to one if and only if T is onto.

Examples

1) Define a linear operator
$T \; \mathbb{R}^3 \longrightarrow \mathbb{R}^3$
such that $\operatorname{Im} T = \langle (4,5,6), (1,2,3) \rangle$

Solution

we define $T(1,0,0) = (4,5,6)$
$T(0,1,0) = (1,2,3)$
$T(0,0,1) = (0,0,0)$

we get
$T(x,y,z) = x\,T(1,0,0) + y\,T(0,1,0) + z\,T(0,0,1)$

$= (4x, 5x, 6x) + (y, 2y, 3y)$
$= (4x+y, 5x+2y, 6x+3y)$

$\operatorname{Im} T = \langle (4,5,6), (1,2,3) \rangle$

142)

since $T(x,y,z) = x(4,5,6) + y(1,2,3)$

it is clear that

$$\operatorname{Im} T = \langle (4,5,6), (1,2,3) \rangle$$

2) Define an operator on $\mathbb{R}^4$ such that

$$\operatorname{Ker} T = \langle (0,1,1,1), (1,2,3,4) \rangle$$

take

$$B = \{ (1,2,3,4), (0,1,1,1), (0,0,1,1), (0,0,0,1) \}$$
$$= \{ v_1, v_2, v_3, v_4 \}$$

B is a basis for $\mathbb{R}^4$.

Let $v \in \mathbb{R}^4$

$$v = \alpha v_1 + \beta v_2 + \gamma v_3 + \delta v_4$$

Define $Tv = \gamma v_3 + \delta v_4$

if $v = (x,y,z,t) = \alpha(1,2,3,4) + \beta(0,1,1,1)$
$+ \gamma(0,0,1,1) + \delta(0,0,0,1)$

we get

143)

$$\begin{cases} x=\alpha \\ y=2\alpha+\beta \\ z=3\alpha+\beta+\gamma \\ t=4\alpha+\beta+\gamma+\delta \end{cases}$$

we deduce

$$\begin{cases} \alpha=x \\ \beta=y-2x \\ \gamma=z-y-x \\ \delta=t-x-z \end{cases}$$

Thus

$$T(x,y,z,t)=\gamma(0,0,1,1)+\delta(0,0,0,1)$$

$$=(z-y-x)(0,0,1,1)+(t-x-z)(0,0,0,1)$$

$$=(0,0,z-y-x,-2x-y+t)$$

<u>Matrix of a linear transformation</u>

Let $T\ V \longrightarrow W$ be linear

$B=\{v_1,\dots,v_n\}$ a basis of V and

and $C = \{w_1, \dots, w_m\}$ a basis of W.

We write

$$Tv_1 = a_{11} w_1 + a_{12} w_2 + \dots + a_{m1} w_m$$

$$\vdots$$

$$Tv_n = a_{n1} w_1 + a_{n2} w_2 + \dots + a_{mn} w_m$$

If $v = \alpha_1 v_1 + \dots + \alpha_n v_m$ we get

$$[Tv]_C = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

We call $(a_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}}$ the matrix of T

and denote it by $[T]_B^C$

$$[T]_B^C = \begin{array}{c c} \begin{matrix} Tv_1 & \cdots & Tv_n \end{matrix} & \\ \begin{bmatrix} \cdot & \cdots & \cdot \\ \cdot \quad \cdot & \cdots & \cdot \\ & & \\ \cdot & & \cdot \end{bmatrix} & \begin{matrix} w_1 \\ \vdots \\ \\ w_m \end{matrix} \end{array}$$

the columns of $[T]_B^C$ are

$[Tv_1]_C, \dots [Tv_n]_C$

we have

$$[Tv]_C = [T]_B^C [v]_B$$

if $T: V \longrightarrow V$ and $B = C$

we get

$$[Tv]_B = [T]_B \cdot [v]_B$$

Examples

1) Let $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$$(x,y) \longrightarrow (2x+3y, x+y, y-2x)$$

the matrix of T in the bases

$B = \{(1,0), (0,1)\}$ and $C = \{(1,0,0), (0,1,0), (0,0,1)\}$

is $\begin{pmatrix} 2 & 3 \\ 1 & 1 \\ -2 & 1 \end{pmatrix}$

146)

because

$T(1,0) = (2,1,-2)$

$T(0,1) = (3,1,1)$

2) Let $T\ \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be defined by

$T(x,y,z) = (x+y+z, 2x-y+z)$

$B = \{(1,0,0), (0,1,0), (0,0,1)\}$

$C = \{(1,1), (-1,1)\} = \{w_1, w_2\}$

$T(1,0,0) = (1,2) = \alpha(1,1) + \beta(-1,1)$

gives $\begin{cases} \alpha - \beta = 1 \\ \alpha + \beta = 2 \end{cases}$ so $\alpha = \frac{3}{2}$, $\beta = \frac{1}{2}$

$T(1,0,0) = (1,2) = \frac{3}{2} w_1 + \frac{1}{2} w_2$

$T(0,1,0) = (1,-1) = -w_2$

$T(0,0,1) = (1,1) = w_1$

147)

So $$[T]_B^C = \begin{pmatrix} \frac{3}{2} & 0 & 1 \\ \frac{1}{2} & -1 & 0 \end{pmatrix}$$

3) Let $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be defined by

$$(x,y) \longrightarrow (2x+y, 2y, y-x)$$

and $\mathcal{B} = \{(1,1), (-1,1)\}$

$$C = \{(1,1,0), (1,0,1), (0,1,1)\}$$

Let us find $[T]_B^C$

$$T(1,1) = (3,2,0) = \alpha(1,1,0) + B(1,0,1) + \gamma(0,1,-1)$$

we get $\begin{cases} \alpha + B = 3 \\ \alpha + \gamma = 2 \\ B + \gamma = 0 \end{cases} \Rightarrow \begin{cases} \alpha = \frac{5}{2} \\ B = \frac{1}{2} \\ \gamma = -\frac{1}{2} \end{cases}$

So $T(1,1) = \frac{5}{2}(1,1,0) + \frac{1}{2}(1,0,1) - \frac{1}{2}(0,1,1)$

Similarly

$$T(-1,1) = -\frac{1}{2}(1,1,0) - \frac{1}{2}(1,0,1) + \frac{5}{2}(0,1,1)$$

we deduce $[T]_B^C$

148)

$$[T]_B^C = \begin{pmatrix} \frac{5}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{5}{2} \end{pmatrix}$$

4) If $T\ \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is linear

and $[T]_B = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$

where $B = \{(1,1), (-1,1)\}$

find $T(x,y)$ for all $(x,y) \in \mathbb{R}^2$

write $(x,y) = \alpha(1,1) + \beta(-1,1)$

$= (\alpha - \beta, \alpha + \beta)$

to get $\begin{cases} \alpha - \beta = x \\ \alpha + \beta = y \end{cases}$ so $\begin{cases} \alpha = \frac{1}{2}(x+y) \\ \beta = \frac{1}{2}(y-x) \end{cases}$

so if $v = (x,y) \in \mathbb{R}^2$

$$[v]_B = \begin{pmatrix} \frac{1}{2}(x+y) \\ \frac{1}{2}(y-x) \end{pmatrix}$$

thus

149)

$$[T\,v]_B = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2}(x+y) \\ \frac{1}{2}(y-x) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3}{2}x - \frac{1}{2}y \\ \frac{1}{2}x + \frac{3}{2}y \end{pmatrix}$$

So $T(x,y) = \left(\frac{3}{2}x - \frac{1}{2}y\right)(1,1) + \left(\frac{1}{2}x + \frac{3}{2}y\right)(-1,1)$

$$= (x-2y,\ 2x+y)$$

5) if $T\ P_2 \longrightarrow \mathbb{R}^2$

$$ax^2+bx+c \longrightarrow (a+2b,\ a-b+c)$$

and $C = \{(1,0),(0,1)\}$ and $B = \{1, x, x^2\}$

find $[T]_B^C$

$T(1) = (0,1)$, $T(x) = (2,-1)$, $T(x^2) = (1,1)$

So $[T]_B^C = \begin{pmatrix} 0 & 2 & 1 \\ 1 & -1 & 1 \end{pmatrix}$

If $T: V \longrightarrow W$ is linear and B a basis of V, C a basis of W we have $[T(v)]_C = [T]_B^C [v]_B$

Example

$T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$(x, y) \longrightarrow (x+y, x-y)$

$B = \{(1,1), (1,-1)\} = \{v_1, v_2\}$

$C = \{(1,0), (1,1)\} = \{w_1, w_2\}$

if $v = (2,1)$ find $[T(v)]_C$

we have two methods

the direct method: $T(2,1) = (3,1)$

$(3,1) = \alpha(1,0) + \beta(1,1) = (\alpha+\beta, \beta)$

so $\beta = 1$, $\alpha = 2$

so $[T(v)]_C = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

the second method uses the equality

$[T(v)]_C = [T]_B^C [v]_B$

151)

We compute $[T]_B^C$

$Tv_1 = T(1,1) = (2,0) = 2w_1$

$Tv_2 = T(1,-1) = (0,2) = a w_1 + b w_2$

$= (a+b, b)$

so $\begin{cases} b = 2 \\ a = -2 \end{cases}$

$Tv_2 = -2w_1 + 2w_2$

Thus $[T]_B^C = \begin{array}{cc} Tv_1 & Tv_2 \end{array}$ $\begin{array}{c} \\ \left(\begin{array}{cc} 2 & -2 \\ 0 & 2 \end{array}\right) \end{array} \begin{array}{l} w_1 \\ w_2 \end{array}$

Let us find $[v]_B$

$v = (2,1) = \alpha(1,1) + \beta(1,-1)$

$= (\alpha+\beta, \alpha-\beta)$

$\begin{cases} \alpha+\beta = 2 \\ \alpha-\beta = 1 \end{cases} \Rightarrow \begin{cases} \alpha = 3/2 \\ \beta = 1/2 \end{cases} \Rightarrow [v]_B = \begin{pmatrix} 3/2 \\ 1/2 \end{pmatrix}$

So $[Tv]_C = \begin{pmatrix} 2 & -2 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} \frac{3}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Proposition

Let $T: V \longrightarrow V$ be a linear operator ($\dim V < +\infty$)

Let $\mathcal{B}$ and C be two bases of V

then

$$[T]_C = {}_CP_B \cdot [T]_B \cdot {}_BP_C$$

<u>Examples</u>

1) $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$$(x,y,z) \longrightarrow (x+y, y+z, z-2x)$$

$\mathcal{B} = \{(1,0,0), (0,1,0), (0,0,1)\}$
$= \{v_1, v_2, v_3\}$

$C = \{(1,1,0), (1,0,1), (0,1,0)\} = \{w_1, w_2, w_3\}$

find $[T]_C$ and $[T]_B$

$T(1,0,0) = (1,0,-2)$, $T(0,1,0) = (1,1,0)$

$T(0,0,1) = (0,1,1)$

$\frac{(53)}{80}$

$$[T]_B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -2 & 0 & 1 \end{pmatrix}$$

$$_BP_C = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$_CP_B = {_BP_C}^{-1} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{pmatrix}.$$

We can find $_CP_B$ directly

$$_CP_B = \begin{array}{c} \begin{matrix} v_1 & v_2 & v_3 \end{matrix} \\ \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \end{array} \begin{matrix} w_1 \\ w_2 \\ w_3 \end{matrix}$$

$$v_1 = (1,0,0) = \alpha w_1 + \beta w_2 + \gamma w_3$$
$$= \alpha(1,1,0) + \beta(1,0,1) + \gamma(0,1,0)$$

$$\begin{cases} \alpha + \beta = 1 \\ \alpha + \gamma = 0 \\ \beta = 0 \end{cases} \Longrightarrow \begin{cases} \alpha = 1 \\ \gamma = -1 \\ \beta = 0 \end{cases}$$

154)

So $v_1 = w_1 - w_3$

Similarly $v_2 = (0,1,0) = w_3$

$$v_3 = (0,0,1) = -w_1 + w_2 + w_3$$

So $$_C P_B = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{pmatrix}$$

Finally

$$[T]_C = {}_C P_B \, [T]_B \, {}_B P_C$$

$$= \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 2 & 1 \\ -2 & -1 & 0 \\ -3 & -1 & 0 \end{pmatrix}$$

Note we can compute $[T]_C$ directly

155)

$$[T]_C = \begin{array}{c} \begin{array}{ccc} Tw_1 & Tw_2 & Tw_3 \end{array} \\ \begin{pmatrix} 4 & \cdot & \cdot \\ -2 & \cdot & \cdot \\ -3 & \cdot & \cdot \end{pmatrix} \end{array} \begin{array}{c} w_1 \\ w_2 \\ w_3 \end{array}$$

$$Tw_1 = T(1,1,0) = (2,1,-2)$$
$$= a(1,1,0) + b(1,0,1) + c(0,1,0)$$

$$\Rightarrow \begin{cases} a+b = 2 \\ a+c = 1 \\ b = -2 \end{cases} \quad \text{So} \quad \begin{cases} a = 4 \\ b = -2 \\ c = -3 \end{cases}$$

Similarly we can find the two other columns of $[T]_C$.

2) Let $T \; P_2 \longrightarrow P_2$ be linear, and $B = \{1, 1-x, x-x^2\}$
$= \{v_1, v_2, v_3\}$

and $C = \{1+x^2, 1-x^2, 2x\} = \{w_1, w_2, w_3\}$

Assume

156)

$$[T]_B = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ -1 & 0 & 3 \end{bmatrix}$$

find $[T]_C$.

we have $[T]_C = {}_CP_B \cdot [T]_B \cdot {}_BP_C$

first we compute ${}_CP_B$

$v_1 = 1 = a w_1 + b w_2 + c w_3$

$= a(1+x^2) + b(1-x^2) + 2c\,x$

$= a+b + (a-b)x^2 + 2c\,x$

So $\begin{cases} a-b=0 \\ c=0 \\ a+b=1 \end{cases} \Rightarrow \begin{cases} a=\frac{1}{2} \\ b=\frac{1}{2} \\ c=0 \end{cases}$ $\quad v_1 = \frac{1}{2}w_1 + \frac{1}{2}w_2$

Similarly

$v_2 = 1-x = \frac{1}{2}w_1 + \frac{1}{2}w_2 - \frac{1}{2}w_3$

$v_3 = x - x^2 = -\frac{1}{2}w_1 + \frac{1}{2}w_2 + \frac{1}{2}w_3$

So $\quad {}_C P_B = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$

We compute ${}_B P_C$ directly (We can also use ${}_B P_C = {}_C P_B^{-1}$)

$$w_1 = 1 + x^2 = a v_1 + b v_2 + c v_3$$
$$= a.1 + b(1-x) + c(x-x^2)$$
$$= a+b + (c-b)x - cx^2$$

So $\begin{cases} a+b=1 \\ c-b=0 \\ -c=1 \end{cases} \longrightarrow \begin{cases} a=2 \\ b=-1 \\ c=-1 \end{cases}$

$$w_1 = 2v_1 - v_2 - v_3$$

Similarly $w_2 = 1-x^2 = v_2 + v_3$

$w_3 = 2x = 2v_1 - 2v_2$

158)

so $$ {}_{B}P_{C} = \begin{pmatrix} 2 & 0 & 2 \\ -1 & 1 & -2 \\ -1 & 1 & 0 \end{pmatrix} $$

$$ [T]_C = {}_{C}P_{B}\,[T]_B\,{}_{B}P_{C} $$

$$ = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ -1 & 0 & 3 \end{pmatrix} \cdot {}_{B}P_{C} $$

$$ = \begin{pmatrix} \frac{9}{2} & -\frac{1}{2} & 4 \\ -\frac{1}{2} & \frac{5}{2} & 2 \\ -4 & 1 & -2 \end{pmatrix} $$

Eigenvalues Eigenvectors and diagonalization

Definition

Let A be a square matrix of order n

If for some vector $x \neq 0$ we have

$Ax = \lambda x$ for some real number λ

we say x is an eigen vector

and λ is an eigenvalue.

x is the eigenvector corresponding

to the eigen value λ.

Remark

1) the condition $x \neq 0$ is important

since $A0 = \lambda 0$ for any λ.

2) If $Ax = \lambda x$ then $Acx = \lambda cx$

for any $c \neq 0$, so cx is also an eigenvector.

3) If x is an eigenvector for the eigenvalue λ we have

$Ax = \lambda x$ so $(\lambda I - A)x = 0$

so $\ker(\lambda I - A) \neq \{0\}$ and thus

$\det(\lambda I - A) = 0$

Conversely if $\det(\lambda I - A) = 0$ then $\ker(\lambda I - A) \neq \{0\}$ and there exists $x \neq 0$ $(\lambda I - A)x = 0$ ie $Ax = \lambda x$.

<u>Definition</u>

The polynomial $p(\lambda) = \det(\lambda I - A)$ is called the characteristic polynomial of the matrix A. The solutions of the equation $p(\lambda) = 0$ are the eigenvalues of A.

Example

Find the eigenvalues and eigen vectors of the matrix

$$A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

$$\lambda I - A = \begin{pmatrix} \lambda-1 & +1 & 0 \\ 1 & \lambda-2 & 1 \\ 0 & 1 & \lambda-1 \end{pmatrix}$$

$$P(\lambda) = \begin{vmatrix} \lambda-1 & 1 & 0 \\ 1 & \lambda-2 & 1 \\ 0 & 1 & \lambda-1 \end{vmatrix} = (\lambda-1)\begin{vmatrix} \lambda-2 & 1 \\ 1 & \lambda-1 \end{vmatrix}$$

$$- \begin{vmatrix} 1 & 0 \\ 1 & \lambda-1 \end{vmatrix}$$

$$= (\lambda-1)\big((\lambda-1)(\lambda-2)-1\big) - (\lambda-1)$$

$$= (\lambda-1)(\lambda^2-3\lambda) = \lambda(\lambda-1)(\lambda-3)$$

so we have three eigenvalues

16.2

$\lambda_1 = 0, \lambda_2 = 1 \quad \lambda_3 = 3$

Let us find the corresponding eigenvectors

for $\lambda_1 = 0$ we solve

$$A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} x - y = 0 \\ -x + 2y - z = 0 \\ -y + z = 0 \end{cases} \Leftrightarrow x = y = z$$

so $(x, y, z) = (x, x, x) = x(1, 1, 1)$

so $\{(1,1,1)\}$ is a basis for the subspace corresponding to $\lambda_1 = 0$

Similarly we solve

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 1 \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} x - y = x \\ -x + 2y - z = y \\ -y + z = z \end{cases} \Leftrightarrow y = 0 \text{ and } z = -x$$

so $(x, y, z) = (x, 0, -x) = x(1, 0, -1)$

all eigenvectors corresponding to $\lambda_2 = 1$ are of the form $x(1,0,-1)$ $x \in \mathbb{R}$

For $\lambda_3 = 3$ we solve $A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3 \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} x - y = 3x \\ -x + 2y - z = 3y \\ -y + z = 3z \end{cases} \Leftrightarrow \begin{cases} 2x + y = 0 \\ x + y + z = 0 \\ y + 2z = 0 \end{cases}$$

$$\Longleftrightarrow \begin{cases} y = -2x \\ z = x \end{cases}$$

So the eigenvectors are of the form $x(1,-2,1) = (x,-2x,x)$

In general we denote by E_λ the subspace $E_\lambda = \{x,\ Ax = \lambda x\}$ and we call it the eigenspace corresponding to the eigenvalue λ.

Proposition

If $x_1, \ldots x_k$ are eigenvectors corresponding to different eigenvalues $\lambda_1, \ldots \lambda_k$ then $\{x_1, \ldots x_k\}$ is linearly independent

Proposition

If a matrix A has n different eigenvalues $\lambda_1, - \lambda_n$ then there exists a basis $\{v_1, - v_n\}$ of eigenvectors, that is the exists a basis of eigenvectors for $\mathbb{R}^n$

Example

$$A = \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}$$

$$\det(\lambda I - A) = \begin{vmatrix} \lambda+1 & -4 & 2 \\ 3 & \lambda-4 & 0 \\ 3 & -1 & \lambda-3 \end{vmatrix} = (\lambda-1)(\lambda-2)(\lambda-3)$$

We can show that $X_1=(1,1,1)$ the $X_2=(2,3,3)$ and $X_3=(1,3,4)$ are corresponding eigen vectors

Example

Find the eigenvalues and the eigenspaces of the matrix

$$A = \begin{pmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{pmatrix}$$

$$p(\lambda) = \det(\lambda I - A) = \begin{vmatrix} \lambda-5 & -4 & -2 \\ -4 & \lambda-5 & -2 \\ -2 & -2 & \lambda-2 \end{vmatrix}$$

$$= \lambda^3 - 12\lambda^2 + 21\lambda - 10$$

$$= (\lambda-1)(\lambda^2 - 11\lambda + 10)$$

$$= (\lambda-1)^2(\lambda-10)$$

so we have two eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 10$

we find the eigenspaces

for $\lambda_1 = 1$ we solve $A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 1 \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

167)

to get

$$\begin{cases} 5x+4y+2z = x \\ 4x+5y+2z = y \\ 2x+2y+2z = z \end{cases}$$

$$\Leftrightarrow \begin{cases} 4x+4y+2z=0 \\ 4x+4y+2z=0 \\ 2x+2y+z=0 \end{cases} \Leftrightarrow 2x+2y+z=0$$

so $E_1 = \{(x,y,z),\ 2x+2y+z=0\}$

$= \{(x,y,-2x-2y)\quad x,y\in\mathbb{R}\}$

$= \{x(1,0,-2)+y(0,1,-2),\ x,y\in\mathbb{R}\}$

$E_1 = \langle (1,0,-2),(0,1,-2)\rangle$

For $\lambda_2=10$ we solve $A\begin{pmatrix}x\\y\\z\end{pmatrix} = 10\begin{pmatrix}x\\y\\z\end{pmatrix}$

we get $\begin{cases} 5x+4y+2z=10x \\ 4x+5y+2z=10y \\ 2x+2y+2z=10z \end{cases}$

$$\Leftrightarrow \begin{cases} -5x+4y+2z=0 \\ 4x-5y+2z=0 \\ x+y-4z=0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x=y \\ z=\frac{x}{2} \end{cases}$$

$$E_{10} = \left\{ (x,y,z),\ y=x,\ z=\frac{x}{2}\quad x\in\mathbb{R} \right\}$$

$$= \left\{ (x,x,\tfrac{x}{2}),\ x\in\mathbb{R} \right\}$$

$$= \left\{ x(1,1,\tfrac{1}{2})\quad x\in\mathbb{R} \right.$$

$$E_{10} = \langle (1,1,\tfrac{1}{2}) \rangle$$

<u>Definition</u>:

We say that a matrix A is diagonalizable if there exists an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix

Example

Show that $A = \begin{pmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{pmatrix}$

is diagonalizable

$$p(\lambda) = \begin{vmatrix} \lambda+1 & -4 & 2 \\ 3 & \lambda-4 & 0 \\ 3 & -1 & \lambda-3 \end{vmatrix}.$$

$$= 2 \begin{vmatrix} 3 & \lambda-4 \\ 3 & -1 \end{vmatrix} + (\lambda-3) \begin{vmatrix} \lambda+1 & -4 \\ 3 & \lambda-4 \end{vmatrix}$$

$$= 2(-3-3\lambda+12) + (\lambda-3)\left((\lambda+1)(\lambda-4)+12\right)$$

$$= 6(3-\lambda) + (\lambda-3)(\lambda^2-3\lambda+8)$$

$$= (\lambda-3)(\lambda^2-3\lambda+2) = (\lambda-1)(\lambda-2)(\lambda-3)$$

$p(\lambda)=0 \iff \lambda=1$ or $\lambda=2$ or $\lambda=3$

so A has three eigenvalues

170)

We find E_1

We solve $A\begin{pmatrix}x\\y\\z\end{pmatrix} = 1.\begin{pmatrix}x\\y\\z\end{pmatrix}$

$$\Leftrightarrow \begin{cases} -x+4y-2z = x \\ -3x+4y = y \\ -3x+y+3z = z \end{cases}$$

$$\Leftrightarrow \begin{cases} -2x+4y-2z = 0 \\ -3x+3y = 0 \\ -3x+y+2z = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x = y \\ z = x \end{cases}$$

$$E_1 = \{(x,x,x),\ x \in \mathbb{R}\}$$
$$= \{x(1,1,1),\ x \in \mathbb{R}\}$$

We find E_2

$A\begin{pmatrix}x\\y\\z\end{pmatrix} = 2\begin{pmatrix}x\\y\\z\end{pmatrix} \Leftrightarrow$

171)

$$\Longleftrightarrow \begin{cases} -x+4y-2z=2x \\ -3x+4y=2y \\ -3x+y+3z=2z \end{cases}$$

$$\Longleftrightarrow \begin{cases} -3x+4y-2z=0 \\ -3x+2y=0 \\ -3x+y+z=0 \end{cases}$$

$$\Longleftrightarrow \begin{cases} y=\frac{3}{2}x \\ z=\frac{3}{2}x \end{cases}$$

$$E_2 = \left\{ \left(x, \tfrac{3}{2}x, \tfrac{3}{2}x\right), x \in \mathbb{R} \right\}$$

$$E_2 = \left\langle \left(1, \tfrac{3}{2}, \tfrac{3}{2}\right) \right\rangle = \langle (2,3,3) \rangle$$

For E_3 we solve $A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$\Longleftrightarrow \begin{cases} -x+4y-2z=3x \\ -3x+4y=3y \\ -3x+y+3z=3z \end{cases} \Longleftrightarrow \begin{cases} y=3x \\ z=4x \end{cases}$$

$$E_3 = \{ (x, 3x, 4x), x \in \mathbb{R} \} = \langle (1,3,4) \rangle$$

172

put $v_1 = (1,1,1)$
$v_2 = (2,3,3)$ $\quad C = \{v_1, v_2, v_3\}$
$v_3 = (1,3,4)$

We have $Av_1 = v_1$
$Av_2 = 2v_2$
$Av_3 = 3v_3$

C is a basis for $\mathbb{R}^3$ and

$$[A]_C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

If $B = \{(1,0,0), (0,1,0), (0,0,1)\}$

$A = [A]_B$

$$[A]_C = {}_CP_B [A]_B \, {}_BP_C$$

$${}_BP_C = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 4 \end{pmatrix} \quad \text{so}$$

$$[A]_{c} = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 4 \end{pmatrix}^{-1} \begin{pmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

<u>Proposition:</u>

1) If A is a matrix of order n and $\lambda_1, \dots, \lambda_n$ its eigenvalues then $\det A = \lambda_1 \cdot \lambda_2 \cdots \lambda_n$

2) A and A^T have the same eigenvalues

3) If A is invertible and λ is an eigenvalue of A then $\frac{1}{\lambda}$ is an eigen value of A^{-1}

4) If P is an invertible matrix A and $P^{-1}AP$ have the same eigenvalues.

Proposition

If A is of order n then A is diagonalizable If and only if it has n independent eigenvectors

Example

$$A = \begin{pmatrix} 4 & 4 & 4 \\ 6 & 6 & 5 \\ -6 & -6 & -5 \end{pmatrix}$$

show that A is diagonalizable

$$p(\lambda) = \begin{vmatrix} \lambda-4 & -4 & -4 \\ -6 & \lambda-6 & -5 \\ 6 & 6 & \lambda+5 \end{vmatrix} = \begin{vmatrix} \lambda-4 & -4 & -4 \\ -6 & \lambda-6 & -5 \\ 0 & \lambda & \lambda \end{vmatrix}$$

$$= (\lambda-4)\begin{vmatrix} \lambda-6 & -5 \\ \lambda & \lambda \end{vmatrix} + 6 \cdot \begin{vmatrix} -4 & -4 \\ \lambda & \lambda \end{vmatrix}$$

$$= (\lambda-4)(\lambda(\lambda-6)+5\lambda) + 6(-4\lambda+4\lambda)$$

$$= (\lambda-4)(\lambda^2-\lambda) = \lambda(\lambda-1)(\lambda-4)$$

175)

We find three eigen values

$\lambda_0 = 0$, $\lambda_1 = 1$, $\lambda_2 = 4$

Let us find the eigenspaces

for E_0 we solve

$$A\begin{pmatrix}x\\y\\z\end{pmatrix} = 0\cdot\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\Longleftrightarrow \begin{cases} 4x+4y+4z=0 \\ 6x+6y+5z=0 \\ -6x-6y-5z=0 \end{cases} \Longleftrightarrow \begin{cases} y=-x \\ z=0 \end{cases}$$

$$E_0 = \{(x,-x,0),\ x\in\mathbb{R}\}$$

$$= \langle (1,-1,0) \rangle$$

for E_1 we solve $A\begin{pmatrix}x\\y\\z\end{pmatrix} = 1\cdot\begin{pmatrix}x\\y\\z\end{pmatrix}$

$$\Longleftrightarrow \begin{cases} 4x+4y+4z=x \\ 6x+6y+5z=y \\ -6x-6y-5z=z \end{cases} \Longleftrightarrow \begin{cases} x=0 \\ y=-z \end{cases}$$

$$E_1 = \{(0,y,-y),\ y\in\mathbb{R}\} = \langle (0,1,-1) \rangle$$

176)

for E_4 we solve

$$A\begin{pmatrix}x\\y\\z\end{pmatrix}=4\begin{pmatrix}x\\y\\z\end{pmatrix}$$

$$\Leftrightarrow \begin{cases} 4x+4y+4z=4x \\ 6x+6y+5z=4y \\ -6x-6y-5z=+4z \end{cases}$$

$$\Leftrightarrow \begin{cases} y=2x \\ z=-2x \end{cases}$$

$$E_4=\{(x,2x,-2x),\ x\in\mathbb{R}\}$$

$$=\langle(1,2,-2)\rangle$$

A has three independant eigen vectors $v_0=(1,-1,0)$, $v_1=(0,1,-1)$ and $v_3=(1,2,-2)$

so A is diagonalizable and if $P=\begin{pmatrix}1&0&1\\-1&1&2\\0&-1&-2\end{pmatrix}$ then $P^{-1}AP=\begin{pmatrix}0&0&0\\0&1&0\\0&0&4\end{pmatrix}$

177)

Example

Let $A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$

is A diagonalizable?

$$\lambda I - A = \begin{pmatrix} \lambda-1 & -2 & 1 \\ 0 & \lambda+1 & -3 \\ 0 & 0 & \lambda-1 \end{pmatrix}$$

$$p(\lambda) = \det(\lambda I - A)$$
$$= (\lambda-1)^2(\lambda+1)$$

$$Av = v \Leftrightarrow A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} x+2y-z = x \\ -y+3z = \cancel{y} \\ z = z \end{cases} \Leftrightarrow \begin{cases} z = 2y \\ 3z = 2y \end{cases} \Leftrightarrow \{ y = z = 0$$

so $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is an eigenvector

$$E_1 = \{ (x,0,0) \ x \in \mathbb{R} \} = \{ x(1,0,0) \ x \in \mathbb{R} \}$$

$$Av = -v \Leftrightarrow$$

178)

$$A\begin{pmatrix}x\\y\\z\end{pmatrix} = -\begin{pmatrix}x\\y\\z\end{pmatrix}$$

$$\Leftrightarrow \begin{cases} x+2y-z=-x \\ -y+3z=-y \\ z=-z \end{cases} \Leftrightarrow \begin{cases} y=-x \\ z=0 \end{cases}$$

so $E_{-1} = \{(x,-x,0) \quad x\in\mathbb{R}\}$

$= \{x(1,-1,0) \quad x\in\mathbb{R}\}$

so we find two independent eigenvectors $(1,0,0)$ and $(1,-1,0)$

Thus A is not digonalizable.

<u>Remark</u> if

$$P^{-1}AP = D = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ & \ddots & & \\ 0 & \cdots & 0 & d_m \end{pmatrix}$$

then $$P^{-1}A^mP = D^m = \begin{pmatrix} d_1^m & 0 & \cdots & 0 \\ & \ddots & & \vdots \\ 0 & \cdots & 0 & d_n^m \end{pmatrix}$$

and

$$A^m = P D^m P^{-1}$$

<u>Example</u>

if $A = \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}$

find A^4

$$p(\lambda) = \det(\lambda I - A) = \begin{vmatrix} \lambda+1 & -4 & 2 \\ 3 & \lambda-4 & 0 \\ 3 & -1 & \lambda-3 \end{vmatrix}$$

$$= (\lambda-1)(\lambda-2)(\lambda-3)$$

we find three eigenvectors

$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix} \quad v_3 = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$

$\{v_1, v_2, v_3\}$ is a basis of $\mathbb{R}^3$

if $P = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 4 \end{pmatrix} \qquad P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

$$P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} = D$$

So $A^4 = P D^4 P^{-1}$

$$A^4 = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 81 \end{pmatrix} \begin{pmatrix} 3 & -5 & 3 \\ -1 & 3 & -2 \\ 0 & -1 & +1 \end{pmatrix}$$

$$= \begin{pmatrix} -29 & 10 & 20 \\ -45 & -104 & 150 \\ -45 & -185 & 231 \end{pmatrix}$$

<u>Definition</u>

Let $P(\lambda)$ be a polynomial

if $P(\lambda) = (\lambda - \lambda_1)^m Q(\lambda)$ and $Q(\lambda_1) \neq 0$

we say that λ_1 has multiplicity m

Example

$P(\lambda) = \lambda(\lambda - 1)^3 (\lambda+1)^2 (\lambda - 3)(\lambda^2 + 1)$

$\lambda=0$, $\lambda=3$ have multiplicity 1
$\lambda=-1$ has multiplicity 2
$\lambda=1$ has multiplicity 3

<u>Definition</u>

1) If λ_0 is an eigenvalue of A then the multiplicity of λ_0 is called algebraic multiplicity of λ_0 as a root of the equation $P(\lambda)=0$
2) If λ_0 is an eigenvalue of A we call $\dim E_{\lambda_0} = \dim \operatorname{Ker}(\lambda_0 I - A)$ the geometric multiplicity of λ_0.

<u>Example</u>

$$A=\begin{pmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{pmatrix}$$

182)

We saw that

$$p(\lambda) = (\lambda-1)^2(\lambda-10)$$

so the algebraic multiplicity of 1 is 2 and the algebraic multiplicity of 10 is 1.

We can check that (we saw this)

$$A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \iff 2x+2y+z=0$$

$$\begin{aligned} E_1 &= \{(x,y,z),\ x+y+\tfrac{1}{2}z=0\} \\ &= \{(x,y,-2x-2y)\ x,y\in\mathbb{R}\} \\ &= \{x(1,0,-2)+y(0,1,-2)\ x,y\in\mathbb{R}\} \end{aligned}$$

$\dim E_1 = 2 =$ geometric multiplicity of 1

Similarly we check $\dim E_{10} = 1$

so the geometric multiplicity of 10 is 1.

Theorem

1) Let A be a matrix of order n
A is diagonalizable if and only if the geometric multiplicity of each eigen value equals the algebraic multiplicity

2) The geometric multiplicity is less than or equal to the algebraic multiplicity.

Example

is $A = \begin{pmatrix} 5 & -3 & 0 & 9 \\ 0 & 3 & 1 & -2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$ diagonalizable?

For $p(\lambda) = (\lambda-5)(\lambda-3)(\lambda-2)^2$ we have three eigen values $\lambda_1=5$, $\lambda_2=3$ and $\lambda_3=2$

184)

for $\lambda_1 = 5$

$\dim E_5 = \dim \operatorname{Ker}(5I - A)$

$= 4 - \operatorname{rank}(5I - A)$

$$= 4 - \operatorname{rank}\begin{pmatrix} 0 & 3 & 0 & -9 \\ 0 & 2 & -1 & 2 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

$= 4 - 3 = 1$

for $\lambda_2 = 3$

$\dim E_3 = 4 - \operatorname{rank}(3I - A)$

$$= 4 - \operatorname{rank}\begin{pmatrix} -2 & 3 & 0 & -9 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$= 4 - 3 = 1$

For $\lambda_3 = 2$

$$\dim E_2 = \dim \ker(2I-A)$$

$$= 4 - \operatorname{rank}(2I-A)$$

$$= 4 - \operatorname{rank}\begin{pmatrix} -3 & 3 & 0 & -9 \\ 0 & -1 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= 4 - 2 = 2$$

so for all three eigenvalues the algebraic multiplicity equals the geometric multiplicity

A is diagonalizable.

Example let $A = \begin{pmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{pmatrix}$

is A diagonalizable?

18b)

$$\lambda I - A = \begin{pmatrix} \lambda-2 & -2 & 1 \\ -1 & \lambda-3 & 1 \\ 1 & 2 & \lambda-2 \end{pmatrix}$$

$$\det(\lambda I - A) = \begin{vmatrix} \lambda-2 & -2 & 1 \\ -1 & \lambda-3 & 1 \\ 1 & 2 & \lambda-2 \end{vmatrix}$$

$$= \begin{vmatrix} \lambda-2 & -2 & 1 \\ -1 & \lambda-3 & 1 \\ 0 & \lambda-1 & \lambda-1 \end{vmatrix}$$

$$= (\lambda-1) \begin{vmatrix} \lambda-2 & -2 & 1 \\ -1 & \lambda-3 & 1 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= (\lambda-1) \begin{vmatrix} \lambda-2 & -3 & 1 \\ -1 & \lambda-4 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= (\lambda-1) \begin{vmatrix} \lambda-2 & -3 \\ -1 & \lambda-4 \end{vmatrix}$$

187)

$$= (\lambda-1)\left((\lambda-2)(\lambda-4)-3\right)$$

$$= (\lambda-1)(\lambda^2-6\lambda+5)$$

$$= (\lambda-1)^2(\lambda-5)$$

$$v = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in E_1 \iff Av = v \iff$$

$$A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \iff \begin{cases} 2x+2y-z = x \\ x+3y-z = y \\ -x-2y+2z = z \end{cases}$$

$$\iff \begin{cases} x+2y-z = 0 \\ \end{cases}$$

$$E_1 = \{(x, y, x+2y),\ x, y \in \mathbb{R}\}$$

$$= \{x(1,0,\cancel{2}1) + y(0,1,2)\ \ x, y \in \mathbb{R}\}$$

$$= \langle (1,0,1), (0,1,2) \rangle$$

$\dim E_1 = 2 =$ algebraic multiplicity of 1

188)

For $\lambda_2 = 5$

$$A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 5\begin{pmatrix} x \\ y \\ z \end{pmatrix} \iff \begin{cases} 2x+2y-z=5x \\ x+3y-z=5y \\ -x-2y+2z=5z \end{cases}$$

$$\iff \begin{cases} -3x+2y-z=0 \\ x-2y-z=0 \\ -x-2y-3z=0 \end{cases} \iff \begin{cases} y=x \\ z=-x \end{cases}$$

$$E_5 = \{(x, \cancel{x}, -x) \quad x \in \mathbb{R}\}$$
$$= \{x(1,1,-1), \; x \in \mathbb{R}\}$$

$\dim E_5 = 1 =$ algebraic multiplicity of $\lambda_2 = 5$

So A is diagonalizable.

if $P = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & -1 \end{pmatrix}$ $\quad P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix}$

Example

If $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

is A digonalizable?

$$\lambda I - A = \begin{pmatrix} \lambda-1 & 0 & 0 \\ -1 & \lambda-1 & 0 \\ -1 & 0 & \lambda \end{pmatrix}$$

$$\det(\lambda I - A) = (\lambda-1)^2 \cdot \lambda = \lambda(\lambda-1)^2$$

$$A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Leftrightarrow \begin{cases} x = x \\ x+y = y \\ x = z \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ z = 0 \end{cases}$$

$$\text{so } E_1 = \{(0, y, 0) \quad y \in \mathbb{R}\}$$
$$= \{y(0,1,0) \quad y \in \mathbb{R}\}$$
$$= \langle (0,1,0) \rangle$$

$\dim E_1 = 1$, algebraic multiplicity of 1 is equal to 2

so A is not digonalizable.

Remark:

when the algebraic multiplicity is 1 then the geometric multiplicity is necessarily equal to 1.